Entropy Region and Convolution

– a manuscript by František Matúš and Lászlo Csirmaz[1]

Yunshu Liu

Drexel University ECE Department

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Outline

Preliminaries on Matroid theory

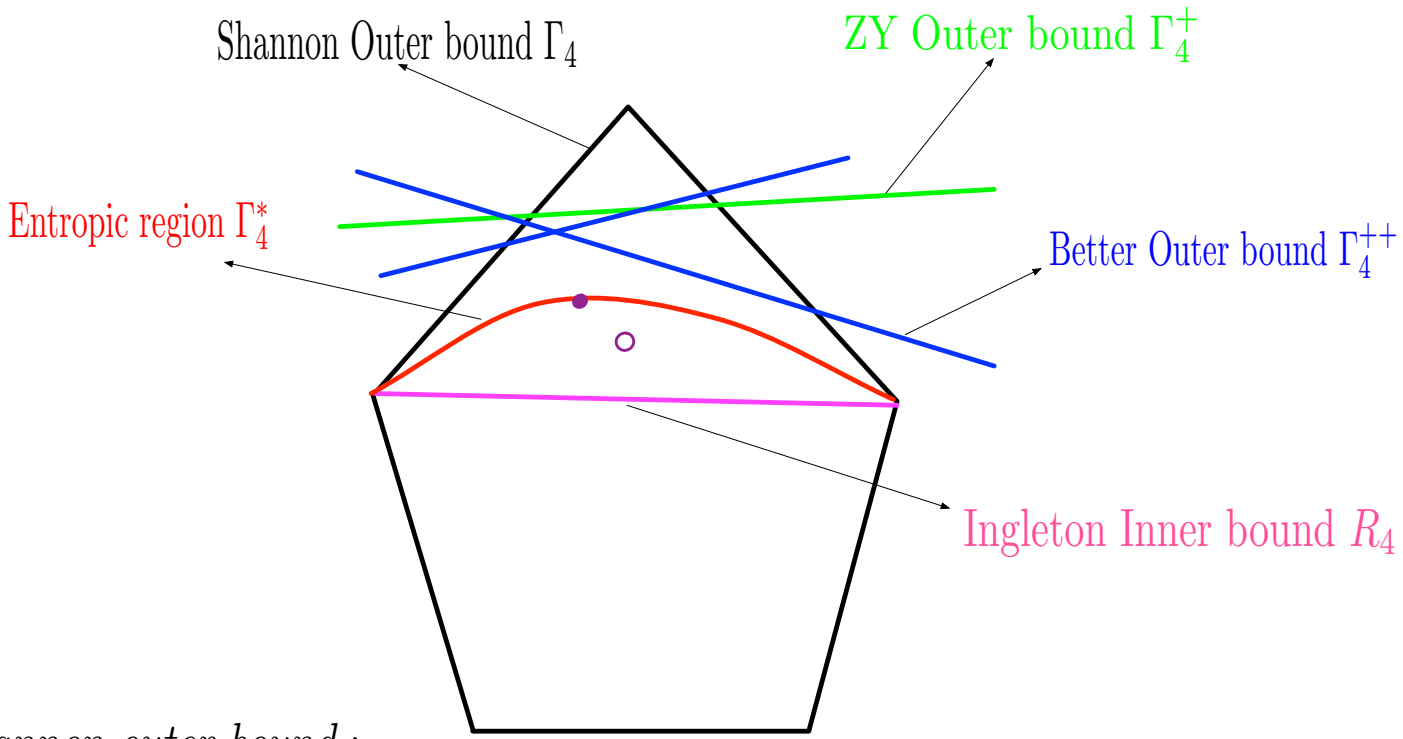
Decomposition into Tight and Modular polymatroids

Entropy region of four variables

Symmetrization of F_{ij}

Ingleton Score

Motivation



Shannon outer bound :
Shannon Type Inequalities Γ_4
Better outer bound :
Shannon Type + ZY-Non Shannon Γ_4^+
Shannon Type + other Non Shannon Γ_4^{++}

Ingleton inner bound R_4:
Ingleton Inequalities + Shannon Type
Better inner bound :
More rays from the gap

References:

- [1] F. Matúš and L. Csirmaz, Entropy Region and Convolution, http://arxiv.org/abs/1310.5957
- [2] F. Matúš, Two Contructions on Limits of Entropy Functions, *IEEE Trans. Inf. Theory, Vol. 53, No. 1, Jan 2007, page 320-330*.
- [3] F. Matúš, Infinitely Many Information Inequalities, *IEEE Int. Symp. Information Theory (ISIT), 2007, page 41-44*.
- [4] F. Matúš and M. Studený, Conditional Independences among Four Random Variables I, *Combinatorics, Probability and Computing, 1995, page 269-278*.

