Network Coding: Computation, Symmetry, and Hierarchy

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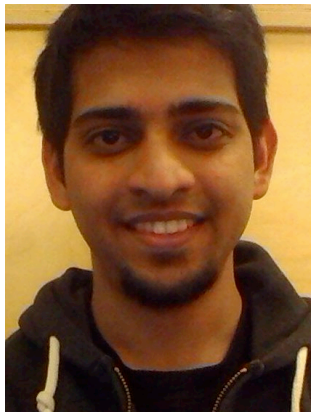
rate region database
network operators
forbidden net. minors



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network symmetry
software: ITAP & CHM

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co-advisor for
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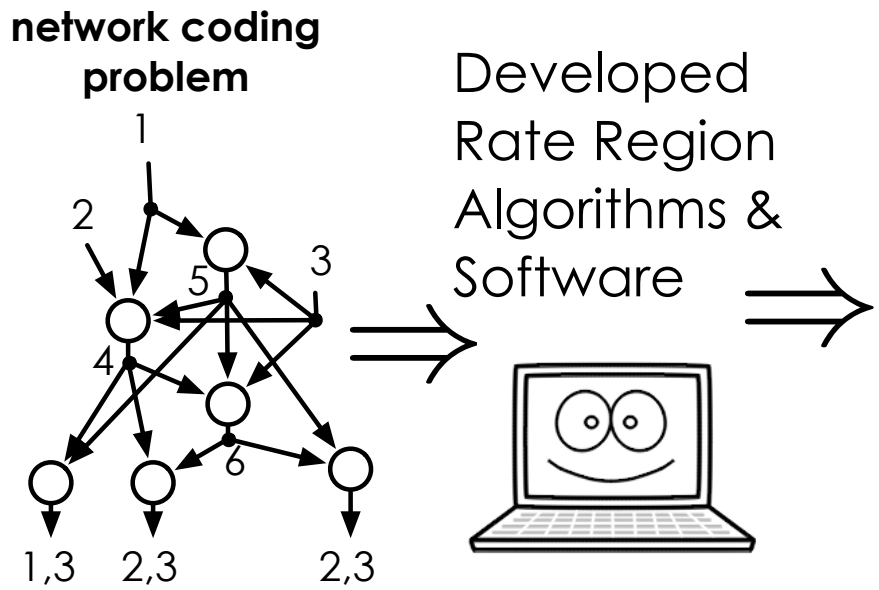
Ph.D. Candidate
ASPITRG,
Drexel University

nonlinear entropic
vectors & codes

Outline

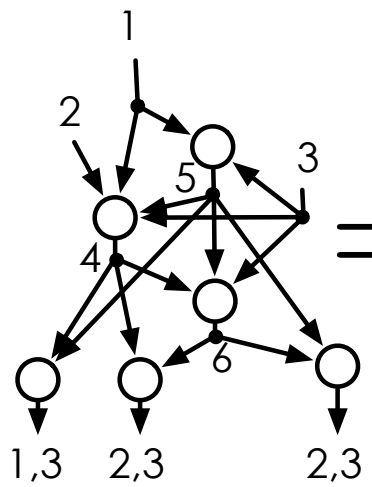
1. Problem Statement. Minimality & Symmetry
2. Symmetry Exploiting Rate Region Computation
   (a) Converse: Symmetry Exploiting CHM
   (b) Achievability:
      i. Linear Code Enumeration
      ii. Nonlinear Code (Partition) Enumeration
3. Rate Region Database
4. Hierarchy: Network Operators
   (a) Big $\rightarrow$ Small: Embedding Operators
   (b) Small $\rightarrow$ Big: Combinations Operators
   (c) Operator Concatenation
   (d) Coverage

Computationally Enabled Research Agenda – Motivation



Computationally Enabled Research Agenda – Motivation

network coding problem



Developed Rate Region Algorithms & Software

inequality description

$$\min\{R_4, R_6\} \geq H(Y_2)$$
$$R_4 + R_6 \geq H(Y_2) + H(Y_3)$$
$$R_5 + R_6 \geq H(Y_2) + H(Y_3)$$
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rate region

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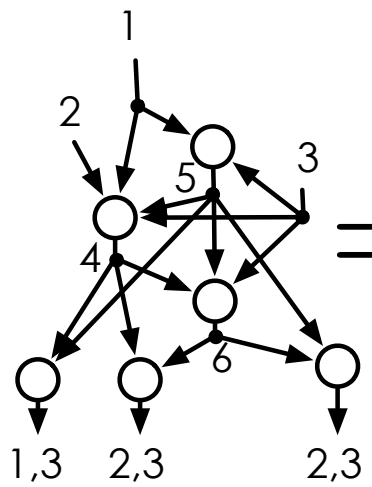

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extreme ray description

$H(Y_1)$	1	0	0	0	0	0	0	1	0	0	0
$H(Y_2)$	0	1	0	0	0	0	1	0	0	0	0
$H(Y_3)$	0	1	2	1	1	0	0	0	0	1	0
R_4	0	1	1	1	1	1	1	1	0	0	0
R_5	1	1	1	0	1	0	0	0	0	1	1
R_6	0	1	1	1	0	0	1	0	1	1	0

rate region

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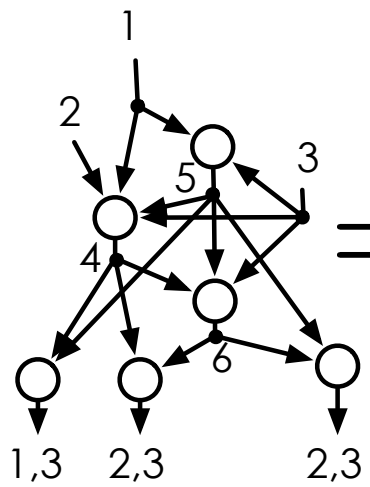

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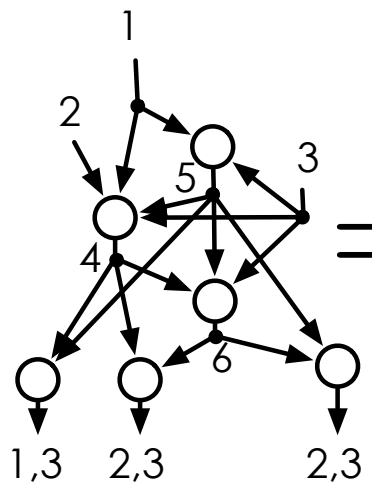

