Matroid Bounds on the Region of Entropic Vectors

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Outline

1. Problem and motivation
2. Background
3. Main theorems on matroid bounds
4. Application

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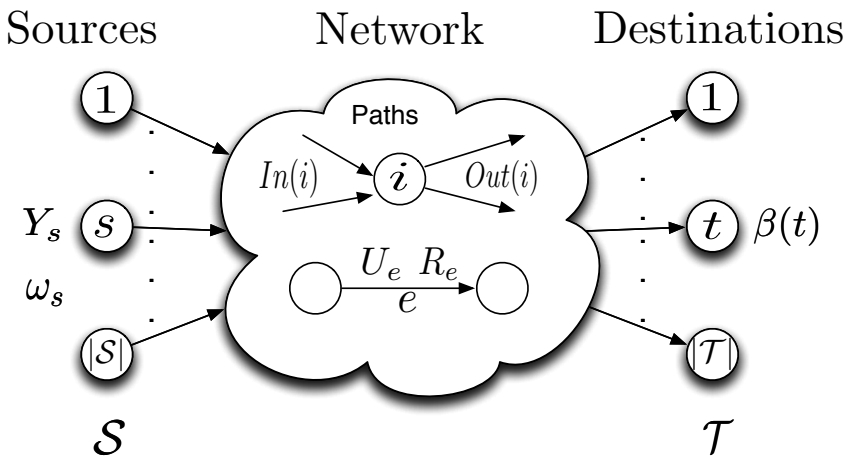
Rate Region

Collect $N = \{Y_s, U_e\}$. Suppose we know the region Γ_N^* of entropic vectors. Here Γ_N^* is a $2^N - 1$ dimensional cone and any point in it is **entropic** (has a valid joint distribution associated with the N variables) [4].



Constraints: (Hyperplanes)

$$\begin{aligned}
\mathcal{L}_1 &= \{\mathbf{h} \in \mathcal{H}_N : h_{Y_s} \geq \omega_s\} \\
\mathcal{L}_2 &= \{\mathbf{h} \in \mathcal{H}_N : h_{Y_\mathcal{S}} = \Sigma_{s \in \mathcal{S}} h_{Y_s}\} \\
\mathcal{L}_3 &= \{\mathbf{h} \in \mathcal{H}_N : h_{X_{\mathrm{Out}(k)}|Y_s} = 0\} \\
\mathcal{L}_4 &= \{\mathbf{h} \in \mathcal{H}_N : h_{X_{\mathrm{Out}(i)}|X_{\mathrm{In}(i)}} = 0\} \\
\mathcal{L}_5 &= \{\mathbf{h} \in \mathcal{H}_N : h_{Y_{\beta(t)}|U_{\mathrm{In}(t)}} = 0\}
\end{aligned}$$

$$\mathcal{R} = \mathrm{Ex}(\mathrm{proj}_{U_e}(\overline{\mathrm{con}(\Gamma_N^* \cap \mathcal{L}_{234})} \cap \mathcal{L}_{15}))$$

Bounds on Rate Region

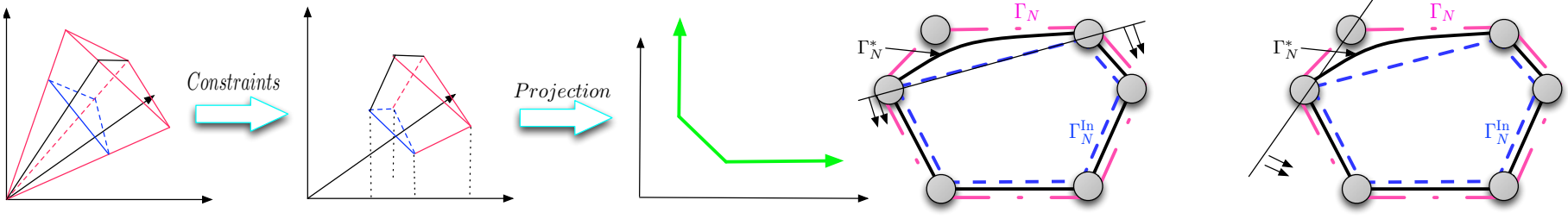
- Replace Γ_N^* with closed outer bound Γ_N^{Out}:

$$\mathcal{R}_{out} \quad = \quad \text{Ex}(\text{proj}_{U_e}(\Gamma_N^{Out} \cap \mathcal{L}_{12345}))$$

- Replace Γ_N^* with a closed inner bound Γ_N^{In}:

