Fundamental rate delay tradeoffs in multipath routed and network coded networks

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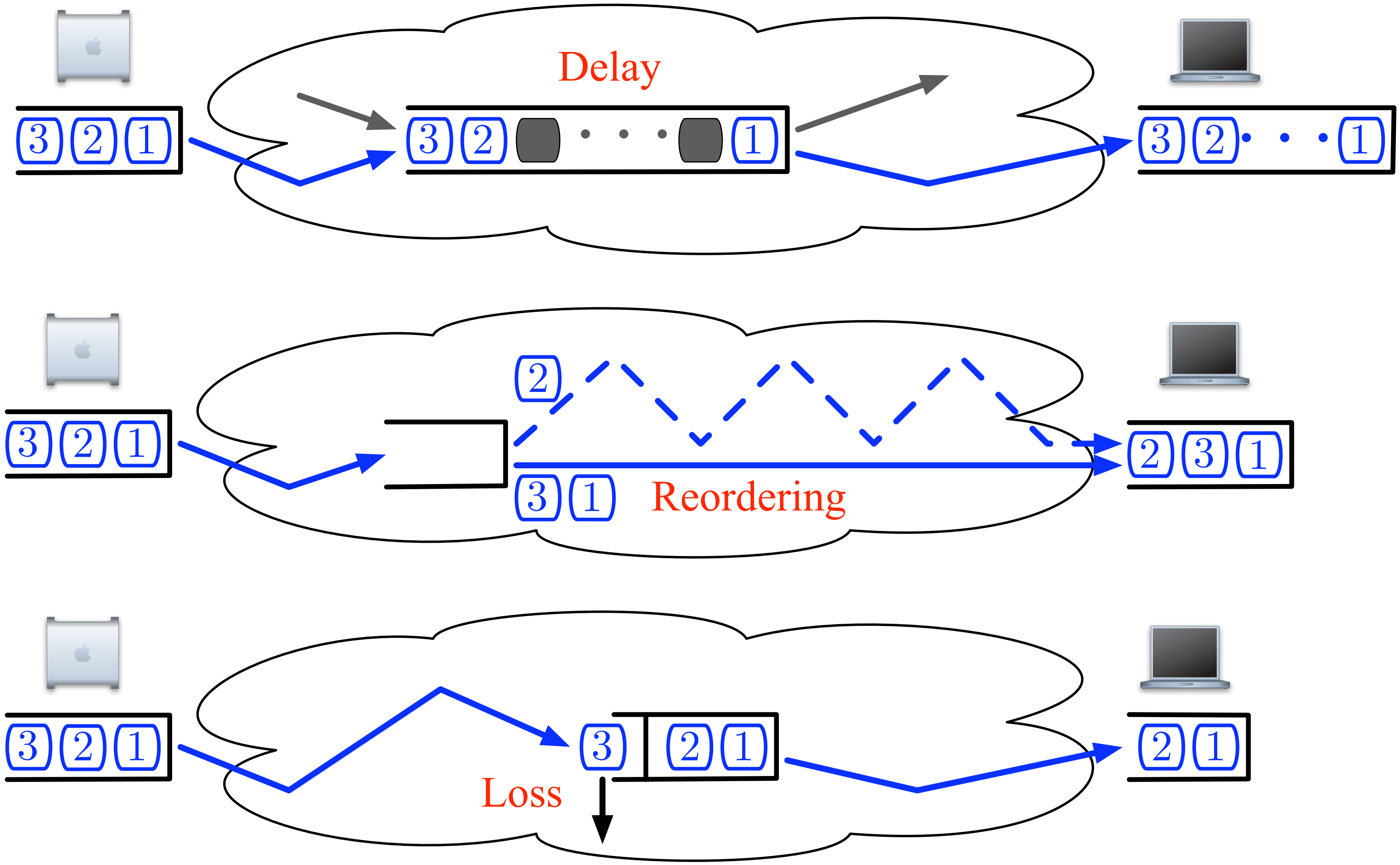
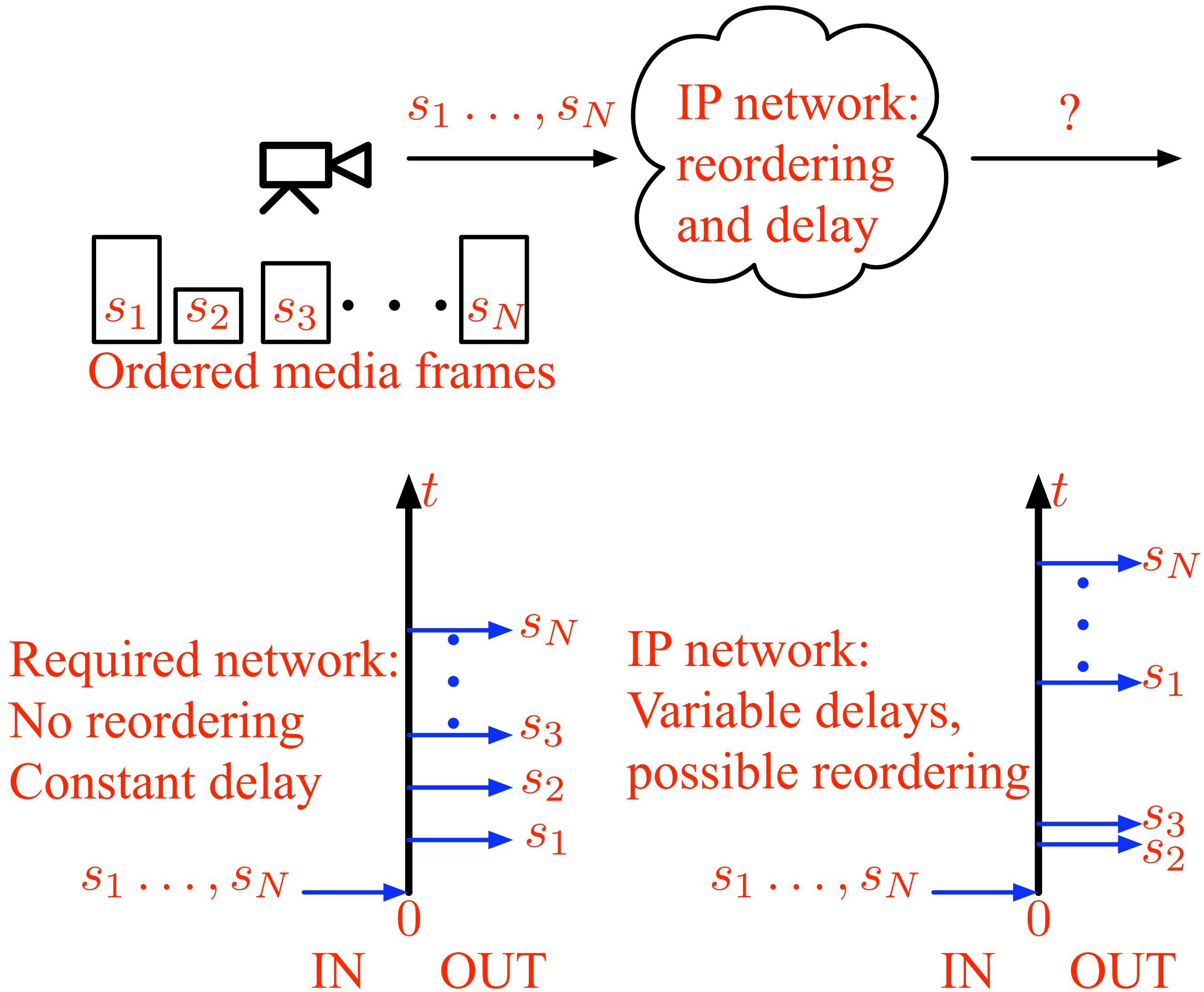
IP networks subject packets to delay, reordering, loss