Result: There exist point in $\overline{\Gamma_4^*}$ give lower Ingleton score than the four atom distribution.

Main results of paper [1]

- Decompose entropic vector: $h = h^{ti} + h^m$ such that if $h \in \overline{\Gamma_4^*}$ then $h^{ti} \in \overline{\Gamma_4^*}$
- Ingleton score of h^{ti} is always equal or less than that of h.
- There exist a distribution D_0 with entropic vector h_0 such that h_0^{ti} give lower Ingleton score than four atom distribution P(0000 0110 1010 1111) = $\{\alpha \quad 0.5-\alpha \quad 0.5-\alpha \quad \alpha\}$ for $\alpha \approx 0.350457$.

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Definitions of Matroid

What is Matroid?

Matroid is an independence structure that captures and generalizes the notion of linear independence in vector spaces.

Rank Function based definition of Matroid

A rank function is a function from subsets of ground set E to integers satisfying the following conditions:

- If $X \subseteq E$ then $0 \leqslant r(X) \leqslant |X|$
- If $X \subseteq Y \subseteq E$, then $r(X) \leqslant r(Y)$
- If $X, Y \subset E$, then $r(X) + r(Y) \geqslant r(X \cup Y) + r(X \cap Y)$

Examples of $|E| = 4$

For $\mathbf{r} = [r_1\ r_2\ r_{12}\ r_3\ r_{13}\ r_{23}\ r_{123}\ r_4\ r_{14}\ r_{24}\ r_{124}\ r_{34}\ r_{134}\ r_{234}\ r_{1234}]$

- $\mathbf{r}_1 = [0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]$ is a matroid
- $\mathbf{r}_2 = [2\ 1\ 2\ 1\ 2\ 2\ 2\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2]$ is not a matroid
- $\mathbf{r}_3 = [2\ 2\ 3\ 2\ 3\ 3\ 4\ 2\ 3\ 3\ 4\ 4\ 4\ 4\ 4]$ is not a matroid

Polymatroids and Matroids

Polymatroidal axioms

Let f map subsets of ground set E to nonnegative real numbers, the following conditions are called Polymatroidal axioms:

- $f(\emptyset) = 0$
- If $X \subseteq Y \subseteq E$, then $r(X) \leqslant r(Y)$
- If $X, Y \subset E$, then $r(X) + r(Y) \geqslant r(X \cup Y) + r(X \cap Y)$

Examples: $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ all correspond to polymatroids.

Relationship between Matroids and Polymatroids

Matroids are Polymatroids with integer rank function and cardinality constrain.

Examples of Polymatroids and Matroids

Matroids that are extreme rays of Γ_4

For $\forall N \in \{1, 2, 3, \cdots\}$, with $I \subseteq N$ and $0 \leq t \leq |N \backslash I|$, define

$$r_t^I(J) = \min\{t, |J \backslash I|\} \textit{ with } J \subseteq N$$

Then r_t^I is the matroid of rank t with loops I

Γ_2 has 3 extreme rays, all of which are matroidal:
$r_1^{\emptyset} = [1\ 1\ 1]$, $r_1^1 = [0\ 1\ 1]$ and $r_1^2 = [1\ 0\ 1]$.

Γ_4 has 27 matroid extreme rays, all of which can be expressed as r_t^I for some t and I

Examples of Polymatroids and Matroids

Non-matroidal extreme rays of Γ_4

For $N = 4$, define the following functions:

$$g_i^{(2)}(J) = \begin{cases} 2 & \text{if } J = i \\ min\{2, |J|\} & \text{if } J \neq i \end{cases}$$

$$g_i^{(3)}(J) = \begin{cases} |J| & \text{if } i \notin J \\ min\{3, |J| + 1\} & \text{if } i \in J \end{cases}$$

$$f_{ij}(K) = \begin{cases} 3 & \text{if } K \in \{ik, jk, il, jl, kl\} \\ min\{4, 2|K|\} & \text{otherwise} \end{cases}$$

Examples of Polymatroids and Matroids

Extreme rays of Γ_4

Γ_4 has 41 extreme rays, including
27 matroid of the form r_t^I for some t and I,
4 extreme rays of the form $g_i^{(2)}$ for $i = 1, 2, 3, 4$,
4 extreme rays of the form $g_i^{(3)}$ for $i = 1, 2, 3, 4$,
6 extreme rays of the form f_{ij} for $i, j \in N$ and $i \neq j$.