Developed Rate Region Algorithms & Software

automated converse proofs

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Proof:

$$R_4 \geq H(U_4)$$

●
●
●

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rate region

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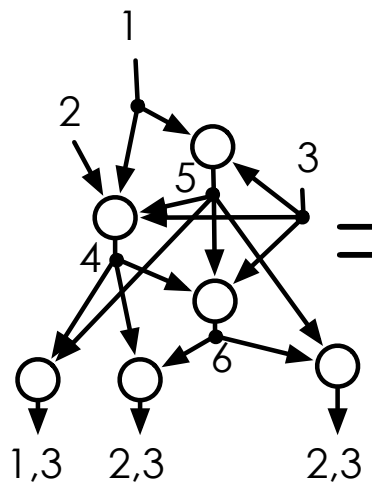

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rate region

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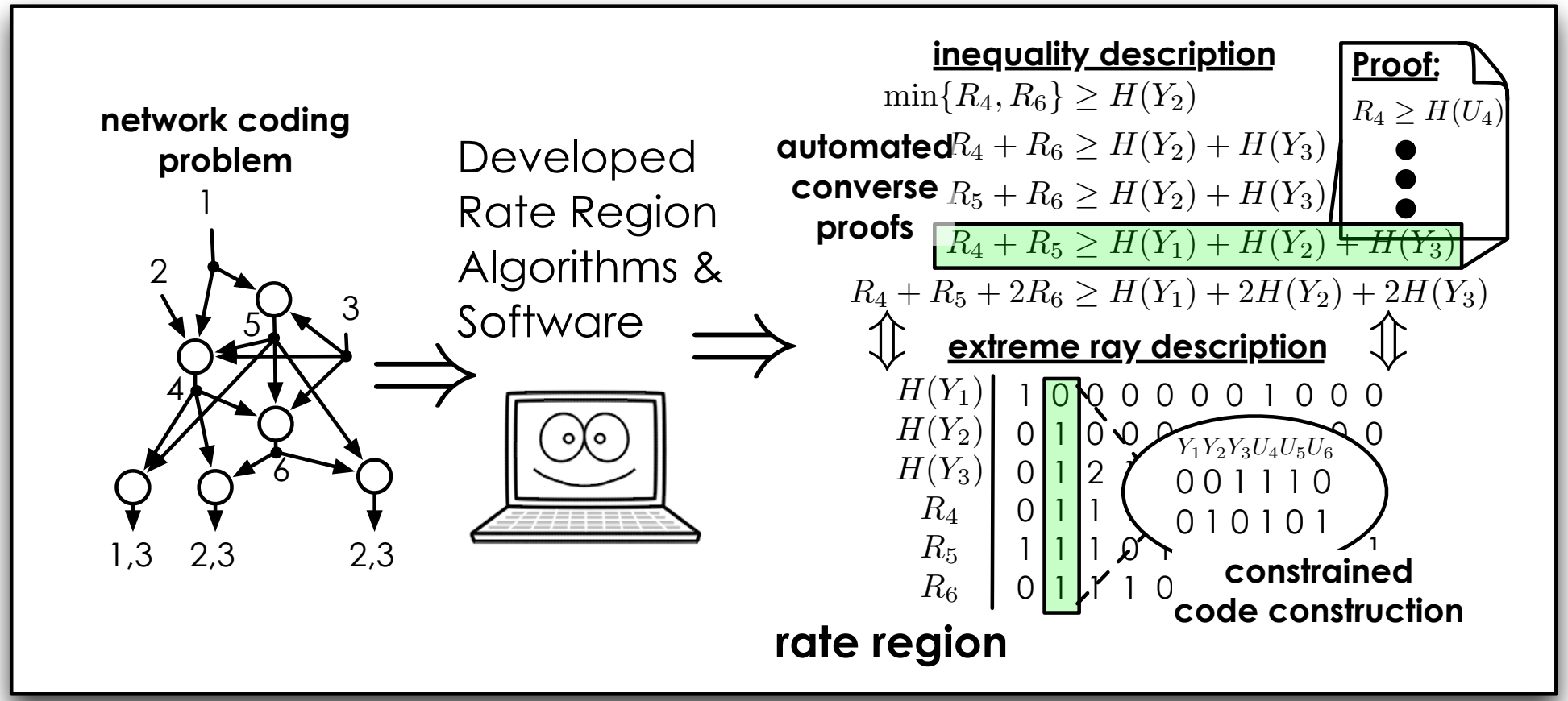