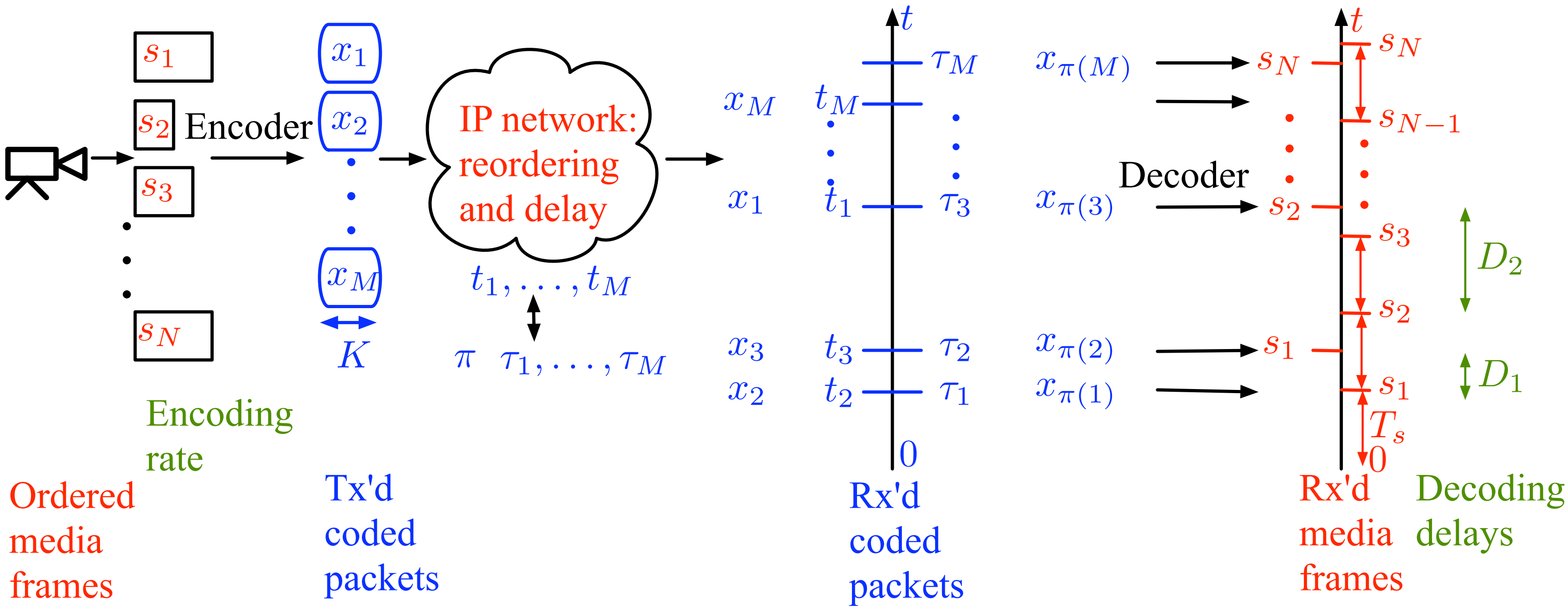


We will focus on delay and reordering...

...are ill-suited to uncoded transfer of ordered content



Competing costs of encoding rate and decoding delay

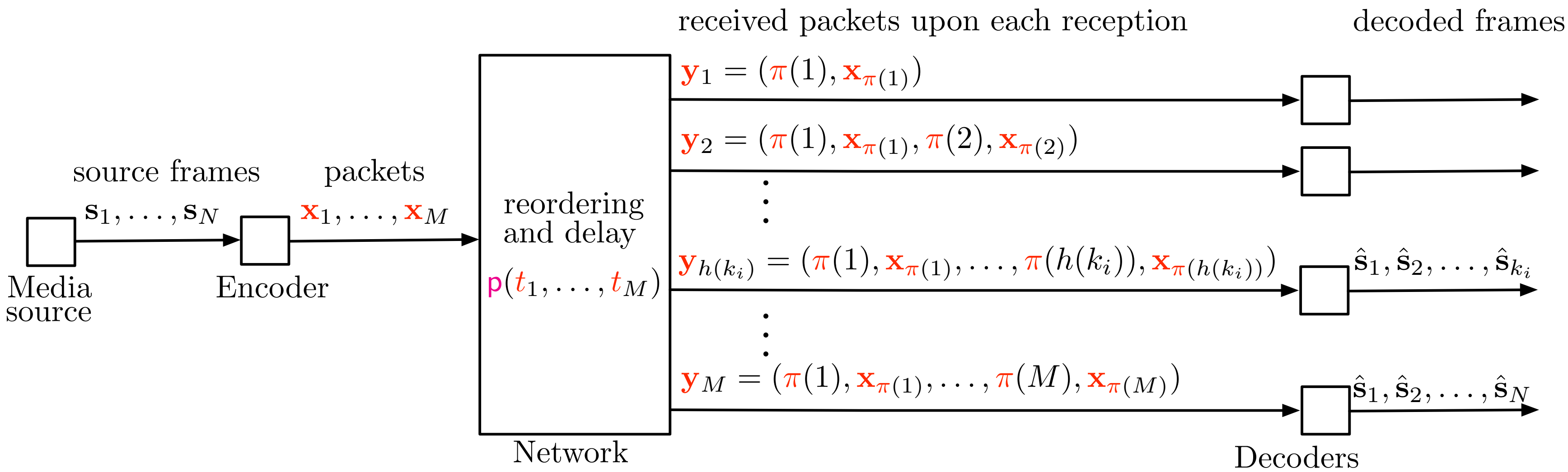


- Encoding rate: cost of adding too much redundancy
- Decoding delay: cost of not adding enough redundancy