Examples of $N = 4$

For $\mathbf{r} = [r_1\ r_2\ r_{12}\ r_3\ r_{13}\ r_{23}\ r_{123}\ r_4\ r_{14}\ r_{24}\ r_{124}\ r_{34}\ r_{134}\ r_{234}\ r_{1234}]$

- $\mathbf{r}_1 = r_1^{12} = [0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]$
- $\mathbf{r}_2 = g_1^{(2)} = [2\ 1\ 2\ 1\ 2\ 2\ 2\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 2]$
- $\mathbf{r}_3 = f_{34} = [2\ 2\ 3\ 2\ 3\ 3\ 4\ 2\ 3\ 3\ 4\ 4\ 4\ 4\ 4]$

Modular and linear polymatroids

Modular polymatroids

- The polymatroid f is *modular* if $\Delta_{I,J} f = 0$ for any I and J disjoint, where $\Delta_{I,J} f = f(I) + f(J) - f(I \cup J) - f(I \cap J)$.
- The above condition is equivalent to $f(I) = \sum_{i \in I} f(i)$, $I \subseteq N$.

Linear polymatroids

- A polymatroid (N, f) is *linear* if there exist subspace E_i, $i \in N$, of a linear space over a field $\mathbb{F}$ such that if $I \subseteq N$ then $f(I)$ equals the dimension of the sum of E_i over $i \in I$.
- For N = 4, linear polymatroids form the Ingleton inner bound R_4.

Operations on matroids and polymatroids

Contraction

- The contraction of a polymatroids (N, f) along $I \subseteq N$ is $(N\backslash I, h)$ where $h(J) = f(J \cup I) - f(I), J \subseteq N\backslash I$.
- The almost entropic polymatroids($\overline{\Gamma^*_N}$) are closed to contractions(due to convexity of $\overline{\Gamma^*_N}$).

Convolution

- The convolution of two polymatroidal rank functions f and g on ground set N is defined as:

$$f * g(I) = \min_{J \subseteq I} \{f(J) + g(I\backslash J)\}, \qquad I \subseteq N.$$

- The almost entropic polymatroids($\overline{\Gamma^*_N}$) are closed to the convolutions with modular polymatroids.[2]

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Decomposition into Tight and Modular polymatroids

Decomposition of polymatroidal rank function

For any polymatroidal rank function h, we can decompose it into h^{ti} and h^m:

$$h^{ti}(I) = h(I) - \sum_{i \in I} [h(N) - h(N \backslash i)]$$

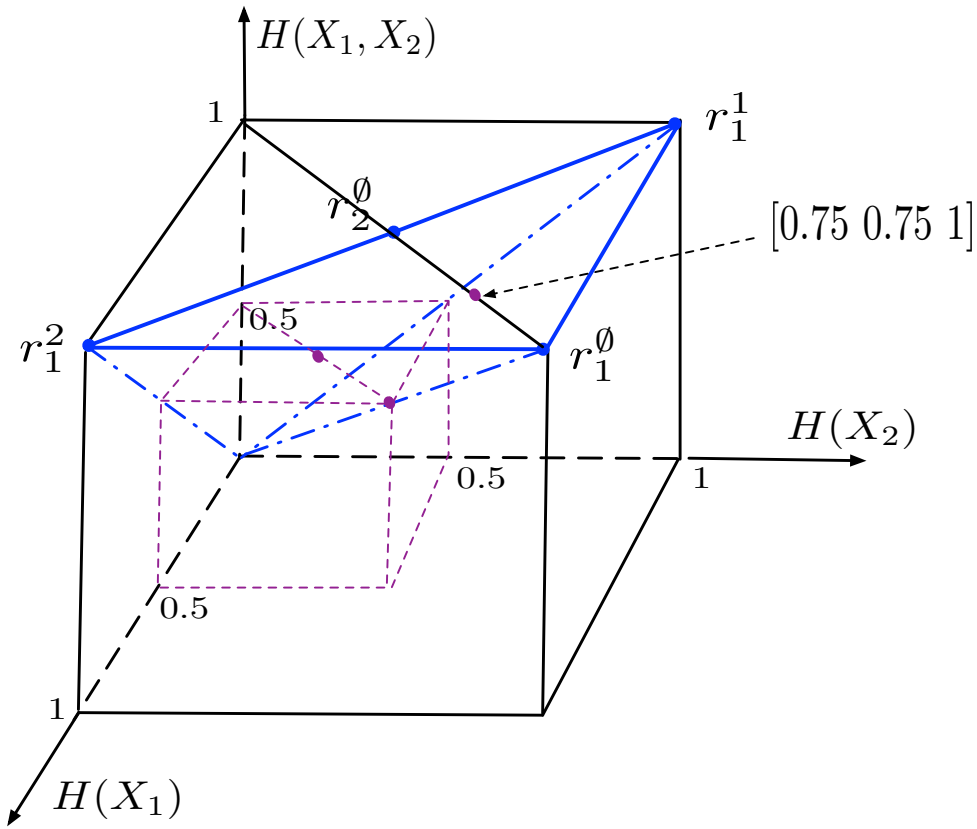
$$h^m(I) = \sum_{i \in I} [h(N) - h(N \backslash i)] \quad I \subseteq N$$