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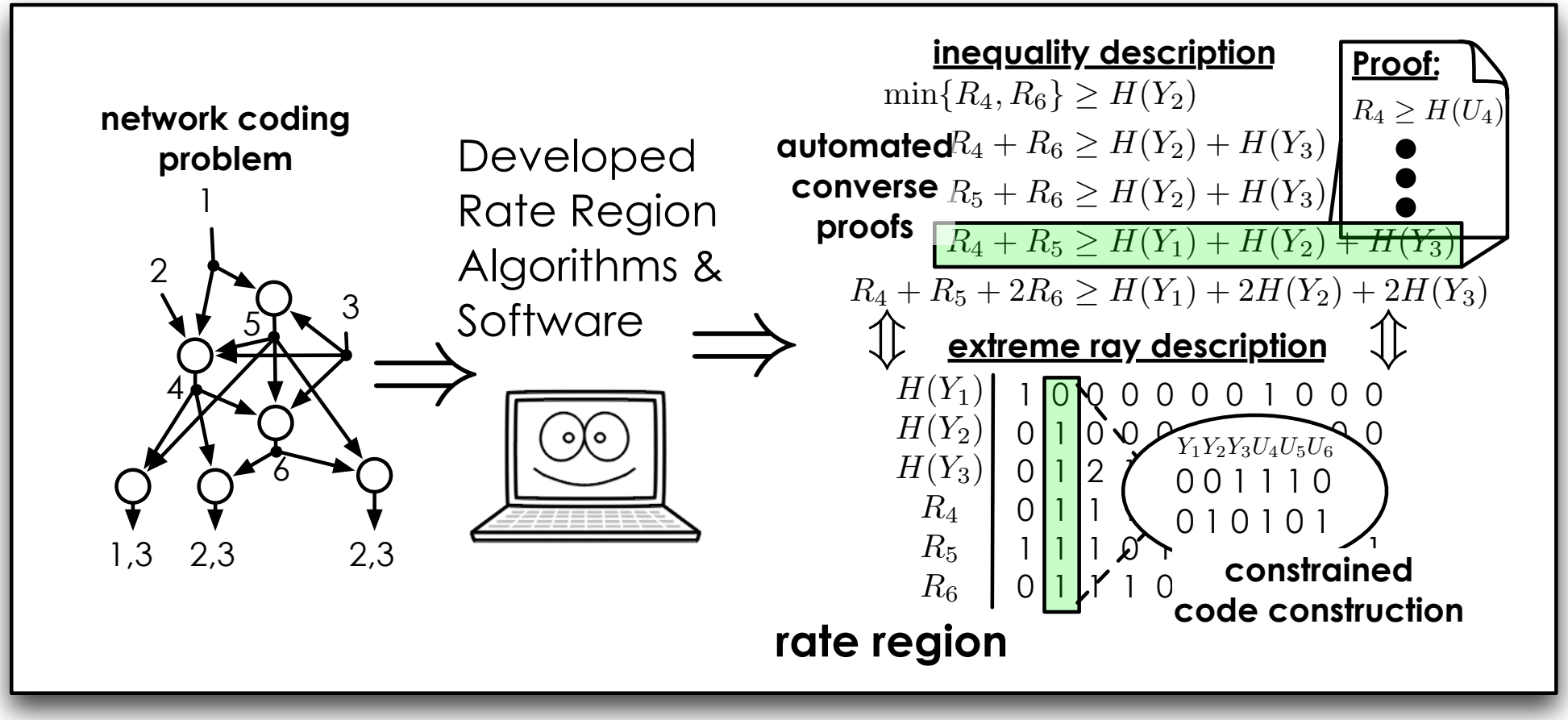


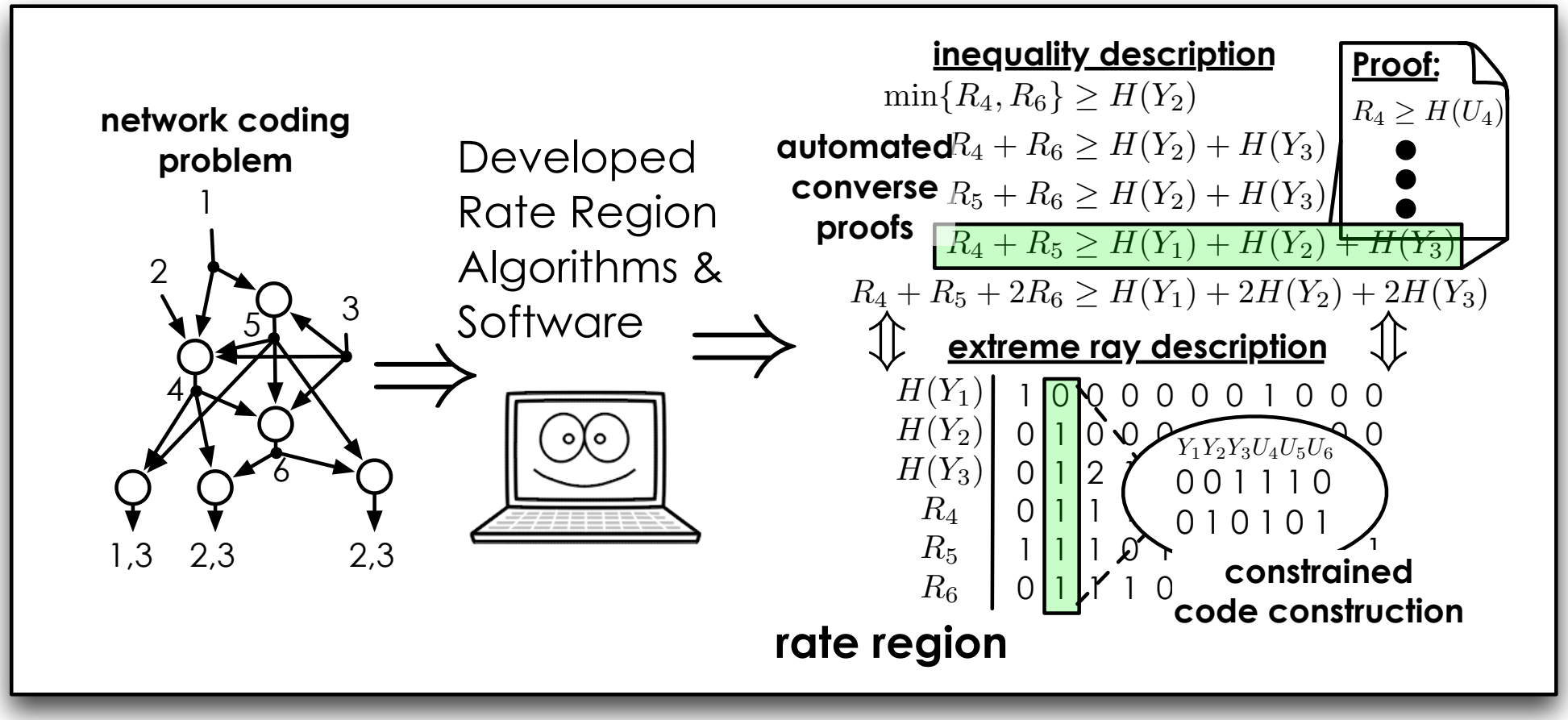
Substitutes outer/inner bounds to Γ_N^* into Yan, Yeung, Zhang '12

$$\mathcal{R}_* = \text{proj}_{R_\mathcal{E}, [H(Y_s)|s\in\mathcal{S}]}(\overline{\text{con}(\Gamma_\text{N}^* \cap \mathcal{L}_{123})} \cap \mathcal{L}_{45}) \tag{1}$$

where $\mathcal{L}_{123} := \left\{ \mathbf{h} \,\middle|\, h_{Y_\mathcal{S}} = \Sigma_{s\in\mathcal{S}} h_{Y_s},\ h_{X_{\text{Out}(k)}|Y_s} = 0,\ h_{X_{\text{Out}(i)}|X_{\text{In}(i)}} = 0 \right\}$ and