Outline

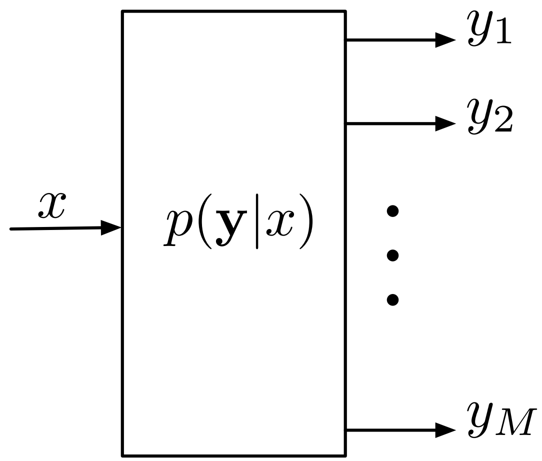
- **Degraded broadcast channels**
- **Tradeoff between rate and delay**
- Relaxation to a calculus of variations problem

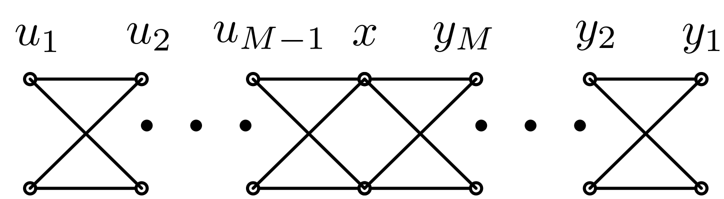
Reordering and delay as a degraded broadcast channel



- Each "receiver" is the cumulative set of arrived packets at each arrival time.
- The channel is degraded since $\mathbf{y}_1 \to \cdots \mathbf{y}_M$ form a Markov chain.
- Each packet is labeled, hence upon the first packet arrival the receiver knows $(\pi(1), x_{\pi(1)})$.

Capacity region of the degraded broadcast channel



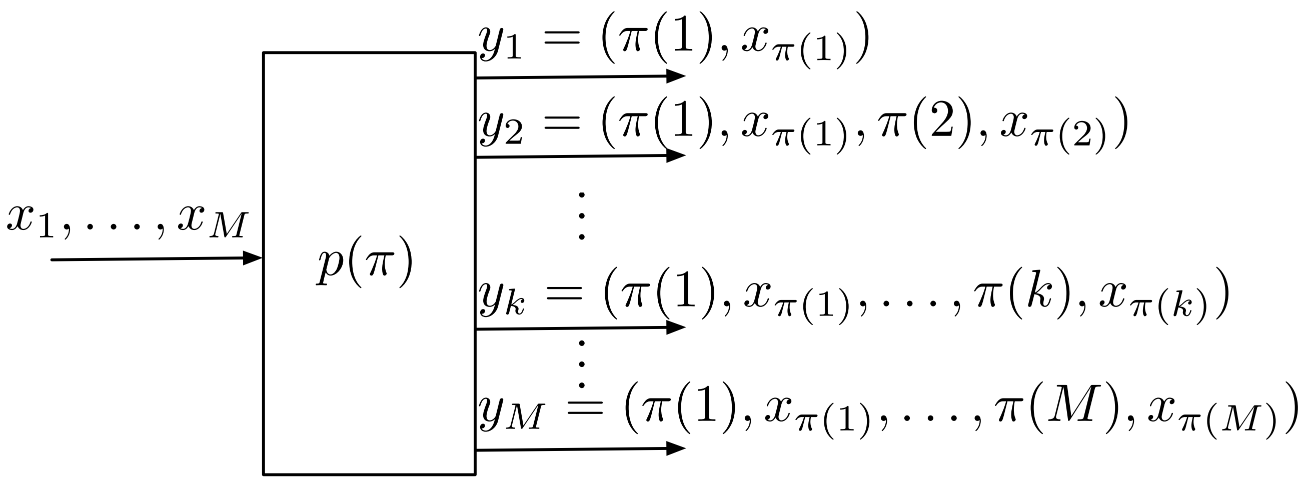


Theorem. (Cover 1972, El Gamal 1978): The capacity region of the degraded broadcast channel is the closure of the convex hull of the region $\mathcal{R}$ of rates satisfying

$$\begin{aligned} R_1 &\leq \mathcal{I}(y_1; u_1) \\ R_2 &\leq \mathcal{I}(y_2; u_2 | u_1) \\ &\vdots \\ R_M &\leq \mathcal{I}(y_M; x | u_1, \ldots, u_{M-1}) \end{aligned}$$

for $u_1 \rightarrow u_2 \rightarrow \cdots \rightarrow u_{M-1} \rightarrow x$ dummy rvs with bounded support.

Capacity region of the permutation broadcast channel



Proposition. The capacity region of the degraded broadcast channel is the closure of the convex hull of the region $\mathcal{R}$ of rates satisfying

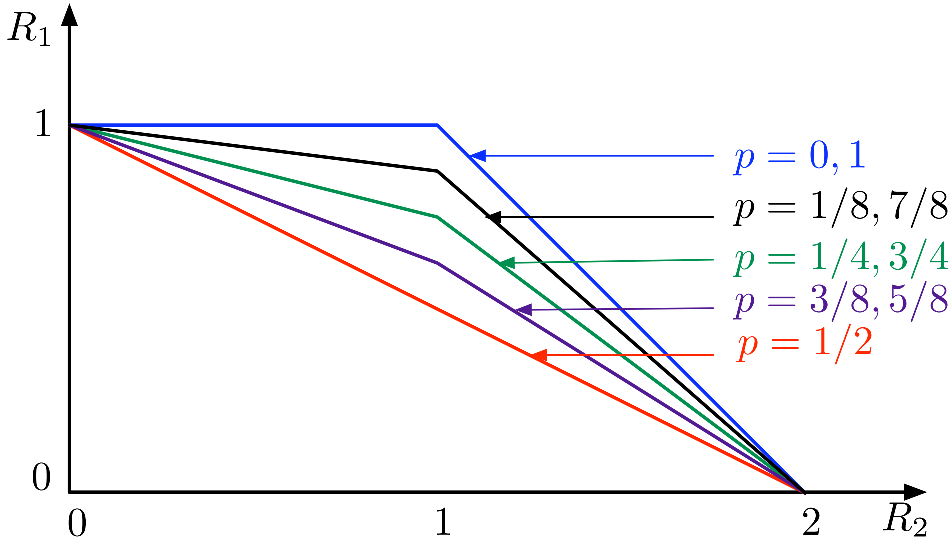
$$R_1 \leq \sum_{\pi} p(\pi) \left(\mathcal{H}(x_{\pi(1)}) - \mathcal{H}(x_{\pi(1)}|u_1) \right)$$

$$R_k \leq \sum_{\pi} p(\pi) \left(\mathcal{H}(x_{\pi(1)}, \ldots, x_{\pi(k)}|u_{k-1}) - \mathcal{H}(x_{\pi(1)}, \ldots, x_{\pi(k)}|u_1)|u_k \right)$$