Thus $h = h^{ti} + h^m$ and Γ_N is decomposed into direct sum of the cone Γ_N^{ti} of tight rank functions and the cone Γ_N^m of modular functions, that is to say $\Gamma_N = \Gamma_N^{ti} \oplus \Gamma_N^{mod}$.

Decomposition into Tight and Modular polymatroids

Example of decomposing Γ_2

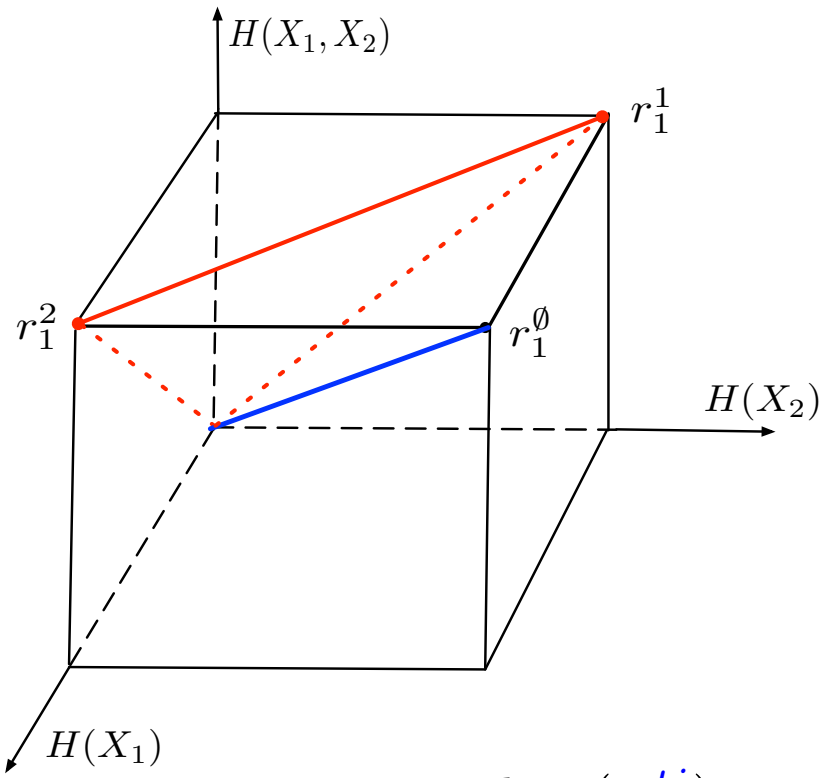
$$r_1^{\emptyset} = [1\ 1\ 1]$$
$$r_1^{1} = [0\ 1\ 1]$$
$$r_1^{2} = [1\ 0\ 1]$$
$$r_2^{\emptyset} = [1\ 1\ 2]$$



$[h_1\ h_2\ h_{12}]$	$[1\ 1\ 1]$	$[1\ 0\ 1]$	$[0.75\ 0.75\ 1]$
	$\|\|$	$\|\|$	$\|\|$
h^{ti}	$[1\ 1\ 1]$	$[0\ 0\ 0]$	$[0.5\ 0.5\ 0.5]$
	$+$	$+$	$+$
h^{m}	$[0\ 0\ 0]$	$[1\ 0\ 1]$	$[0.25\ 0.25\ 0.5]$

Decomposition into Tight and Modular polymatroids

Example of decomposing Γ_2



$$\Gamma_2 = \Gamma_2^{ti} \oplus \Gamma_2^{mod} \qquad dim(\Gamma_2^{ti}) = 1 \quad dim(\Gamma_2^{mod}) = 2$$