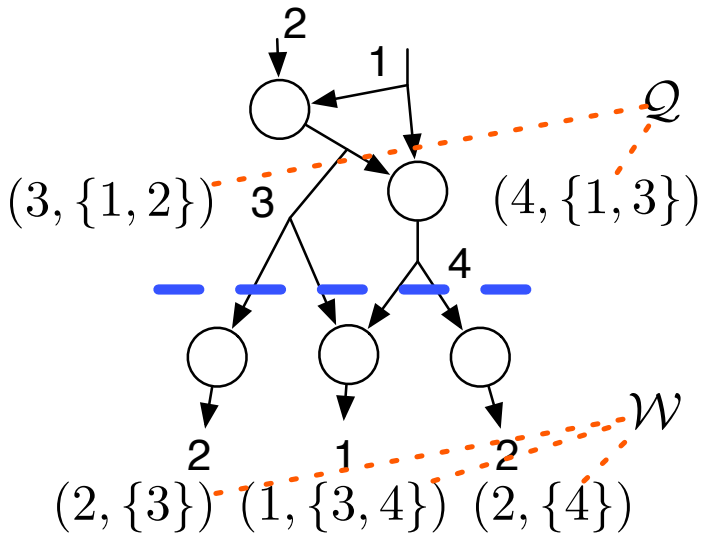
$\mathcal{L}_{45} := \{(\mathbf{h}^T, \mathbf{R}^T)^T \in \mathbb{R}_+^{2^N-1+|\mathcal{E}|} : R_e \geq h_{U_e}, e \in \mathcal{E},\ h_{Y_{\beta(t)}|U_{\text{In}(t)}} = 0\}$

Computationally Enabled Research Agenda – Motivation



- Want software to reach as large networks as possible. $\Longrightarrow$ Define & Exploit Problem Symmetry

Exploiting Symmetry in Rate Region Computation – Definition [Li Arxiv '15]



- Represent a network as $\mathsf{A} := (\mathcal{Q}, \mathcal{W})$:
  - sources: $1 \dots K$
  - edges: $K+1, \dots, K+|\mathcal{E}_U|$
  - Edge definitions $\mathcal{W}$ a set of $(i, \mathcal{A})$, $i \in \{K+1, \dots, K+|\mathcal{E}_U|\}$, $\mathcal{A} \subseteq \{1, \dots, K+|\mathcal{E}_U|\} \setminus \{i\}$;
  - Sink definitions $\mathcal{Q}$ a set of $(i, \mathcal{A})$, $i \in \{1, \dots, K\}$, $\mathcal{A} \subseteq \{1, \dots, K+|\mathcal{E}_U|\} \setminus \{i\}$;
  - Same i can appear in $\mathcal{W}$ but not in $\mathcal{Q}$
- Group $\mathcal{G} := S_{\{1,2,\dots,K\}} \times S_{\{K+1,\dots,K+|\mathcal{E}_U|\}}$ acts naturally on $(\mathcal{Q}, \mathcal{W})$.
- Network Symmetry Group (NSG): stabilizer subgroup $\mathcal{G}_\mathsf{A} = \{g \in \mathcal{G} | g((\mathcal{Q}, \mathcal{W})) = (\mathcal{Q}, \mathcal{W})\}$.
- NSG Stabilizes constraints $\mathcal{L}_{123}$ and $\mathcal{L}_{45}$ in rate region expression. $\pi(\mathcal{R}_*(\mathsf{A})) = \mathcal{R}_*(\mathsf{A})$ for all $\pi \in \mathcal{G}_\mathsf{A}$
- any $g \in \mathcal{G}$ w/ $g \notin \mathcal{G}_\mathsf{A}$ gives an equivalent network $\mathsf{A}' = g\mathsf{A}$ s.t. $g(\mathcal{R}(\mathsf{A})) = \mathcal{R}_*(\mathsf{A}')$.
- Orbit Stabilizer Theorem $\implies$ there are $\frac{|\mathcal{G}|}{|\mathcal{G}_\mathsf{A}|}$ such equivalent networks

Exploiting Symmetry in Rate Region Computation – Converse

Converse: calculate

$$\mathcal{R}_o = \text{proj}_{R_\mathcal{E},[H(Y_s)|s\in\mathcal{S}]}(\Gamma_N^o \cap \mathcal{L}_{12345}) \tag{2}$$

through polyhedral projection.

Extreme Ray/Point Projection

Project Extreme Rays & Points

↓

Redundancy Removal (convex hull)

extremal repr. of projected polyhedron

↓

Representation conversion

↓

inequality repr. of projected polyhedron

Fourier Motzkin Elimination

Successively Eliminates Variables (slow)

inequalities of N dimen. polyhedron

↓

N-1 dim. proj. (elim. 1 var)

↓

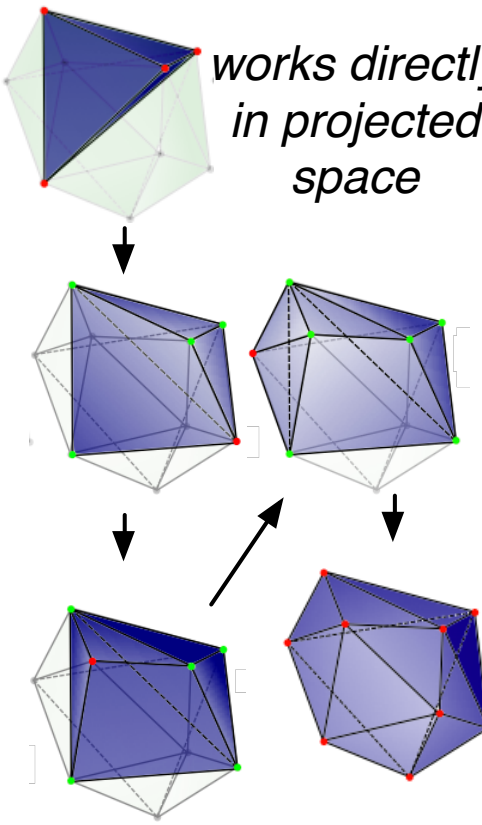
N-2 dim. proj. (elim. 1 var)