$$R_M \leq \mathcal{H}(x_{\pi(1)}, \ldots, x_{\pi(M)}|u_{M-1})$$

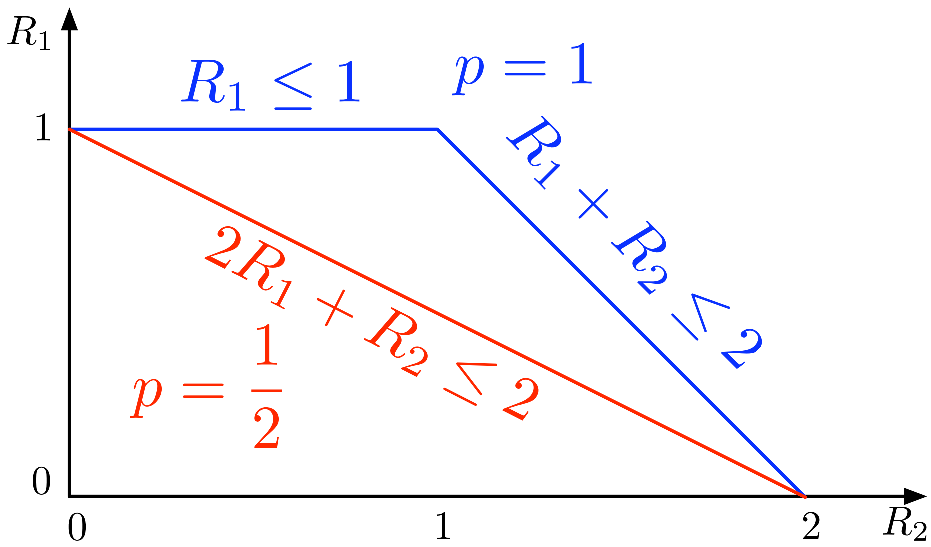
for $u_1 \rightarrow u_2 \rightarrow \cdots \rightarrow u_{M-1} \rightarrow \mathbf{x}$ dummy rvs with bounded support.

Capacity region for $M = 2$ packet permutation BC



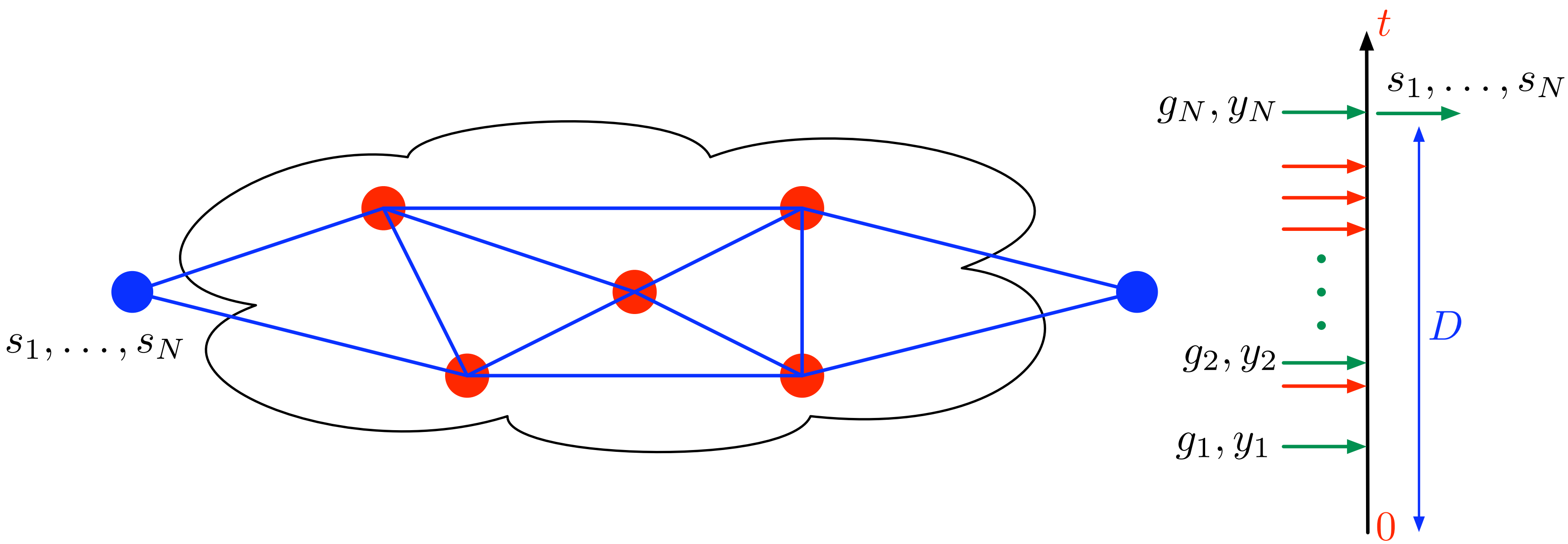
- Packets arrive in order $(1, 2)$ w.p. p, or out of order $(2, 1)$ w.p. $1 - p$.
- Achievable (R_1, R_2) means we may be sure of receiving $R_1 K$ bits upon first arrival and $R_2 K$ new bits upon second arrival.
- $R_1 = 1, R_2 = 0$ always achievable by setting $x_1 = x_2$.
- $R_1 = 0, R_2 = 2$ always achievable by not coding at all.
- When $p = 1/2$, increasing R_1 by 1 bit requires decreasing R_2 by 2 bits, since that bit must be added to both packets.

Capacity region for two specific permutation BCs



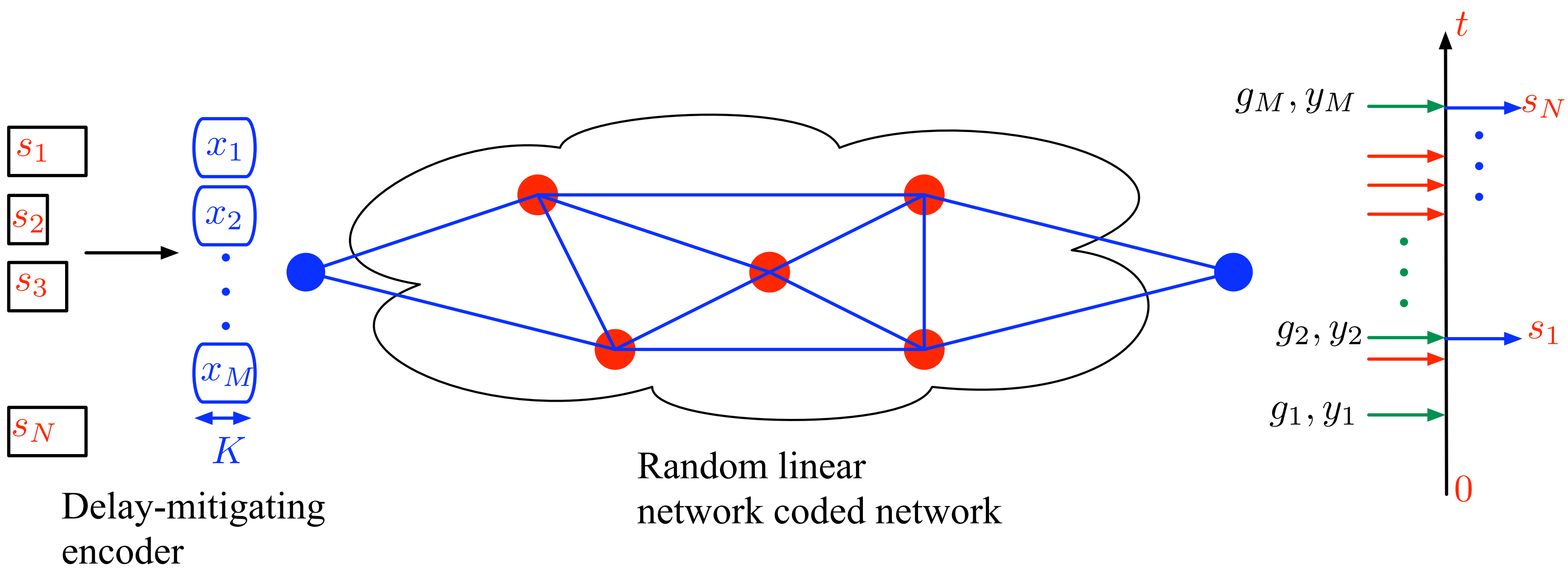
- Uniform channel: if $p(\pi) = 1/M!$ then $\mathcal{R}$ is given by $\sum_{i=1}^{M} R_i/i \leq 1$.
- Single permutation channel: if $p(\pi) = 1$ for some π then $\mathcal{R}$ is given by $\sum_{i=1}^{p} R_i \leq p$ for $p = 1, \ldots, M$.
- These two cases give lower and upper bounds on the region for general $p(\pi)$.

