1. A certain power distribution company can buy electric power from \(N\) different power generation companies. The cost of \(x_n\) units of power bought from company \(n\) is \(f_n(x_n)\) dollars. Due to legislation regarding alternative energy requirements and the abilities of the generation equipment, company \(n\) can only produce between \(a_n\) and \(b_n\) units of power, i.e. \(a_n \leq x_n \leq b_n\). At a particular time, there is demand for a total of \(D\) units of power.

(a) Write the optimization problem for finding the minimum operating cost for the distribution company at this demand.

(b) Determine the analytical form of the Lagrangian dual function for this problem by defining the domain \(X := X_1 \times \cdots \times X_N\) with \(X_i = [a_i, b_i]\) and dualizing the demand constraint. Express the resulting Lagrangian dual as the sum of \(N\) functions of a single Lagrange multiplier variable \(\mu\).

(c) Consider the special case that \(f_i(x_i) = \sqrt{x_i} + 2\sin(x_i)\) and \(a_i = 0, b_i = 2\pi\). Use the first order necessary conditions together with numerical minimization over the stationary points to find the exact solution to this problem for \(N = 10, 100, 500\) as a function of the demand \(D\).

(d) Evaluate and plot the relative duality gap \((f^* - g^*)/f^*\) for the previous problem for \(N = 10, 100, 500\) as a function of \(D/(2\pi N)\) in Matlab.

(Email your Matlab code of (c) and (d) to the TA of this course.)

2. Consider the problem of maximizing the rate of communication over \(N\) parallel additive white Gaussian noise channels with a total power budget of \(P\).

\[
\max_{\mathbf{p}=(p_1, \ldots, p_N) | \mathbf{p} \geq 0, \mathbf{1}^T \mathbf{p} \leq P} \sum_{k=1}^{N} \log_2 \left(1 + \frac{p_k}{N_k} \right) \tag{1}
\]

(a) Find the Lagrangian dual function for this problem. Is there a duality gap?

(b) Analytically solve this optimization problem.