↓

⋮

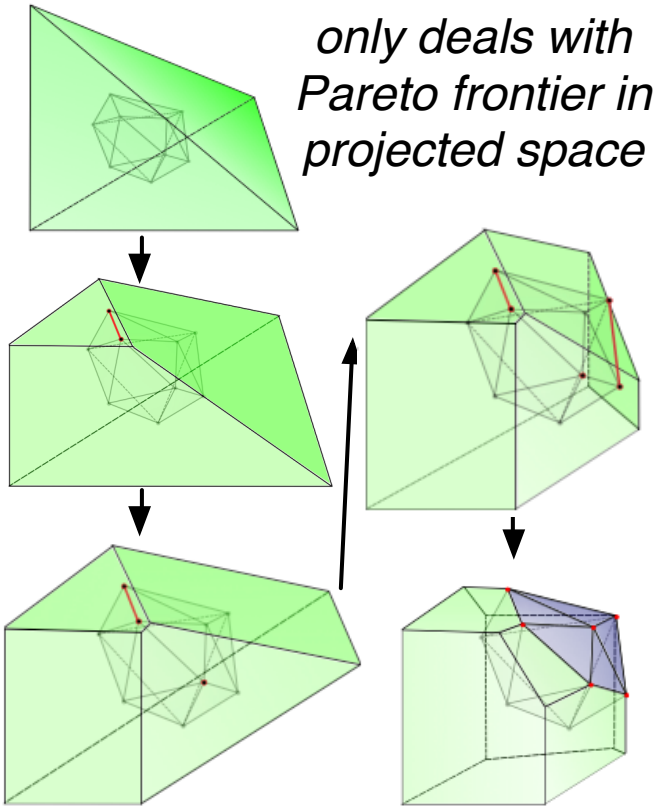
Convex Hull Method

works directly in projected space

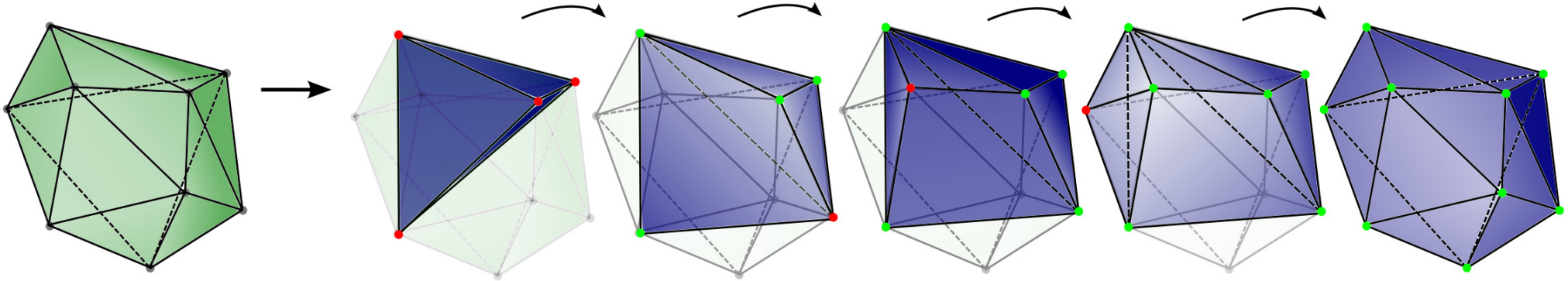


Benson's Outer Approximation Algorithm

only deals with Pareto frontier in projected space

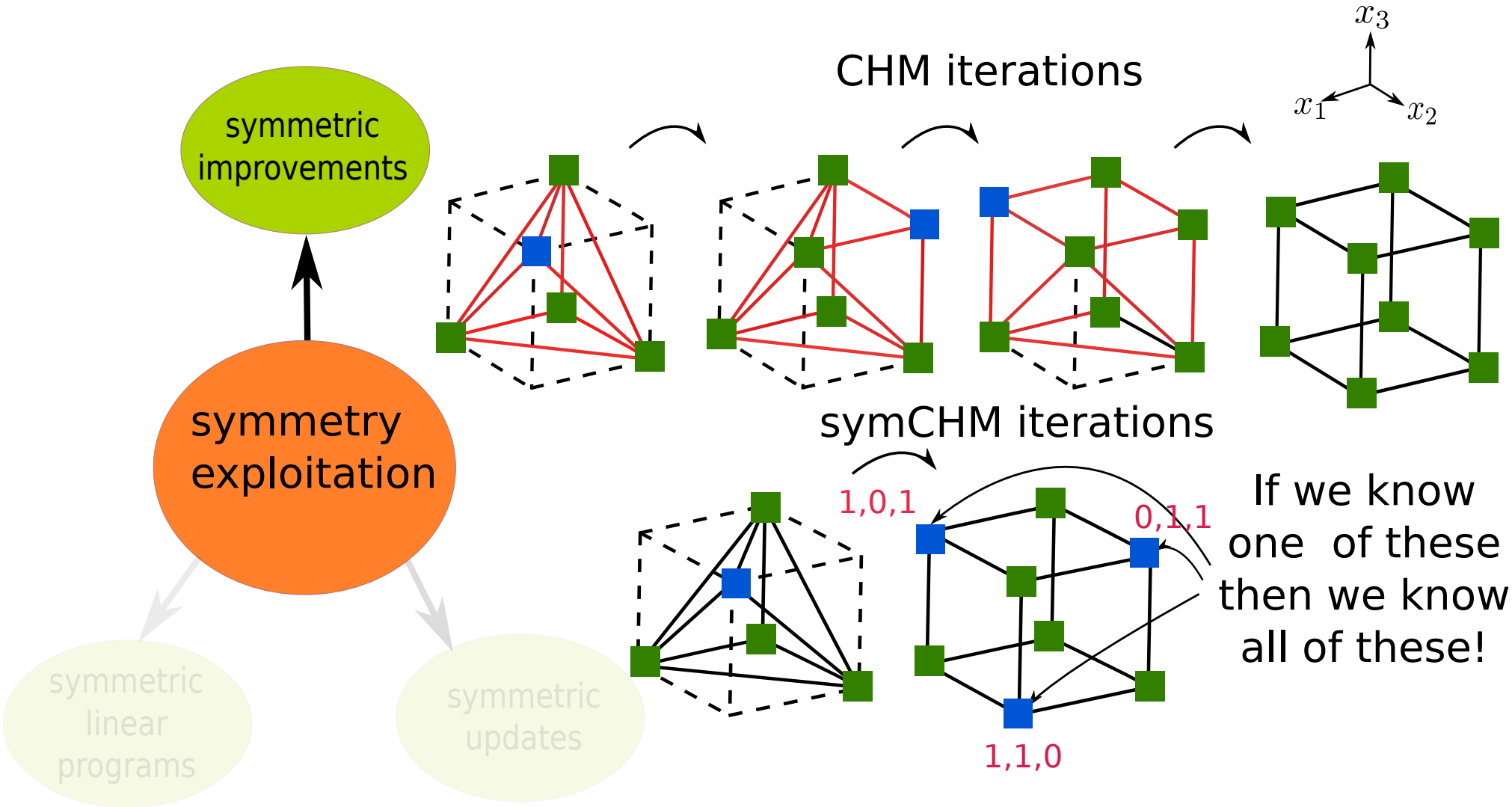


Exploiting Symmetry in Rate Region Computation – Converse



Apte's CHM w/ rational LP solver implementation available from ASPITRG website.

Exploiting Symmetry in Rate Region Computation – Converse [Apte Netcod 2015]



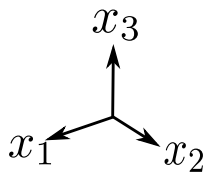
Exploiting Symmetry in Rate Region Computation – Converse [Apte Netcod 2015]