Point to point random linear network coded networks



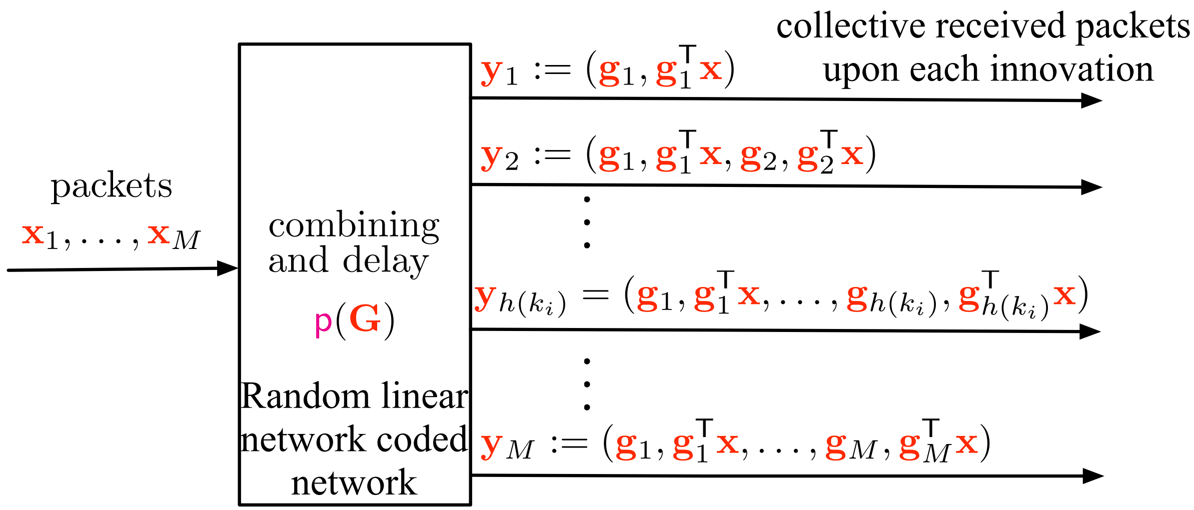
- Each node transmits a random linear combination of the combinations it has received.
- Each packet contains both the linear combination (y_k) and the corresponding encoding vector (g_k).
- All N packets are decodable as soon as N linearly independent combinations are received.
- Efficiency requires N be large $\Rightarrow$ potentially large delay.

Random linear network coded networks with delay mitigating outer codes



- Outer delay mitigating code and inner random linear network coded network.
- Redundancy of outer code permits frames to be decoded before receiving a full-rank matrix of encoding vectors.
- Redundancy reduces code rate, but reduces decoding delay.

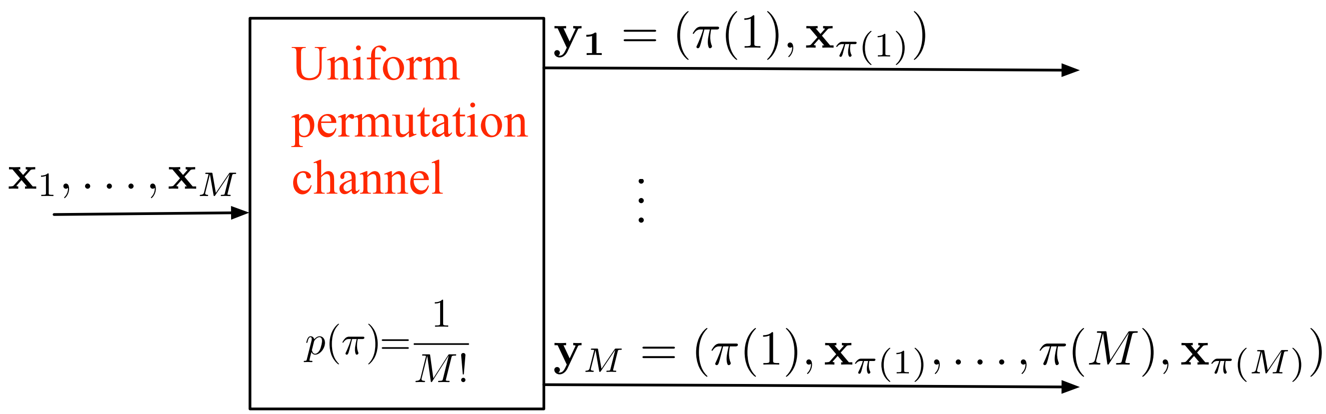
Broadcast channel for a random linear network coded network



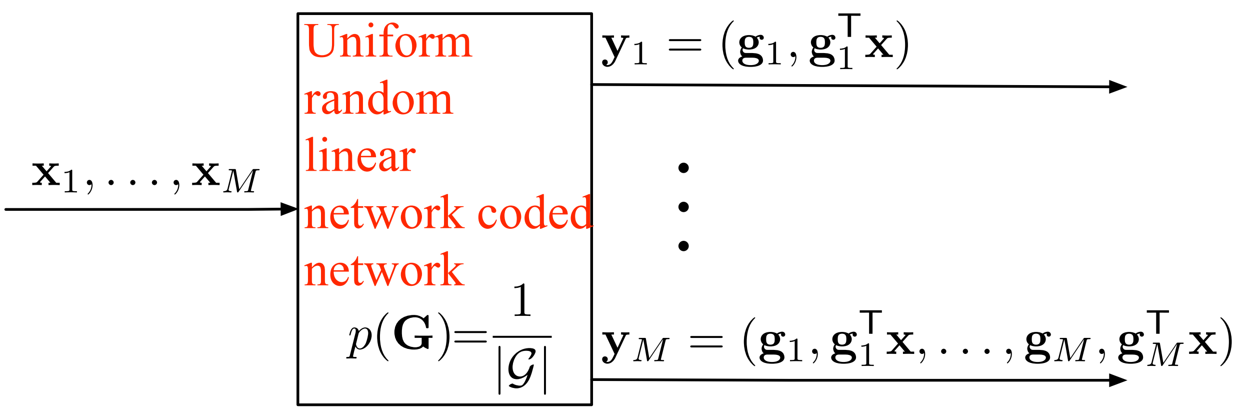
Proposition. The capacity region of the degraded broadcast channel is the closure of the convex hull of the region $\mathcal{R}$ of rates satisfying