Decomposition into Tight and Modular polymatroids

Theorem: If $h \in \overline{\Gamma^*_N}$ then $h^{ti} \in \overline{\Gamma^*_N}$

Let (N, h) be a polymatroid, N' be a disjoint copy of N and $i \mapsto i'$ between them. The polymatroid (N, h) extends to $(N \cup N', f)$ by

$$f(I \cup J') = h(I \cup J), \qquad I, J \subseteq N \quad \text{where } J' = \{j' : j \in J\}.$$

Let $(N \cup N', g)$ be a modular polymatroid, then from definition of convolution, for $I \subseteq N$

$$f * g(I \cup N') = \min_{J \subseteq I, K \subseteq N} [h(J \cup K) + g(I \backslash J) + g(N' \backslash K')]$$

By monotonicity of g, we can restricted the above minimization to $J = K \cap I$

$$f * g(I \cup N') = \min_{K \subseteq N} [h(K) + g(I \backslash K) + g(N' \backslash K')]$$

Decomposition into Tight and Modular polymatroids

(cont.)If $g(i) + g(i') = h(i)$ for $i \in N$ then

$$f * g(I \cup N') = \min_{K \subseteq N} [h(K) + \sum_{i \in I \setminus K} h(i) + g(N' \setminus (I' \cup K'))]$$

By submodularity of h, the minimization can be restricted to $K \supseteq I$, thus

$$f * g(I \cup N') = \min_{I \subseteq K \subseteq N} [h(K) + g(N' \setminus K')], \quad I \subseteq N.$$

If $g(i') = h^m(i) = h(N) - h(N \setminus i) \leqslant h(i), \forall i \in N$

$$f * g(I \cup N') = h^{ti}(I) + h^m(N) \quad I \subseteq N.\text{for when K = I}$$

Thus $h^{ti}(I) = f * g(I \cup N') - f * g(N')$, h^{ti} is the contraction of $f * g$ along N'. If $h \in \overline{\Gamma^*_N}$, then $f \in \overline{\Gamma^*_{2N}}$, after convolution, we have $f * g \in \overline{\Gamma^*_{2N}}$, thus h^{ti}, the contraction of $f * g$ along N' is almost entropic.

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Entropy region of four variables

Ingleton inequality

$$
\begin{aligned}
&Ingleton_{34} = \square_{34} h \\
&= I(X_1; X_2|X_3) + I(X_1; X_2|X_4) + I(X_3; X_4|\emptyset) - I(X_1; X_2|\emptyset) \\
&= h_{12} + h_{13} + h_{23} + h_{14} + h_{24} - h_1 - h_2 - h_{34} - h_{123} - h_{124} \geqslant 0
\end{aligned}
$$

R_4: generated by four variable shannon-type inequalities and 6 Ingleton inequalities

R_4 has 35 extreme rays, including
27 matroid of the form r_t^I for some t and I,
8 extreme rays of the form $g_i^{(2)}$ and $g_i^{(3)}$ for $i = 1, 2, 3, 4$.
Note: $f_{ij} \notin R_4$ for $i, j \in N$ and $i \neq j$

G_4^{ij}: The gap between R_4 and Γ_4

$G_4^{ij} = \{h \in \Gamma_4 | Ingleton_{ij} \leq 0\}$

G_4^{ij} is the convex hull of 15 extreme rays, the V-representation are generated by the 15 linearly independent functions f_{ij}, r_1^{ijk}, r_1^{ijl}, r_1^{ikl}, r_1^{jkl}, $r_1^{\emptyset}$, $r_3^{\emptyset}$, r_1^{i}, r_1^{j}, r_1^{ik}, r_1^{jk}, r_1^{il}, r_1^{jl}, r_2^{k}, r_2^{l}.

Decomposition of G_4^{ij}

We can decompose G_4^{ij} into the cone of tight polymatroids G^{ti} and the cone of modular polymatroids G^m.

- G^m is 4 dimensional, consisting of four extreme rays r_1^{ijk}, r_1^{ijl}, r_1^{ikl} and r_1^{jkl}. ($G^m = \Gamma_4^{mod} \in \overline{\Gamma_4^*}$)
- G^{ti} is 11 dimensional, consisting of eleven extreme rays f_{ij}, $r_1^{\emptyset}$, $r_3^{\emptyset}$, r_1^{i}, r_1^{j}, r_1^{ik}, r_1^{jk}, r_1^{il}, r_1^{jl}, r_2^{k}, r_2^{l}.