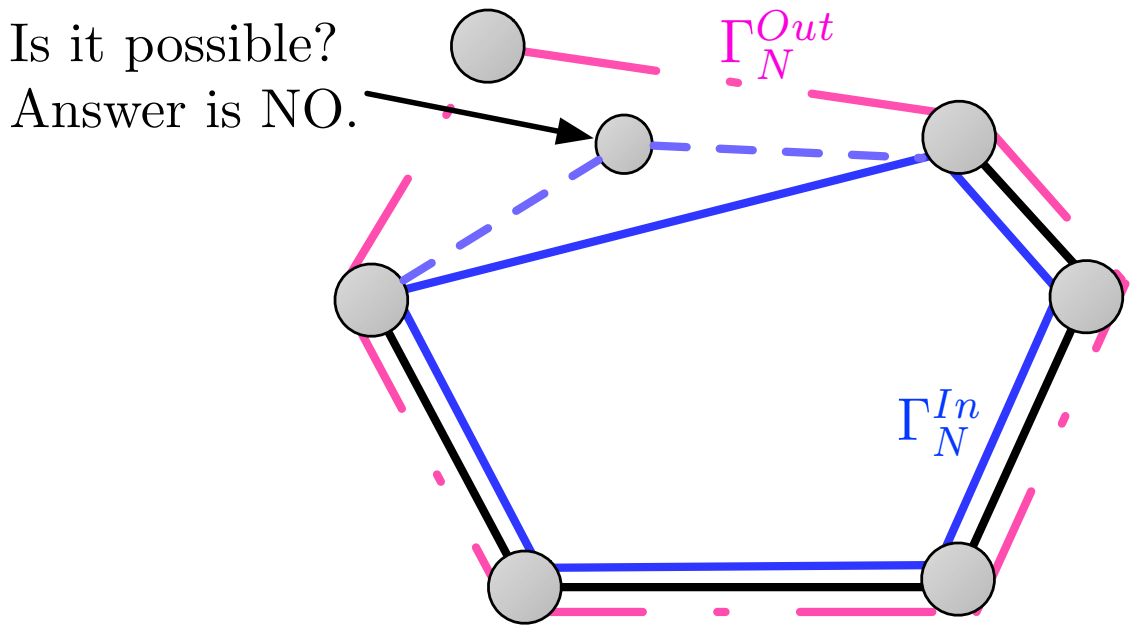
$$\mathcal{R}_{in} \quad = \quad \text{Ex}(\text{proj}_{U_e}(\Gamma_N^{In} \cap \mathcal{L}_{12345}))$$

- It becomes general polyhedral computations [1]:
  - Initial polyhedra $\rightarrow \cap$ constraints $\rightarrow$ projections



Bounds relationships and concern

Different bounds share some extreme rays.
One can reduce the computation complexity if only do some checking on the extreme rays of bigger bounds.
Concern: is it sufficient to only consider extreme rays?



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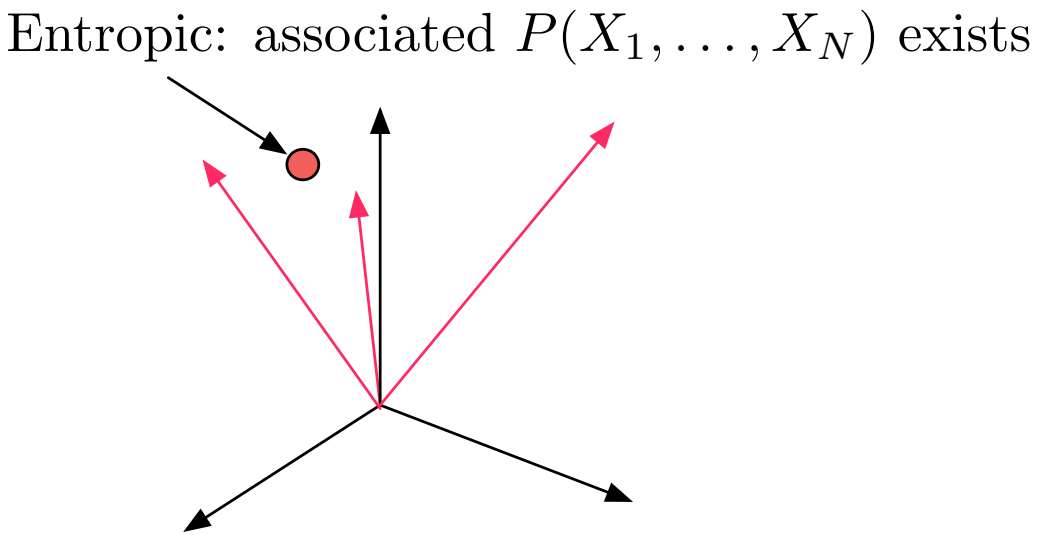
Region of entropic vectors

Region of entropic vectors Γ_N^*:

- A collection $\mathbf{X}$ of N discrete variables $X_1, \ldots X_N$, (joint) entropies of all non-empty subsets A of $\mathbf{X}$ form a $2^N - 1$-dimensional vector $\mathbf{h} = (H(A), \forall A \subseteq \mathbf{X} \backslash \emptyset)$.
- A vector $\mathbf{h}$ is said to be entropic if there exists a joint distribution associated with $\mathbf{h}$.
- Region of entropic vector: $\Gamma_N^* = \{\mathbf{h} : \mathbf{h} \text{ is entropic}\}$.
- Not all points in Euclidean space are entropic.



Natural outer bound: Shannon (Polymatroid)

Region defined by Shannon inequalities
$I(X_i, X_j|X_K) \geq 0, i, j \in \mathbf{X}, K \subseteq \mathbf{X}\backslash\{i, j\}$ naturally form an outer bound on Γ_N^*, which is equivalent to the polymatroid cone:

1. Normalization: $f(\emptyset) = 0$;
2. Monotonicity: if $A \subseteq B \subseteq \mathbf{X}$ then $f(A) \leq f(B)$;
3. Submodularity: if $A, B \subseteq \mathbf{X}$ then $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$.