$$
\begin{aligned}
R_1 &\leq \sum_{\mathbf{G}_1 \in \mathcal{G}_1} p(\mathbf{G}_1) \left(\mathcal{H}(\mathbf{G}_1 \mathbf{x}) - \mathcal{H}(\mathbf{G}_1 \mathbf{x} | u_1) \right) \\
R_k &\leq \sum_{\mathbf{G}_k \in \mathcal{G}_k} p(\mathbf{G}_k) \left(\mathcal{H}(\mathbf{G}_k \mathbf{x} | u_{k-1}) - \mathcal{H}(\mathbf{G}_k \mathbf{x} | u_k) \right) \\
R_M &\leq \mathcal{H}(\mathbf{x} | u_{M-1})
\end{aligned}
$$

Equal capacity regions for the uniform case



$$\sum_{i=1}^{M} \frac{R_i}{i} \leq 1$$



$$\sum_{i=1}^{M} \frac{R_i}{i} \leq 1$$

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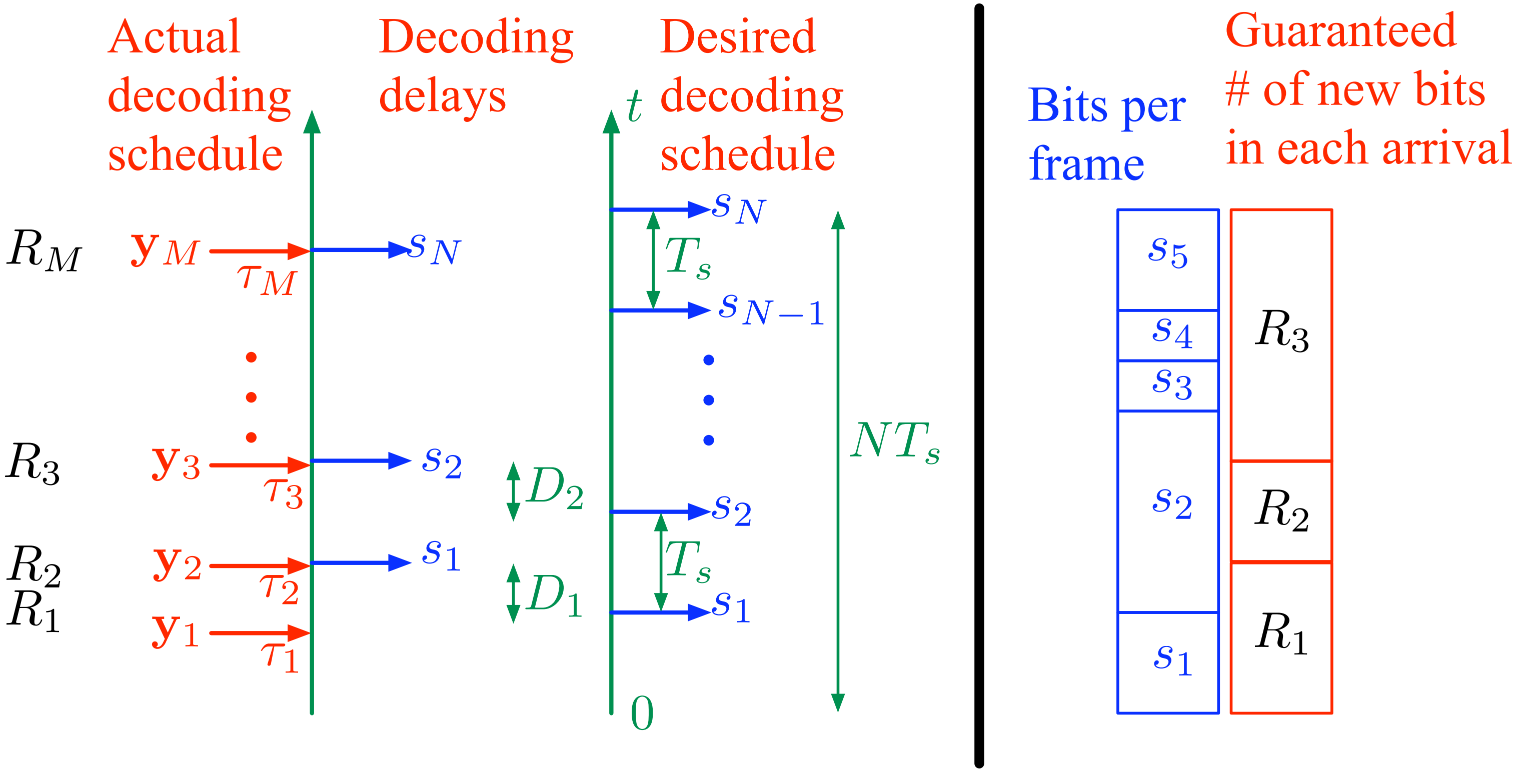
- **Tradeoff between rate and delay**
- Relaxation to a calculus of variations problem

Defining rate and delay



- Rate: $\rho(\mathbf{R}) = (\sum_{k=1}^{M} R_k)/(NT_s)$ (bps)
- Packet arrival for frame decoding: $g(i)=\inf\{n : \sum_{k=1}^{n} R_k \geq \sum_{j=1}^{i} |s_j|\}$
- Delay: $D(\mathbf{R}) = \sum_{i=1}^{N} (\tau_{g(i)} - iT_s)^+$ (total decoding delay violation)

Tradeoff between rate and delay

$$\rho^*(d) = \max_{\mathbf{R} \in \mathcal{R}} \{\rho(\mathbf{R}) \mid \mathbb{E}[D(\mathbf{R})] \leq d\} .$$

- Maximizing rate equivalent to maximizing the resolution of the source. Total number of bits in frames is limited by the rate:

$$\sum_{i=1}^{N} |s_i| \leq \sum_{k=1}^{M} R_k \;\Leftrightarrow\; \frac{1}{NT_s} \sum_{i=1}^{N} |s_i| \leq \rho(\mathbf{R})$$

- Fundamental tension for reordering and delay:
  - Costs less to get bits in later frames than earlier ones ($2R_1 + R_2 \leq 2$).
  - Getting bits later causes decoding delays, sufficiently high delay violates delay bound.