L_{ij}: the tight and almost entropic polymatroids in G_4^{ij}

Tight and almost entropic polymatroids

Let L_{ij} be the tight and almost entropic polymatroids in G_4^{ij}, then $\overline{\Gamma_4^*}$ consist of Ingleton inner bound R_4 and the disjoint union of six convex cones $L_{ij} \bigoplus \Gamma_4^{mod}$, that is to say $G_4^{ij} \cap \overline{\Gamma_4^*} = L_{ij} \bigoplus \Gamma_4^{mod}$.

Every $g \in G^{ti}$ is the unique conic combination of the rank functions f_{ij}, $r_1^{\emptyset}$, $r_3^{\emptyset}$, r_1^i, r_1^j, r_1^{ik}, r_1^{jk}, r_1^{il}, r_1^{jl}, r_2^k, r_2^l.

$$\begin{aligned} g \quad &= -(\square_{ij} g) f_{ij} + (\Delta_{ij|\emptyset} g) r_i + (\Delta_{kl|ij} g) r_3 \\ &+ (\Delta_{kl|i} g) r_1^i + (\Delta_{kl|j} g) r_1^j + (\Delta_{ij|k} g) r_2^l + (\Delta_{ij|l} g) r_2^k \\ &+ (\Delta_{jl|k} g) r_1^{ik} + (\Delta_{il|k} g) r_1^{jk} + (\Delta_{jk|l} g) r_1^{il} + (\Delta_{ik|l} g) r_1^{jl} \end{aligned}$$

Faces of L_{ij}

F_{ij}: the face of L_{ij} such that $\Delta_{ij|\emptyset} g = 0$ and $\Delta_{kl|ij} g = 0$

Let $A_{i,j}$ and $B_{ij,k}$ be two linear mappings defined by

$$A_{i,j} g = g + (\Delta_{i,j|\emptyset} g)(r_1^i - r_1)$$
$$B_{ij,k} g = g + (\Delta_{kl|ij} g)(r_2^k - r_3)$$

$A_{i,j}$ maps g into hyperplane $\Delta_{i,j|\emptyset} g = 0$, $B_{ij,k}$ maps g into $\Delta_{kl|ij} g = 0$ and

$$F_{ij} = A_{i,j} B_{ij,k} g = B_{ij,k} A_{i,j} g = g + (\Delta_{i,j|\emptyset} g)(r_1^i - r_1) + (\Delta_{kl|ij} g)(r_2^k - r_3)$$

More importantly, $A_{i,j}$ and $B_{ij,k}$ leave $\square_{ij}$ unchanged

$$\square_{ij} g = \square_{ij}(A_{i,j} g) = \square_{ij}(B_{ij,k} g) \text{ for } g \in \Gamma_4$$

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Symmetrization of F_{ij}

L_{ij} and F_{ij} are closed to action $h \mapsto h_\pi$ where π is a permutation on N stabilizes the two-element set ij.

Let C_{ij} be the linear mapping given by

$$C_{ij}h \triangleq= |P_{ij}|^{-1} \sum_{\pi \in P_{ij}} h_\pi$$

where P_{ij} denotes the stabilizer of ij, consisting of four permutations. If $h \in G^{ti}$ then

$$\begin{aligned} C_{ij}h \quad &= -(\square_{ij}h)f_{ij} + (\Delta_{ij|\emptyset}g)r_i + (\Delta_{kl|ij}g)r_3 \\ &+ \frac{1}{2}[\Delta_{kl|i}h + \Delta_{kl|j}h][r_1^i + r_1^j] + \frac{1}{2}[\Delta_{ij|k}h + \Delta_{ij|l}h][r_2^l + r_2^k] \\ &+ \frac{1}{4}[\Delta_{jl|k}h + \Delta_{il|k}h + \Delta_{jk|l}h + \Delta_{ik|l}h][r_1^{ik} + r_1^{jk} + r_1^{il} + r_1^{jl}] \end{aligned}$$

The dimension of $C_{ij}L_{ij}$ is 6 and dimension of $C_{ij}F_{ij}$ is 4.

Symmetrization of F_{ij}

$S_{ij} \triangleq \{h \in C_{ij}F_{ij} : h(N) = 1\}$

Let $\bar{\alpha}_h = -4\Box_{ij}h$, $\bar{\beta}_h = \Delta_{kl|i}h + \Delta_{kl|j}h$, $\bar{\gamma}_h = 2\Delta_{ij|k}h + 2\Delta_{ij|l}h$ and $\bar{\delta}_h = \Delta_{jl|k}h + \Delta_{il|k}h + \Delta_{jk|l}h + \Delta_{ik|l}h$.