Matroids

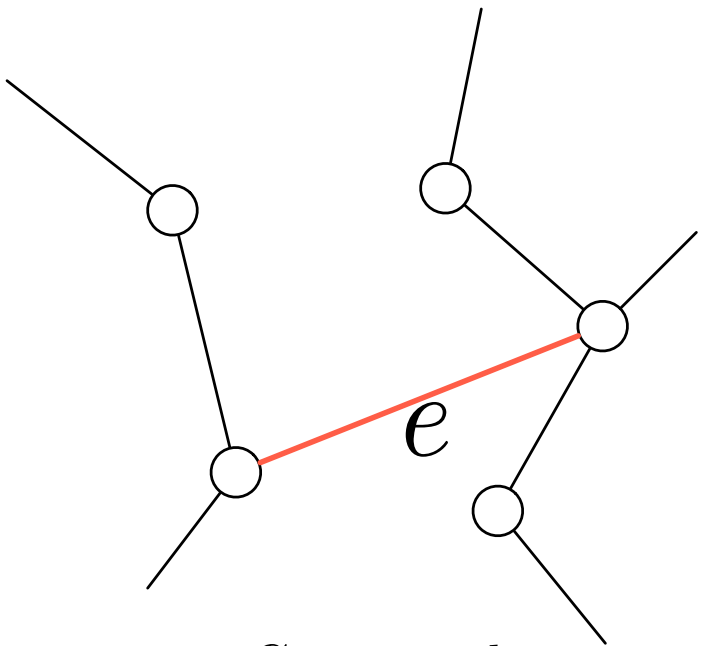
Definition

A set function $r : 2^S \to \{0, \ldots, N\}$ is a rank function of a matroid if it obeys the polymatroid axioms and two more conditions:

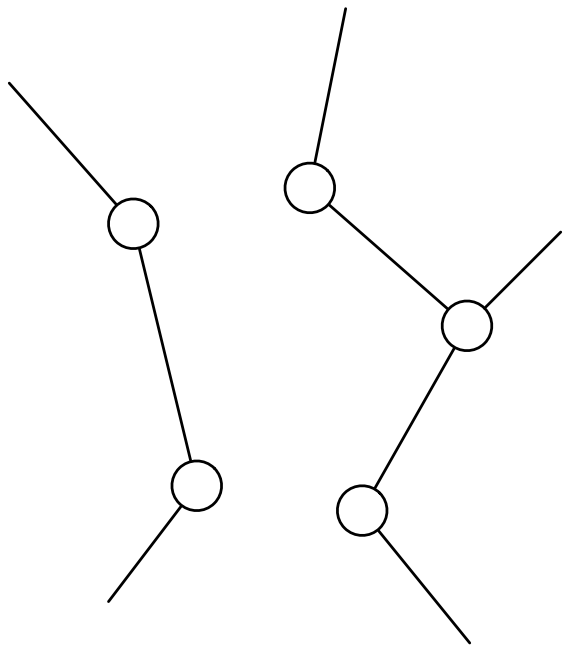
1. Integrality: $r(A)$ is integer-valued;
2. Cardinality: $0 \leq r(A) \leq |A|$.

Connected (poly)matroids

- Connected (poly)matroids: $r(A) + r(S - A) > r(E), \forall A \subset S$
- Disconnected: $\exists A, s.t\ r(A) + r(S - A) = r(E)$.



Connected



Disconnected

Minors of matroids

Definition

If M is a matroid on S and $T \subseteq S$, a matroid M' on T is called a *minor* of M if M' is obtained by any combination of deletion ($\setminus$) and contraction ($/$) of M.

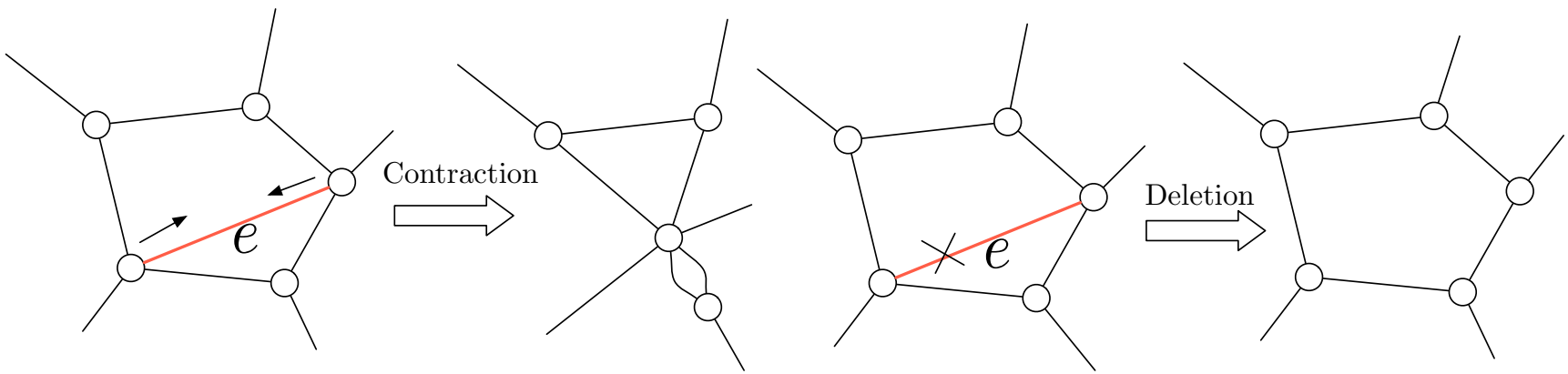
Notation: For any pair of sets $\mathcal{A}, \mathcal{B} \subseteq [[N]]$ the *minor* associated with deleting $[[N]] \setminus (\mathcal{A} \cup \mathcal{B})$ and contracting on $\mathcal{A}$:

$$r_{\mathcal{B}|\mathcal{A}}(\mathcal{C}) := r(\mathcal{C} \cup \mathcal{A}) - r(\mathcal{A}) \quad \forall \mathcal{C} \subseteq \mathcal{B} \tag{1}$$

Contraction and Deletion

Let M/T denote the matroid obtained by contraction of M on $T \subset S$, and let $M \setminus T$ denote the matroid obtained by deletion from M of $T \subset S$. Then, $\forall X \subseteq S - T$

$$\begin{aligned} r_{M/T}(X) &= r_M(X \cup T) - r_M(T) \\ r_{M \setminus T}(X) &= r_M(X) \end{aligned} \tag{2}$$



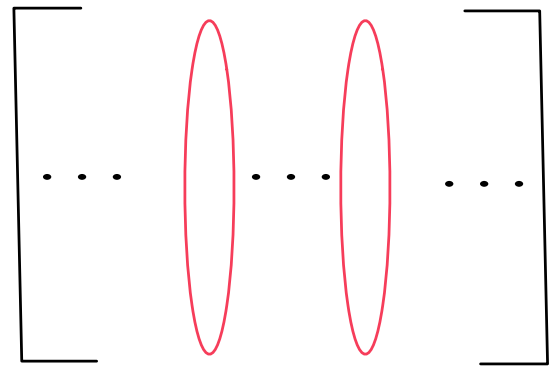
Representable Matroids

Definition

A matroid M with ground set S of size $|S| = N$ and rank $r_M = r$ is representable over a field $\mathbb{F}$ if there exists a matrix $A \in \mathbb{F}^{r \times N}$ such that for each independent set $I \in \mathcal{I}$ the corresponding columns in A, viewed as vectors in $\mathbb{F}^r$, are linearly independent.

$$S_1, \ldots, S_i, \ldots, S_j, \ldots, S_N \overset{Mapping}{\Longleftrightarrow} 1\ 2\ \ldots\ i\ \ldots\ j\ \ldots\ N$$

$$r(S_i, S_j) = rank(i, j\text{-th columns})$$



- Representable matroids usually can be characterized by forbidden minors (cannot contain such minors).
- All matroids with $|S| \leq 7$ are representable over some field.

Inner Bounds from Representable Matroids

Proposition

Conic hull of representable matroids form inner bounds on region of entropic vectors.

Proof.

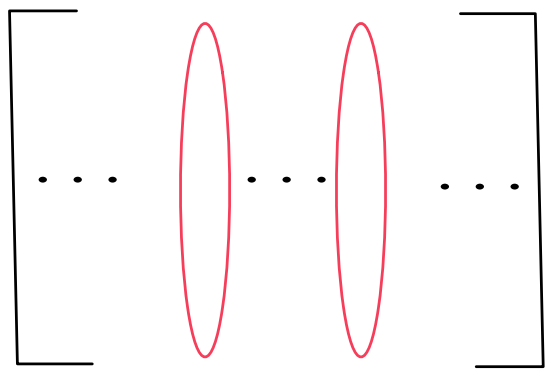
It suffices to show that a rank of representable matroid is entropic. Suppose the associated representation matrix is $\mathbf{A} \in \mathbb{F}_q^{k \times N}$, from which we can create the random variables

$$(X_1, \ldots, X_N) = \mathbf{uA}, \quad \mathbf{u} \sim \mathcal{U}(\mathbb{F}_q^k). \tag{3}$$

$$S_1, \ldots, S_i, \ldots, S_j, \ldots, S_N \overset{Mapping}{\Longleftrightarrow} 1\ 2\ \ldots\ i\ \ldots\ j\ \ldots\ N$$

$$r(S_i, S_j) = rank(i, j\text{-th columns})$$

$$h_{i,j} = rank(i, j\text{-th columns}) * h_{\mathbf{u}} = r(S_i, S_j) \log_2 q$$



