This week you will apply the tracking systems analysis from last week to understand carrier and timing synchronization techniques in digital receivers.

Information in a digital QAM signal is encoded by selecting sequences of symbols $a[n] \in \mathcal{C}$ from a finite set of complex numbers called a constellation $\mathcal{C}$. The simplest QAM constellation $\mathcal{C}$ is for QPSK, for which $\mathcal{C} = \{ \pm \frac{1}{\sqrt{2}} \pm j \frac{1}{\sqrt{2}} \}$.

This is then mapped to a complex baseband signal

$$s(t) = \sum_n a_n p(t - nT)$$

using a pulse shape $p(t)$. The pulse shape $p(t)$ is a continuous time signal designed to have the Nyquist property that

$$p(nT) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad \forall n \in \mathbb{Z}$$

and additionally be band-limited in frequency.

This complex signal $s(t)$ is then modulated to a high carrier frequency $f_c$ via

$$m(t) = \text{Re}\{s(t) \exp(j2\pi f_c t)\}$$

and sent to a receiver. Ideally, the receiver would first demodulate the signal back down to baseband via

$$r_{\text{ideal}}(t) = 2\text{LPF}\{m(t) \exp(-j2\pi f_c t)\}$$

where the LPF is an ideal low pass filter that has bandwidth a little larger than the pulse shape. (This demodulation is usually done in two stages, first to an intermediate frequency, then to baseband, in real receivers.) Then, the ideal receiver would sample the baseband signal at the time instants $t = nT$ for each $n \in \mathbb{Z}$ to reconstruct the original information symbols $a_n$ via

$$\hat{a}_n = r_{\text{ideal}}(nT)$$

Unfortunately, we encounter many non-idealities in real life receivers. The first two to consider reflect the fact that the oscillators providing the carriers $\exp(j2\pi f_c t)$ and the sample times $t = nT$ are not perfectly synchronized at the transmitter and the receiver. There will be unknown differences in the phase of these oscillators so that the actual down-converted and sampled received signal will be created through the equations

$$r(t) = 2\text{LPF}\{m(t) \exp(-j(2\pi f_c t + \theta(t)))\} \approx s(t) \exp(-j\theta(t))$$

and

$$x[n] = r(nT + \tau[n])$$

This means that after down-converting and sampling the received signal we will have no longer have $a_n$ as desired, but instead

$$x[n] = \exp(-j\theta[n]) \sum_{k=-\infty}^{\infty} a[k] p(nT + \tau[n] - kT)$$
1. Show that the idealized receiver operating through equations (1)-(5) has the desired property that $\hat{a}_n = r(nT)$.

2. Describe how one could recreate the signal $a[n]$ from the signal $x[n]$ if $\tau[n]$ and $\theta[n]$ are known constants (i.e. they are not a function of $n$). (You may assume that ideal non-causal digital filters are available.)

3. Write MATLAB code to simulate a random vector of non-ideal received samples $x[n]$ as a function of $\tau[n]$ and $\theta[n]$ according to (8) by selecting the symbols $a[n]$ randomly from the QPSK constellation. Use the ideal sinc pulse shape $p(t) = \frac{T}{\pi t} \sin(\frac{\pi t}{T})$.

4. Write MATLAB code to simulate your recreated signal $\hat{a}[n]$ from question 2 as a function of a vector of samples of $x[n]$ using estimated values $\hat{\tau}[n]$ and $\hat{\theta}[n]$. Show via a plot that the original signal is roughly reproduced when these estimates are exact, i.e. when $\hat{\tau}[n] = \tau[n]$ and $\hat{\theta}[n] = \theta[n]$.

5. **Error signal extraction**: in order to track the carrier phase offset $\theta[n]$ and the baud timing offset $\tau[n]$ (i.e. to create good estimates $\hat{\theta}[n]$ and $\hat{\tau}[n]$), we need a signal that provides us information about the carrier phase error $\theta[n] - \hat{\theta}[n]$ and the baud timing error $\tau[n] - \hat{\tau}[n]$. While these error signals are not directly estimable, we can find signals which are approximately proportional to them when the errors are small.

   (a) **Carrier Phase Error**: Show that when $\theta[n] - \hat{\theta}[n] = \theta$ is constant and $\tau[n] - \hat{\tau}[n] = 0$, the time average (expectation) of the following signals provide information regarding the carrier phase error by plotting it as a function of the phase error $\theta$. Do this by using your MATLAB simulators to calculate their average for various $\theta$ values and plotting this average as a function of $\theta$.

   i. $\angle c^*[n]x[n]$, where $c[n] = \frac{1}{\sqrt{2}} (\text{sign}(\text{Re}\{x[n]\}) + j\text{sign}(\text{Im}\{x[n]\}))$

   ii. $\angle x[n]^4$

   (b) **Timing Error**: Show that when $\tau[n] = \tau$ is constant and $\theta[n] - \hat{\theta}[n] = 0$ the time average (expectation) of the following signals provide information regarding the timing error by plotting it as a function (surf or imagesc plot) of $\tau$ and $\hat{\tau}$. Do this by calculating their average (over a long vector of $x[n]$) for various $\tau$, $\hat{\tau}$ values and plotting this average as a function of $\tau$, $\hat{\tau}$.

   i. $\text{Re}\{c^*[n] - y^*[n]\}z[n]\}$ where $c[n] = \frac{1}{\sqrt{2}} (\text{sign}(\text{Re}\{y[n]\}) + j\text{sign}(\text{Im}\{y[n]\}))$ and

   $$y[n] = \sum_{k=-L}^{L} \frac{T}{\pi(kT-\hat{\tau})} \sin \left( \frac{\pi(kT-\hat{\tau})}{T} \right) x[n-k], \text{ some large } L$$

   and

   $$z[n] = \sum_{k=-L}^{L} \frac{d}{d\tau} \left\{ \frac{T}{\pi(kT-\hat{\tau})} \sin \left( \frac{\pi(kT-\hat{\tau})}{T} \right) \right\} x[n-k], \text{ some large } L$$

6. Build a carrier phase synchronization loop in MATLAB by selecting one of the two error signals from the previous problem, and using the estimate update dynamics from the previous homework, i.e.

   $$\hat{\theta}[n] = \hat{\theta}[n-1] + \mu e_{\theta}[n-1]$$

   where $e_{\theta}$ is one of the two the phase error signals from the previous problem. You may assume that the timing is perfectly synchronized, so that $\tau[n] = \hat{\tau}[n] = 0$.

   (a) Show that you can track a small step $\theta[n] = \alpha u[n]$ for a small $n$.

   (b) Make a plot of your asymptotic phase error when $\theta[n] = \beta n u[n] + \alpha u[n]$, small $\beta$ and small $\alpha$. Does this match your analysis from the previous lab.

   (c) Use your alternate estimate update dynamics from the homework 3 problem 1 instead, and show that you can make the asymptotic phase error go to zero when $\theta[n] = \beta n u[n] + \alpha u[n]$.

7. Repeat the previous problem, but for timing synchronization. (You may assume that the carrier phase is perfectly synchronized.)