Then

$$\bar{\alpha}_h + \bar{\beta}_h + \bar{\gamma}_h + \bar{\delta}_h = h(N) = 1$$

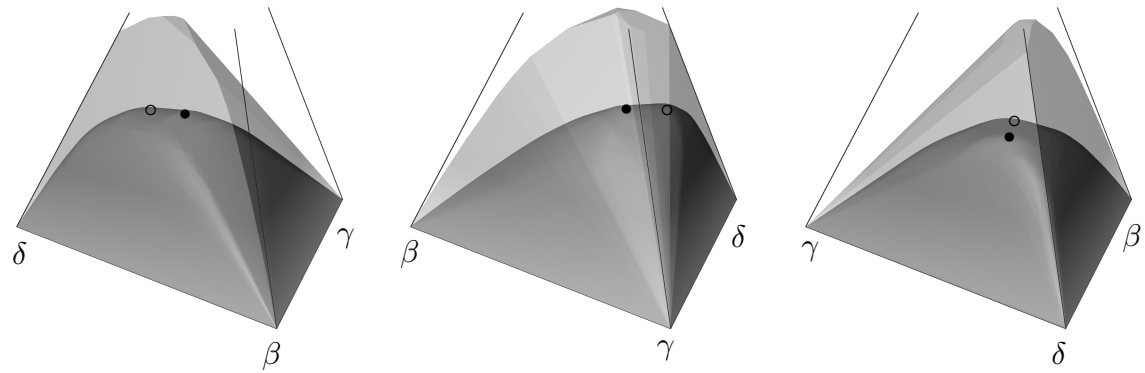
For any $h \in S_{ij}$,

$$\begin{aligned} h &= \bar{\alpha}_h \frac{1}{4} f_{ij} + \bar{\beta}_h \frac{1}{2}[r_1^i + r_1^j] + \bar{\gamma}_h \frac{1}{4}[r_2^l + r_2^k] + \bar{\delta}_h \frac{1}{4}[r_1^{ik} + r_1^{jk} + r_1^{il} + r_1^{jl}] \\ &\triangleq \bar{\alpha}_h \alpha + \bar{\beta}_h \beta + \bar{\gamma}_h \gamma + \bar{\delta}_h \delta \end{aligned}$$

If $f \in \overline{\Gamma_4^*}$, let $g = C_{ij}A_{i,j}B_{ij,k}f^{ti}$, then $h = g/g(N) \in S_{ij}$

Symmetrization of F_{ij}

Inner and outer approximation of S_{ij}



- Dark grey area: Inner bound from distributions take at most 11 values(5 million points and 2200 extreme).
- Ligth grey area: Outer bound from existing non-Shannon Type Inequalities.
- Dark dot: transformation of 4-*atom* distribution(stay tuned).
- Circle: transformation of the distribution refute Four-atom conjecture(stay tuned).