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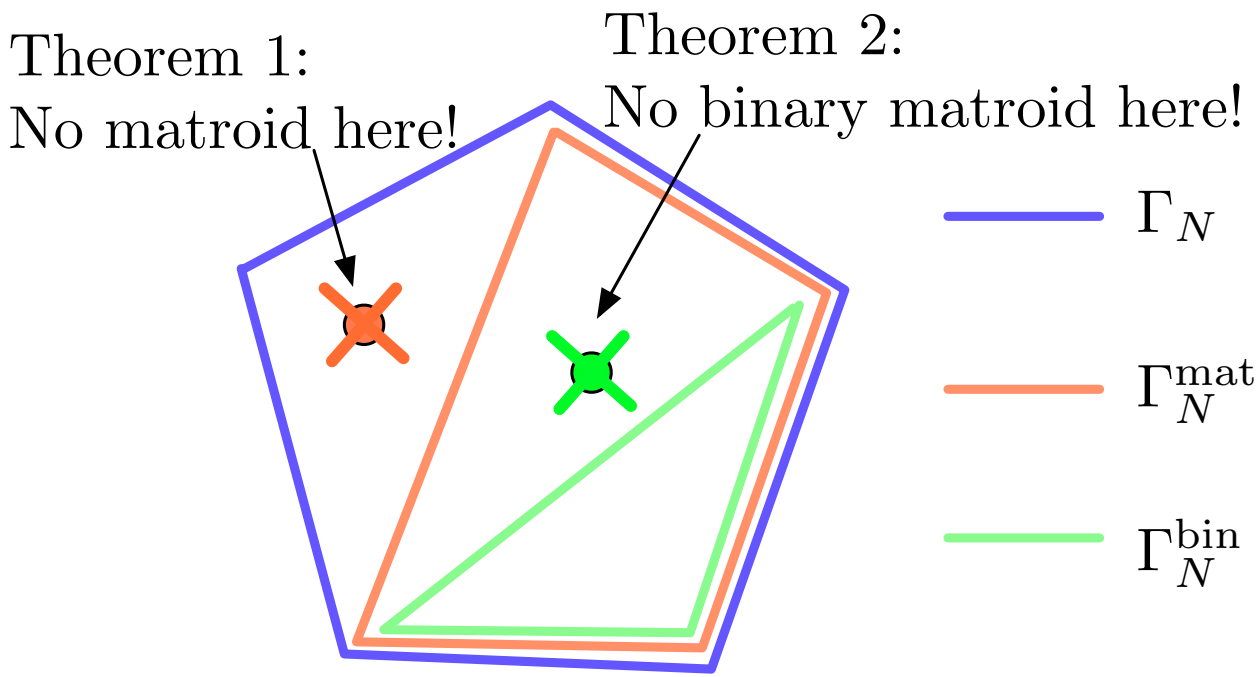
Main results preview

Up to now, we talked about three bounds on region of entropic vectors: polymatroid (Shannon), matroid, representable matroid.
Main theorems in this papers:

- Strict containment of extreme rays of these three bounds.

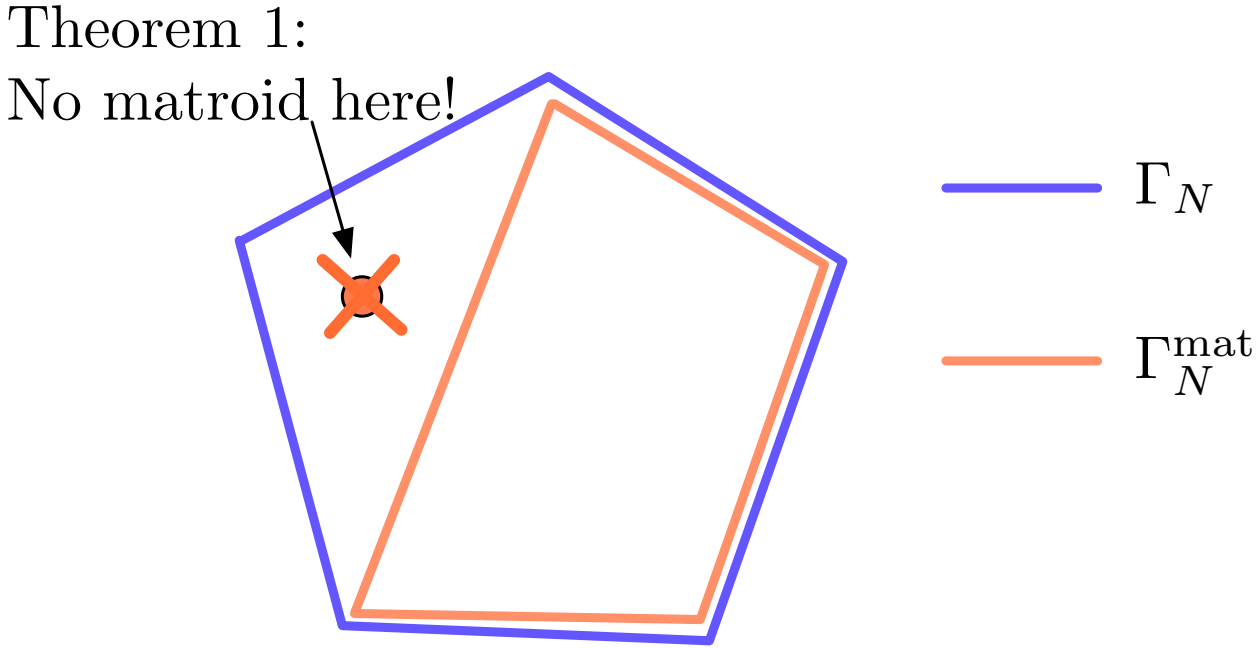


Theorem 1: Extremal matroids are extremal polymatroids

Theorem

[Extremal Matroids are Extremal Polymatroids] Every extreme ray of $\mathrm{cone}(\mathbb{M}_N)$ *is an extreme ray of* Γ_N*. Equivalently,* $\Gamma_N^{\mathrm{mat}} = \mathcal{T}_N^{\mathrm{mat}}$.