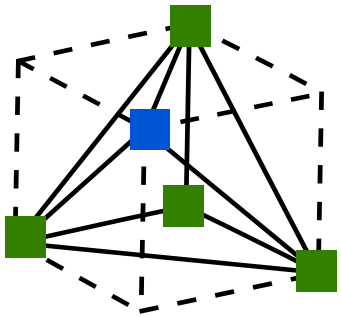
symmetric improvements

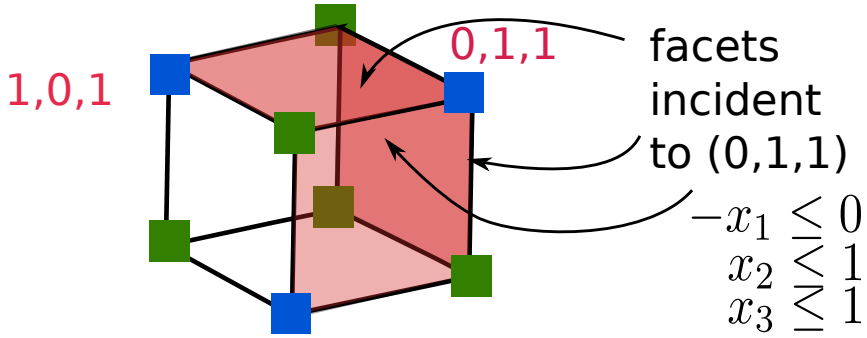
symmetry exploitation

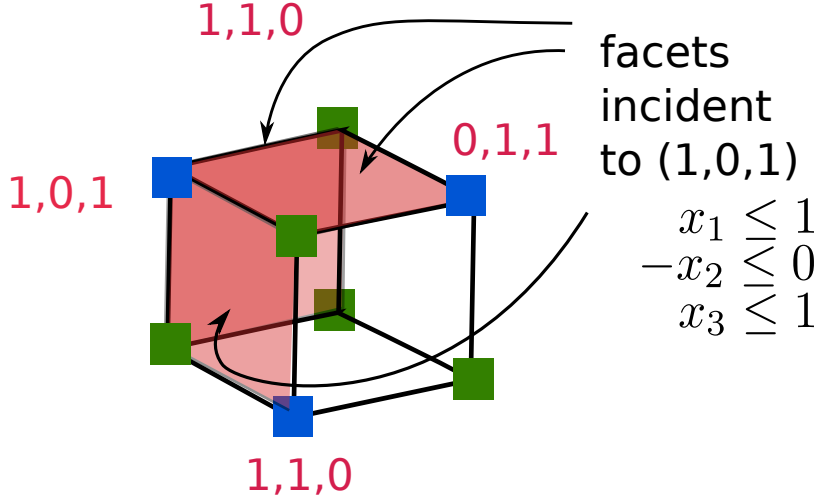
symmetric linear programs

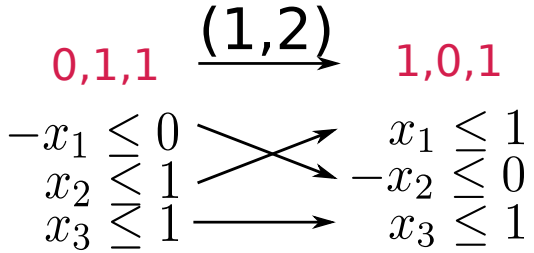