Symmetrization of F_{ij}

Limitations of the numerical experiments

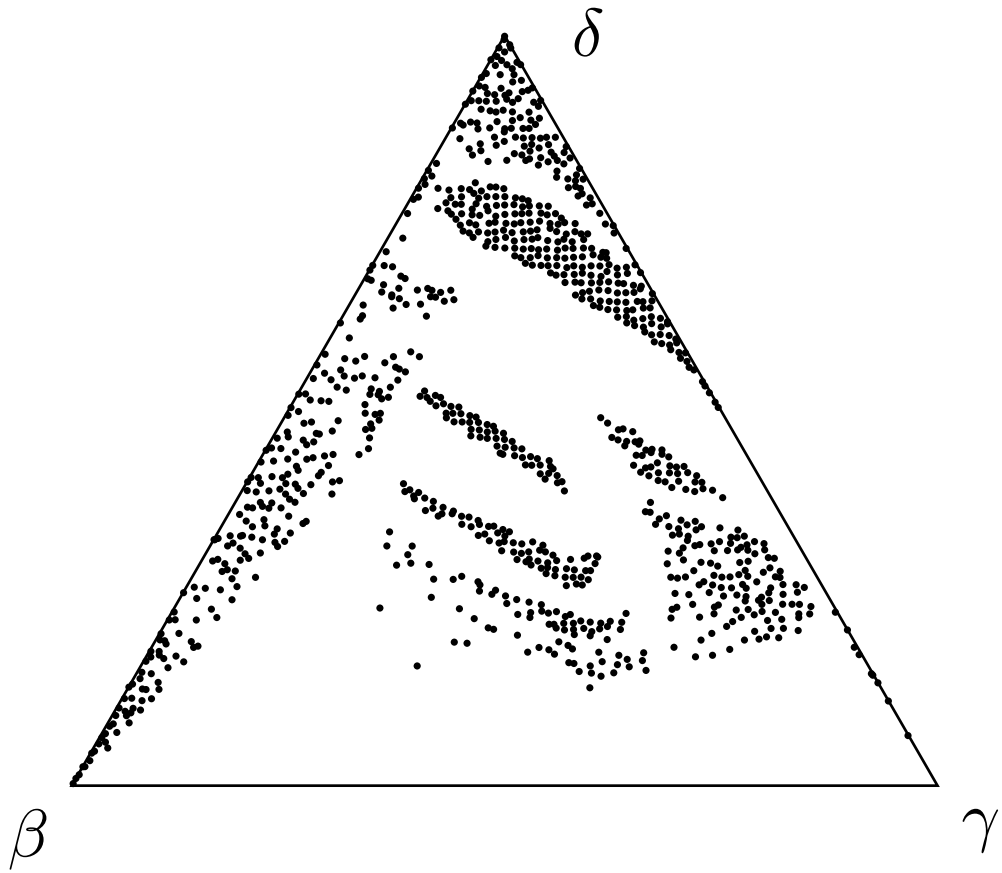


Figure : Extreme points of the dark grey region projected to $\beta\gamma\delta$

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The *Ingleton score* of a polymatroidal rank function $h \neq 0$ is defined as $\mathbb{I}_{ij}(h) = \square_{ij} h / h(N)$. The number

$$\mathbb{I}^* = \inf\{\mathbb{I}_{ij}(h) : 0 \neq h \in \Gamma_4\}$$

is referred to as the *infimal* Ingleton score.

Four-atom conjecture

The lowest possible Ingleton score is approximately -0.0894. It is attained by a four-variable binary distribution, given by

$$P(0,0,0,0) = P(1,1,1,1) \approx 0.3505$$
$$P(0,1,0,1) = P(0,1,1,0) \approx 0.1495$$

Ingleton Score

Minimum Ingleton score of Outer bound

- No copy: -0.2500
- One copy variable: -0.1667
- Two copy variables: -0.1667
- Three copy variables: -0.1590
- Four copy variables with at most three copy steps: -0.1579
- compare to -0.0894 of Four-atom

Ingleton Score

Theorem: $\mathbb{I}^* = \min_{S_{ij}} \mathbb{I}_{ij}$

Since all four extreme rays of Γ_4^{mod} make $\mathbb{I}_{ij} = 0$, if $h \in G_4^{ij}$, then $\mathbb{I}_{ij}(h) \geqslant \mathbb{I}_{ij}(h^{ti})$, thus

$$\mathbb{I}^* = \min\{\mathbb{I}_{ij}(h) : h(N) = 1 \text{ and } h \in L_{ij}\}$$

For any $g \in L_{ij}$,

$$A_{i,j} B_{ij,k} g(N) = g(N) - \Delta_{kl|ij} g = g(ij)$$

Since $g(ij) \leqslant g(N)$ and linear mappings $A_{i,j}$, $B_{ij,k}$ don't change $\square_{ij}$, we have

$$\mathbb{I}_{ij}(g) \geqslant \mathbb{I}_{ij}(A_{i,j} B_{ij,k} g)$$

The rest follows by symmetrization.

Ingleton Score

The distribution lead to the violation of Four-atom conjecture

Let four variables take values in $\{0, 1, 2, 3\}$ and p, q, r, s, t be nonnegative such that $p + q + r + s + t = \frac{1}{8}$

p	q	r	s	t
0000	0210	0011	0010	0001
0101	0321	0120	0121	0100
1010	1100	1002	1000	1012
1212	1332	1230	1232	1210
2121	2001	2103	2101	2123
2323	2233	2331	2333	2321
3232	3012	3213	3212	3233
3333	3123	3322	3323	3332

Ingleton Score

The distribution lead to the violation of Four-atom conjecture

If we let

$$p = 0.09524, q = 0.02494, r = 0.0016, s = t = 0.00161$$

Then $\mathbb{I}_{ij}(f) \approx -0.0783$, however, $\mathbb{I}_{ij}(f^{ti}) \approx -0.0924$.

Furthermore, let $g = A_{i,j}B_{ij,k}f^{ti}$ then $\square_{ij}g = \square_{ij}f$ and $g(N) < f^{ti}(N)$

$$\mathbb{I}_{ij}(g) \approx -0.09243$$

Thanks!

Questions?