- Notation: $\mathbb{M}_N$-all ranks of matroids on N elements; $\mathcal{T}_N^{\mathrm{mat}}$-conic cone of all extremal polymatroids that are matroidal.



Proof of theorem 1

Proof.

1. All extreme matroid rays must be connected, otherwise it has a separator ($A \subsetneq S$) s.t $r(S) = r(A) + r(S \setminus A)$ and $r(A)$ and $r(S \setminus A)$ can be viewed as two different rank functions by adding zeros to other elements, in which case $r(S)$ is not extremal;
2. Nguyen showed (in different terminologies) in [2] that a connected matorid is extremal in the polymatroid cone.
3. Together, we complete the proof.

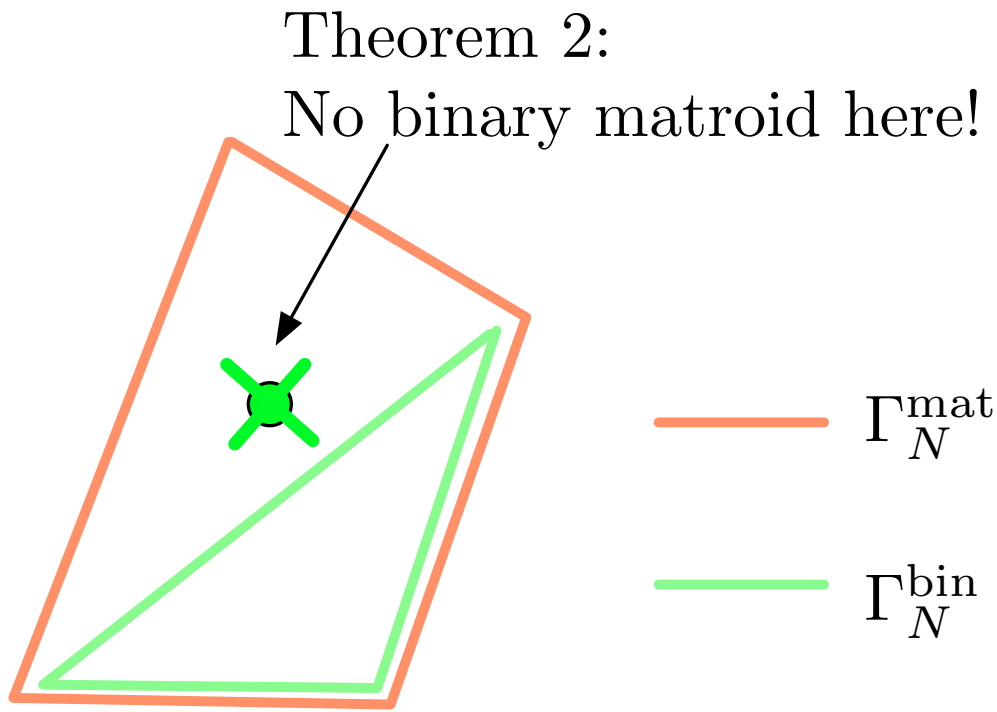
□

Theorem 2: Forbidden minors and extremality

Similarly, we can establish relationship between representable matroid inner bound and general matroid bound.

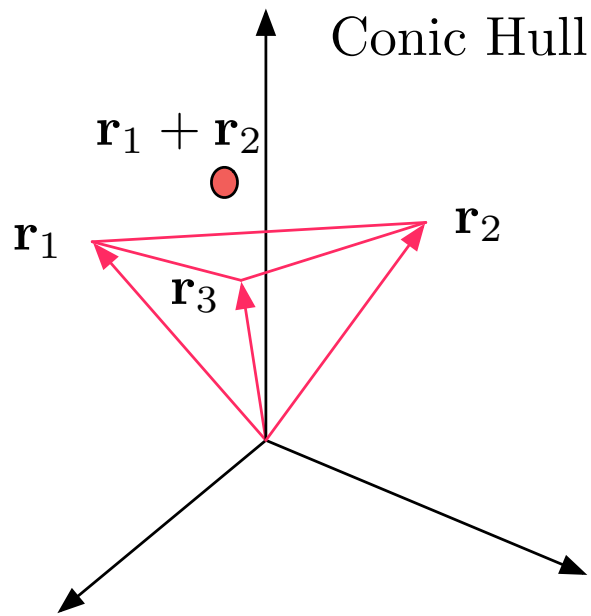
Theorem

[Forbidden Minors and Extremality] Let $\mathcal{V}$ be a set of matroid rank vectors formed by removing from $\mathbb{M}_N$ (exclusively) those with a certain collection of connected forbidden minors. The extreme rays of $\mathrm{cone}(\mathcal{V})$ are all extreme rays of $\mathrm{cone}(\mathbb{M}_N)$.