symmetric updates











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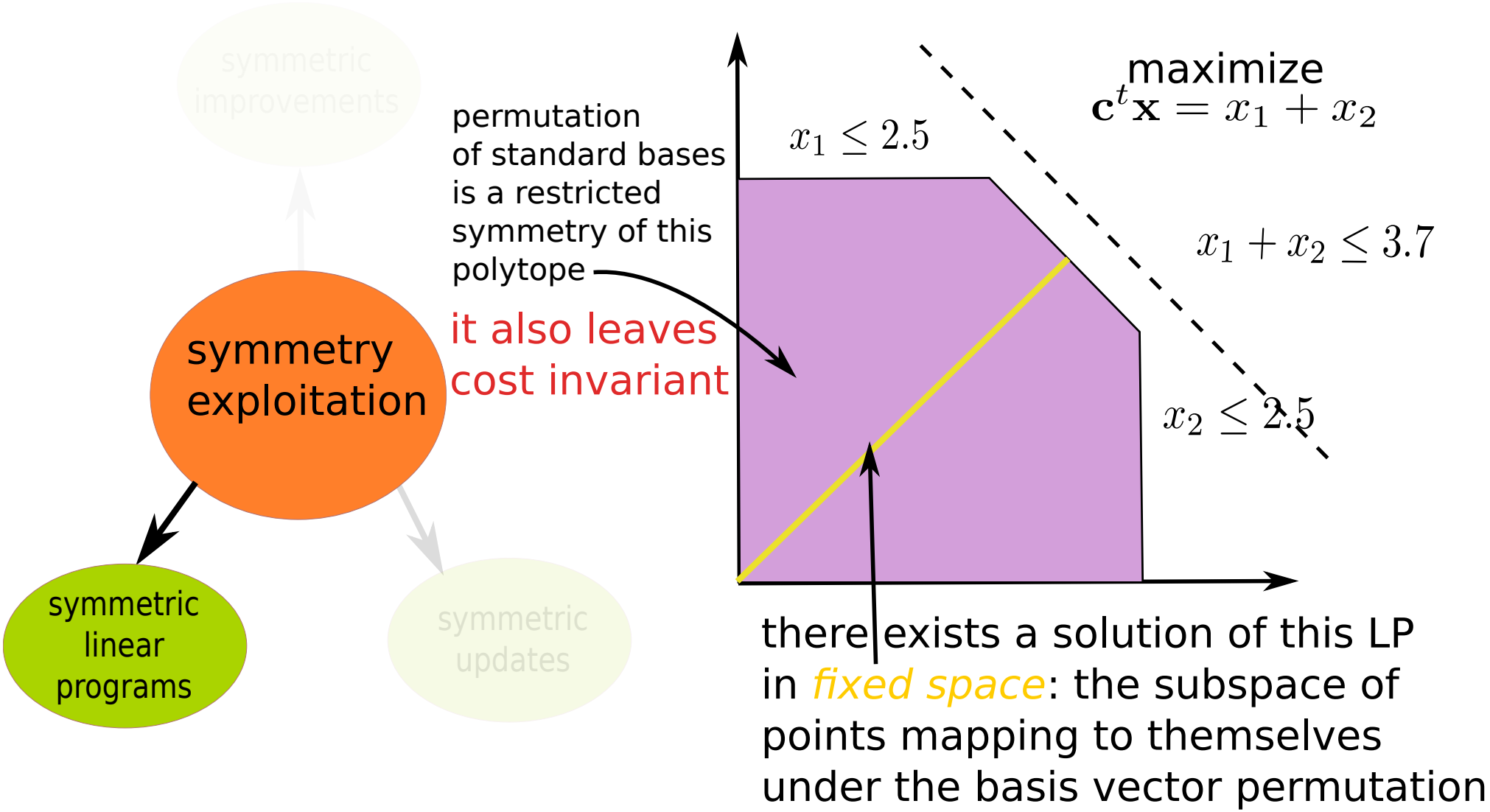

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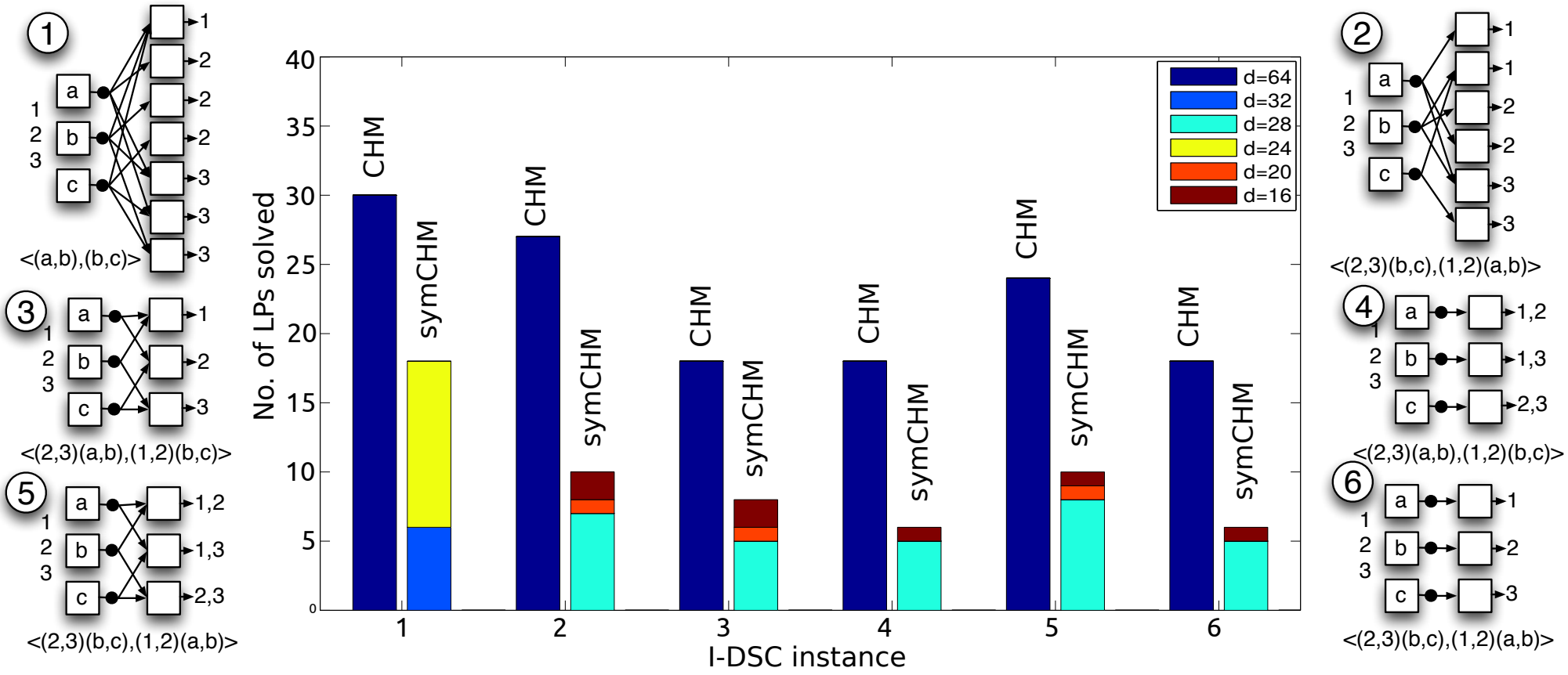