Tradeoff between rate and delay is a combinatorial optimization problem

$$\rho^*(d) = \max_{\{h(i)\} \in \mathcal{H}} \{\rho(\{h(i)\}) \mid \mathbb{E}[D(\{h(i)\})] \leq d\}.$$

where $\{h(i)\} \in \mathcal{H}$ is the set of all non-decreasing sequences from $[N] \to [M]$ giving the number of packet receptions required to decode the first i frames.

Proof. Decompose the optimization:

$$\max_{\mathbf{R} \in \mathcal{R}} \{\rho(\mathbf{R}) \mid \mathbb{E}[D(\mathbf{R})] \leq d\} = \max_{\{h(i)\} \,:\, \mathbb{E}[D(\{h(i)\})] \leq d} \quad \max_{\mathbf{R} \in \mathcal{R} \,:\, \{g(i,\mathbf{R})\} = \{h(i)\}} \rho(\mathbf{R}).$$

Evaluation of the sequence space will be prohibitively difficult for moderate to large M, N.

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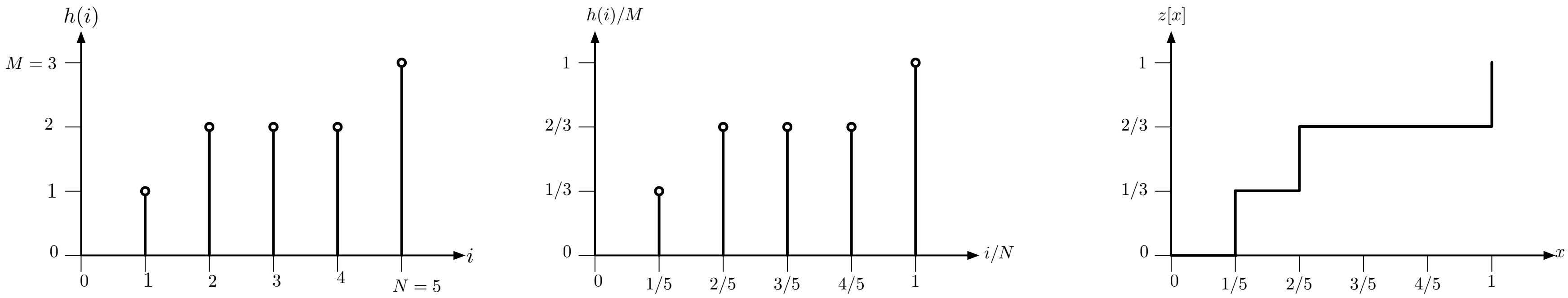
Relaxation of the integrality of the sequence space

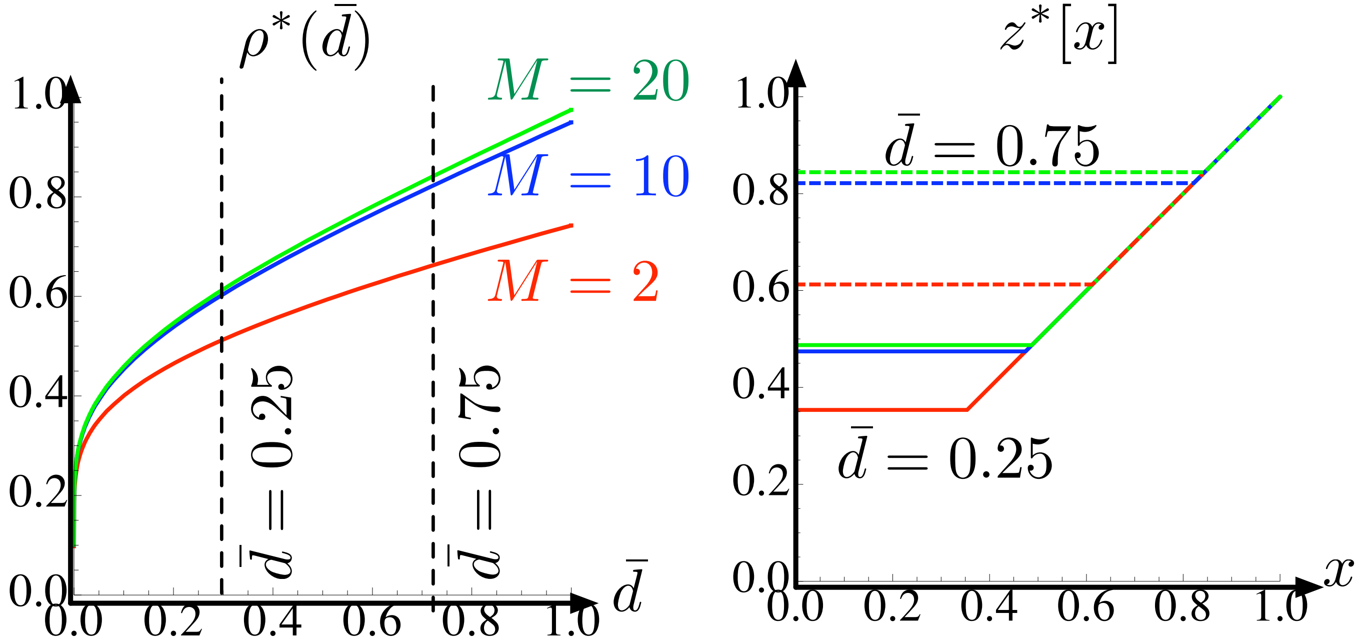


- Combinatorial optimization problem is over $\mathcal{H}$, the set of all non-decreasing sequences $\{h(i)\}$ from $[N] \to [M]$.
- Relaxation of the (normalized) integrality constraint using a zero-order hold yields a piecewise continuous function $z : [0,1] \to [0,1]$:

$$z[i/N] = \frac{h(i)}{M}, i = 1, \ldots, N, \quad z[x] = \sum_{k=1}^{N} \frac{h(x)}{M} \mathbf{1}_{\frac{k-1}{N} \leq x \leq \frac{k}{N}}, \ 0 \leq x \leq 1.$$

- Optimization becomes a calculus of variations problem over all piecewise continuous $z : [0,1] \to [0,1]$.