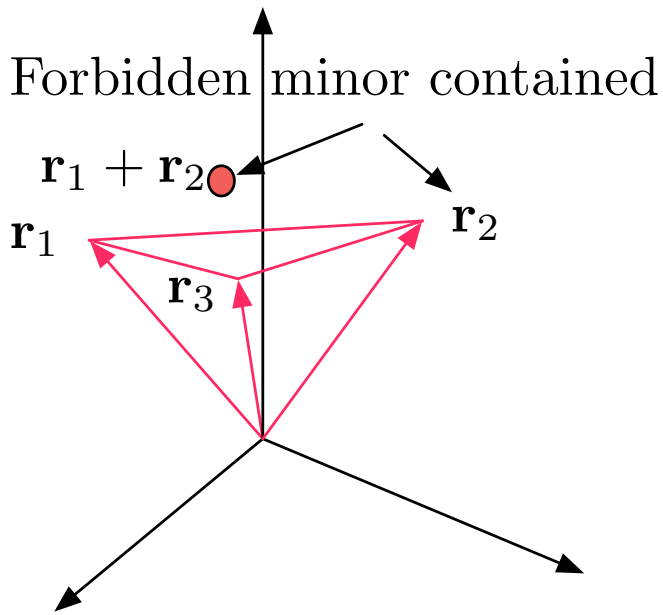
Lemmas needed to prove theorem 2

1. [Conically Dependent Matroids are Simple Sums] Any matroid rank vector which is not an extreme ray of the conic hull $\mathrm{cone}(\mathbb{M}_N)$ is the sum of a collection of matroid rank vectors that are the extreme rays. That is, the coefficients in the conic combination may all be take to be one.



Lemmas needed to prove theorem 2

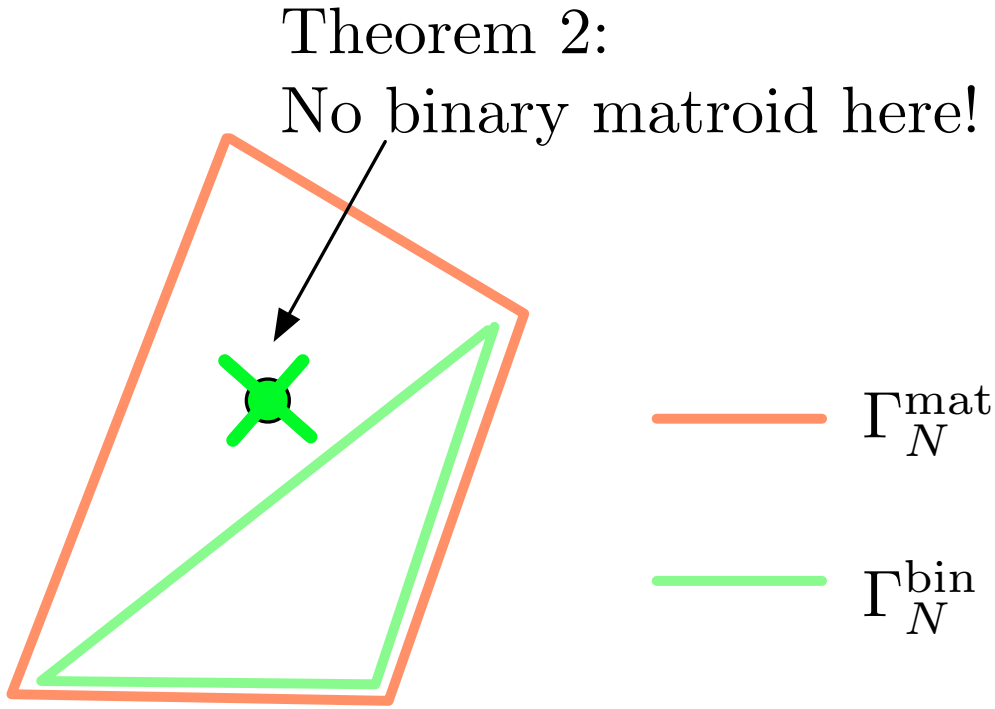
2. A matroid that is not an extreme ray of $\mathrm{cone}(\mathbb{M}_N)$ has a connected forbidden minor if and only if at least one of the extremal matroids that it can be represented as the sum of has the connected forbidden minor.



Proof of theorem 2

Proof.

Suppose $\mathbf{r}_N \in \mathrm{Ext}(\mathrm{cone}(\mathcal{V}))$ but $\mathbf{r}_N \notin \mathrm{Ext}(\mathrm{cone}(\mathbb{M}_N))$. Then $\mathbf{r}_N$ can not have the forbidden minors. $\mathbf{r}_N$ must be expressible as a sum of extremal matroid ranks, all of which do not have the forbidden minors. But this contradicts the extremality with in $\mathrm{cone}(\mathcal{V})$, for these other ranks must be in $\mathcal{V}$ as well (since they do not have the forbidden minors), and now a conic combination of them equals the supposed extreme ray $\mathbf{r}_N$. □



Corollary

Corollary

[Extremal $\mathbb{F}_q$ Linear Representable Matroids are Extremal Matroids for $q \in \{2, 3, 4\}$] Every extreme ray of $\mathrm{cone}(\mathbb{M}_N^q)$ is an extreme ray of $\mathrm{cone}(\mathbb{M}_N)$.

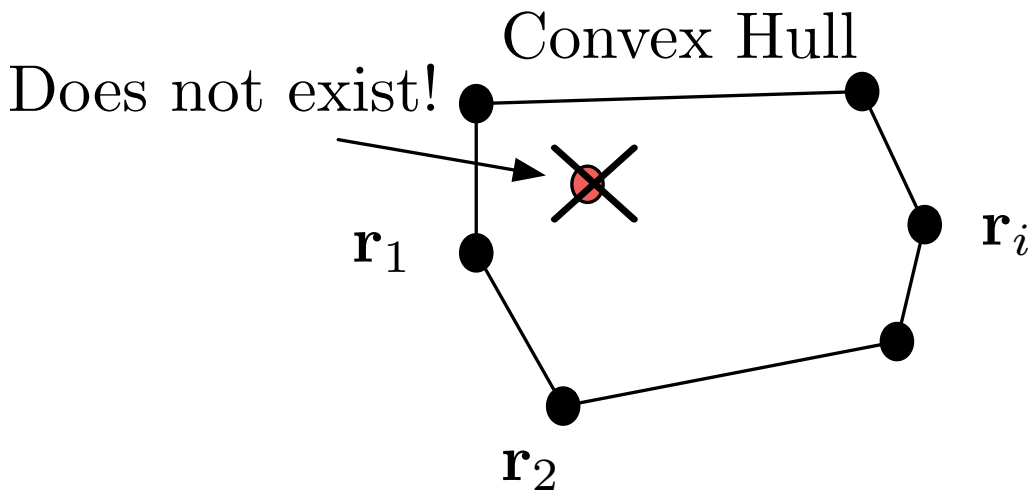
Proof.

Follows from [3] showing the forbidden minor characterizations of these classes of matroids, together with theorem about the extreme rays of $\mathrm{cone}(\mathcal{V})$ are all extreme rays of $\mathrm{cone}(\mathbb{M}_N)$. □

Theorem 3: convex independence

Theorem

[Matroid Rank Vectors are Convex Independent] Any collection of matroid rank vectors $\mathcal{V} \subseteq \mathbb{M}_N$ are convex independent. Equivalently, the set of extreme points of the convex hull $\text{convex}(\mathcal{V})$ *is* $\mathcal{V}$ *itself. Equivalently, a matroid rank vector can not be expressed as a convex combination of any other matroid rank vectors other than itself.*

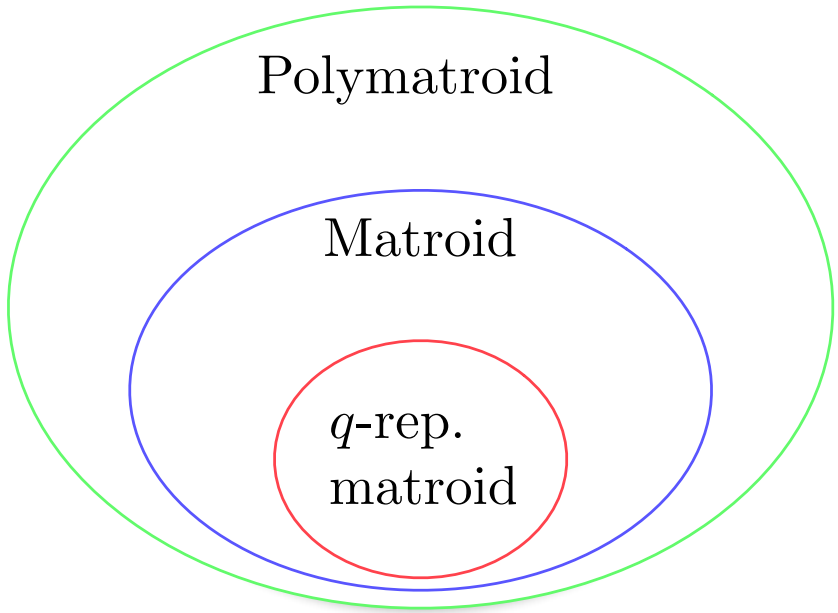


Results summary

Summary of the results we proved:

1. The subset of the extreme rays of Shannon outer bound (the extremal polymatroids) that are matroidal are also the extreme rays of the cone of matroids;
2. The extreme rays of the conic hull of binary/ternary/quaternary representable matroid ranks inner bound are a subset of the extreme rays of conic hull of matroid ranks.
3. All matroid ranks are convex independent.

Extreme rays of different cones



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Application: reduction of complexity in rate region calculation

- The theorems proved provide a simpler way to do the rate region calculation: instead of enumerating all matroid ranks, one only needs to enumerate the extremal (connected) ones with rank less than or equal to the number of source variables (free independent variables).
- For example, the case for $N = 4, 5, 6$ can be found here. Around 50% reduction of complexity.

	$N = 4$	$N = 5$	$N = 6$
$\lvert\mathrm{Ext}(\Gamma_N)\rvert$	41	117984	N/A
$\lvert\mathbb{M}_N\rvert$	68	406	3035
$\lvert\mathrm{Ext}(\Gamma_N^{\mathrm{mat}})\rvert$	27	154	1785
$\lvert\mathrm{Ext}(\Gamma_N^{\mathrm{bin}})\rvert$	26	127	1205

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Thank you

Thank you! Any questions?