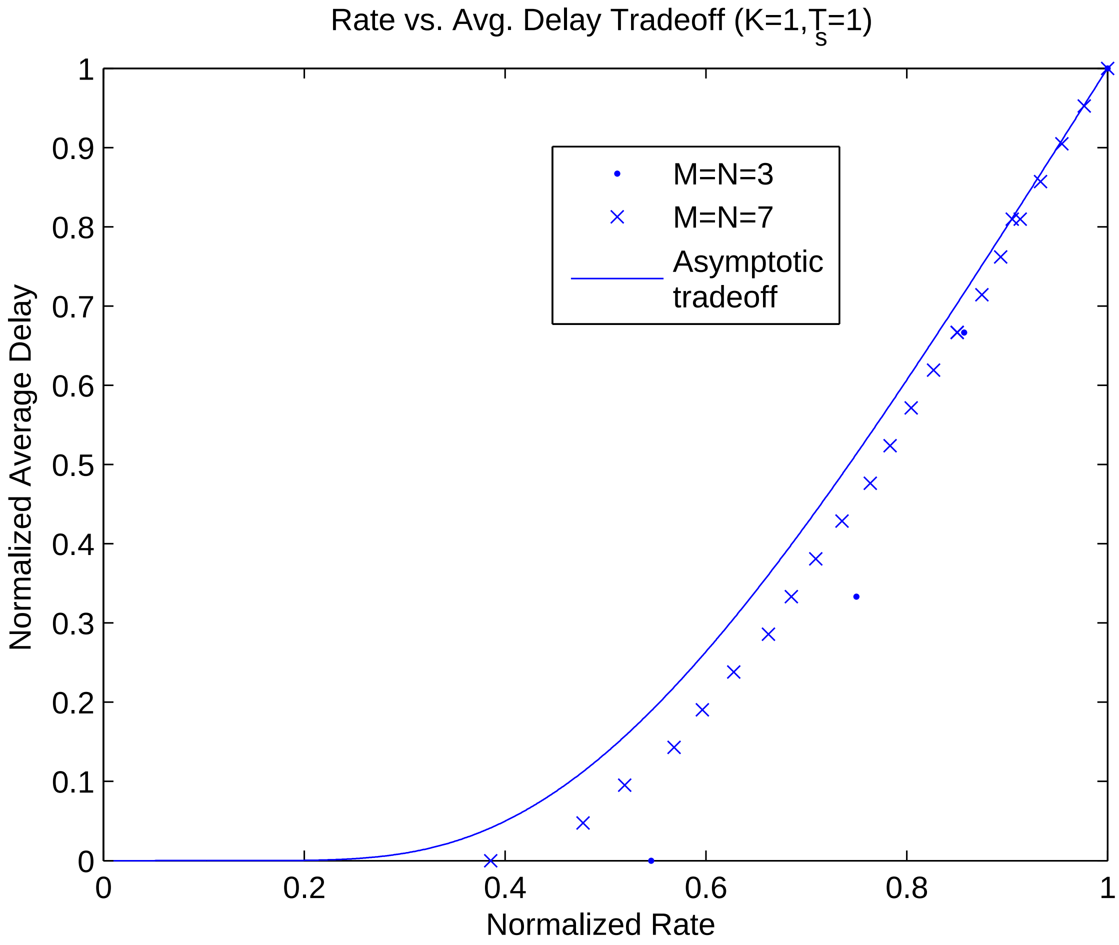
Sample result: constant arrival times ($\tau_i = iT_s$, $N = M$).



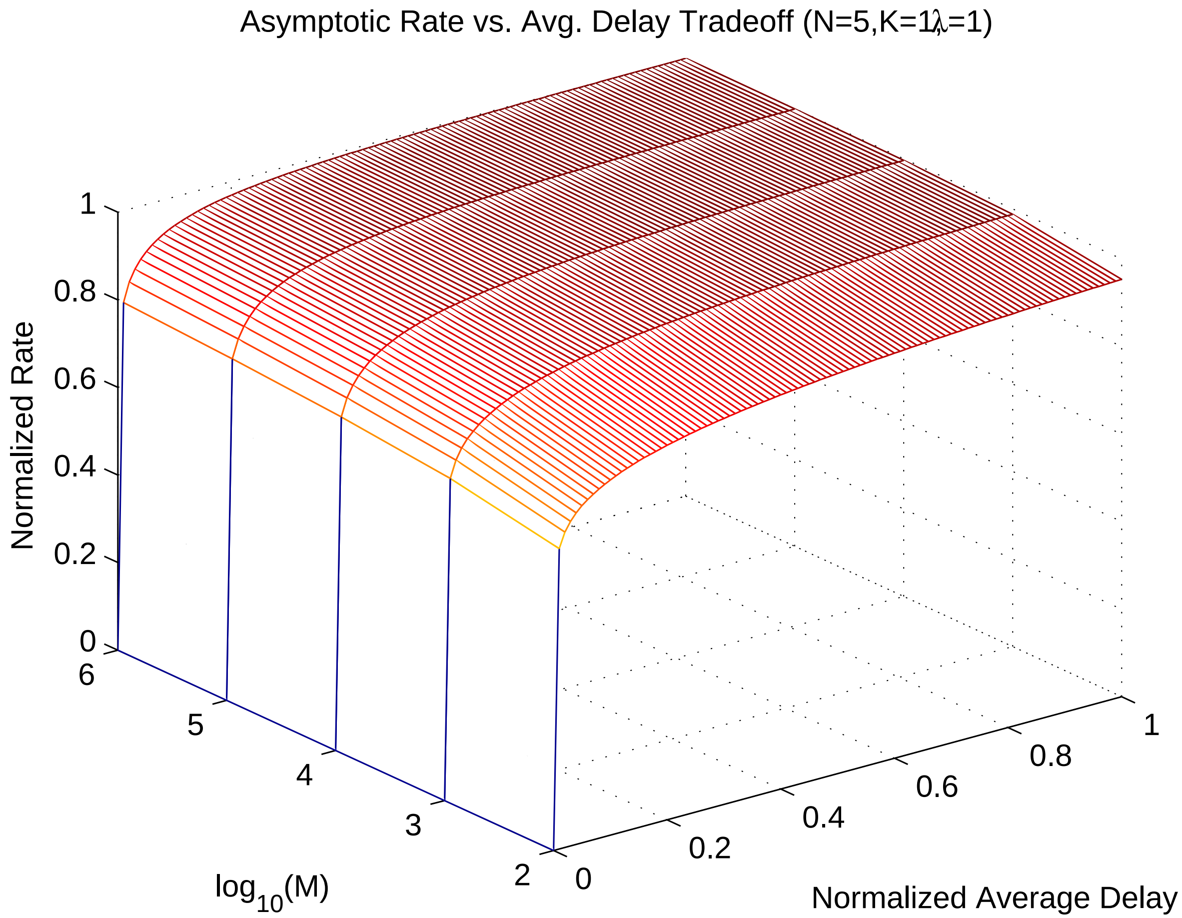
Theorem. Under constant arrival times, the optimal rate delay tradeoff and the optimal decoding deadline function are

$$\rho^*(\bar{d}) = (1 - \log \zeta)^{-1}, \; z^*[x] = \begin{cases} \zeta, & w \leq \zeta \\ w, & w > \zeta \end{cases}, \; \zeta = \sqrt{\frac{M-1}{M}\bar{d}}.$$

Const. arr. times: relaxation accurate for moderate M



Sample result: Exp iid arrival times



Significant delay bounds achievable at moderate rate costs.

Practical delay mitigating codes

- Priority encoded transmission codes (Albanese, Blomer, Edmonds, Luby, Sudan, 1996): time sharing MDS codes to grant different levels of error protection to different parts of a stream.

- Our analysis helps the user identify the time sharing scheme and protection levels for a PET code to achieve the maximum rate subject to a delay bound.

- Feedback (e.g., ARQ, TCP) won't affect tradeoff, since feedback doesn't change the capacity region (El Gamal, 1978).

- Packet loss can be integrated into the model (current work).