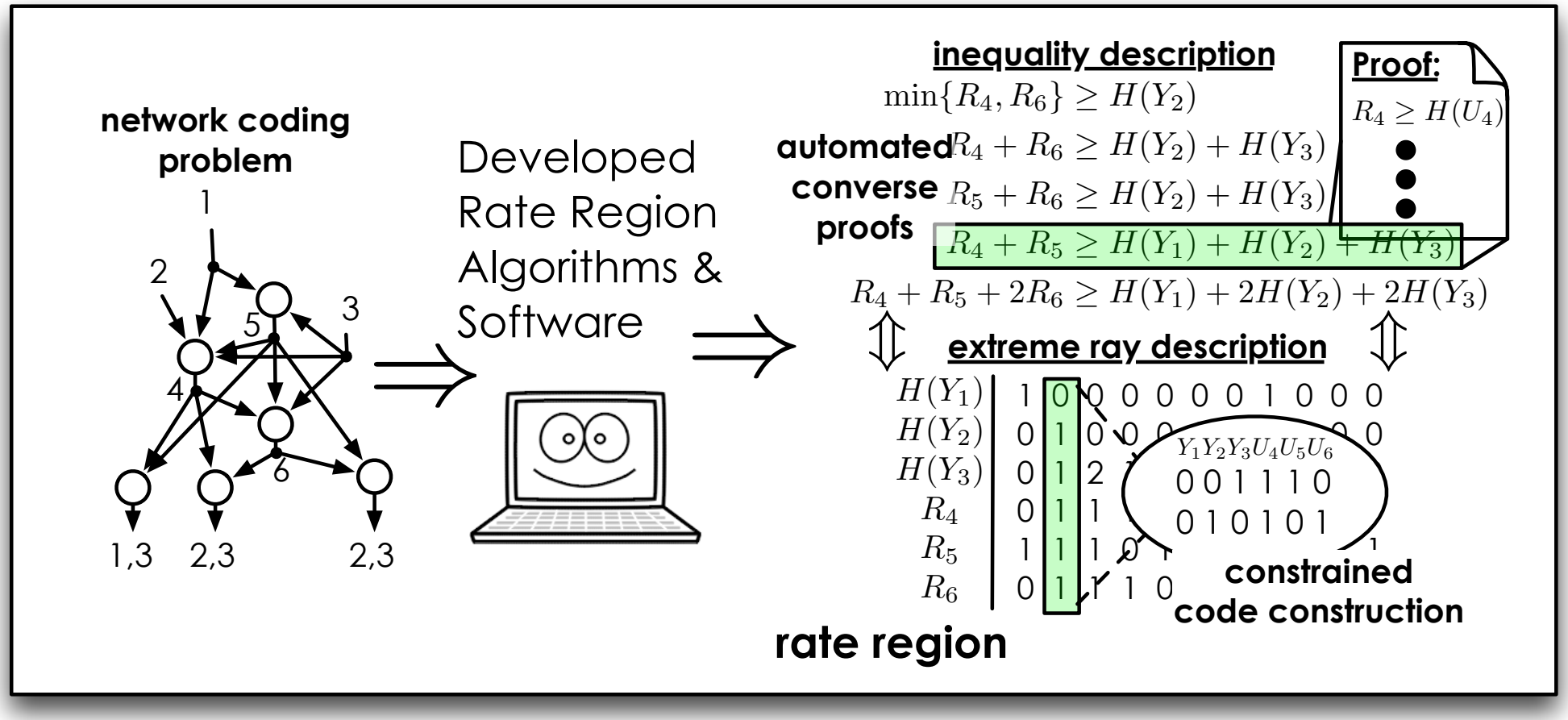
Exploiting Symmetry in Rate Region Computation – Achievability

- Key Idea: linear network code $= (\mathcal{V}, \phi)$
  - $\mathcal{V}$ a *set* of $\ell \leq K + |\mathcal{E}_U|$ subspaces of $GF(q)^r$
  - a map $\phi : \mathcal{V} \to \{1, \ldots, K + |\mathcal{E}_U|\}$ assigning these subspaces to network sources & edges.
- Note: rate region only depends on subset entropies=dimensions/ranks of these subspaces and their sums! (associated polymatroid).
- Two sets $\mathcal{V}$ and $\mathcal{V}'$ will give the same ranks if $\mathcal{V}' = \mathcal{V}g$ for some $g \in P\Gamma L_q^r$!
- Linear Code Enumeration Problem:
  - finding only canonical representatives of orbits in the sets of subspaces $\mathcal{V}$ under $P\Gamma L_q^r$ which can be grouped with a ϕ to obey network constraints (& optionally have specified $R_e, H(Y_s)$ dimension vectors)
  - Algorithm Leiterspiel [Schmalz, Betten Braun & Fripertinger] for finding orbits in the power-set obeying an inherited property recursively in subset-size can be applied to solve this problem.

  - Inherited property: ability to map (partial ϕ or p-map) to a (thus far defined) subset of network variables and obey the constraints they are involved in.
  - NSG (together with stabilizer subgroup of the subspaces $\mathcal{V}$) makes some ϕ equivalent with one another, removing the need to test all of them.

- Method above implemented in our **ITAP** (github) GAP package
- C++ version building on Orbiter [Betten] forthcoming.
- Although enumeration oriented, when used as a verification algorithm (w/ specified rate vector) it can still be faster than the Groebner basis (w/ Singular) based path-gain verification of Subramanian & Thangaraj!
- Nonlinear codes: view discrete random variables (sources & edges) each as partitions of a finite Ω from $(\Omega, \mathcal{F}, \mathbb{P})$. Group is $S_{|\Omega|}$. Have done this for E.V.s [Liu arxiv 2016] but could be coupled with p-maps ϕ to yield nonlinear codes.

Computationally Enabled Research Agenda – Motivation



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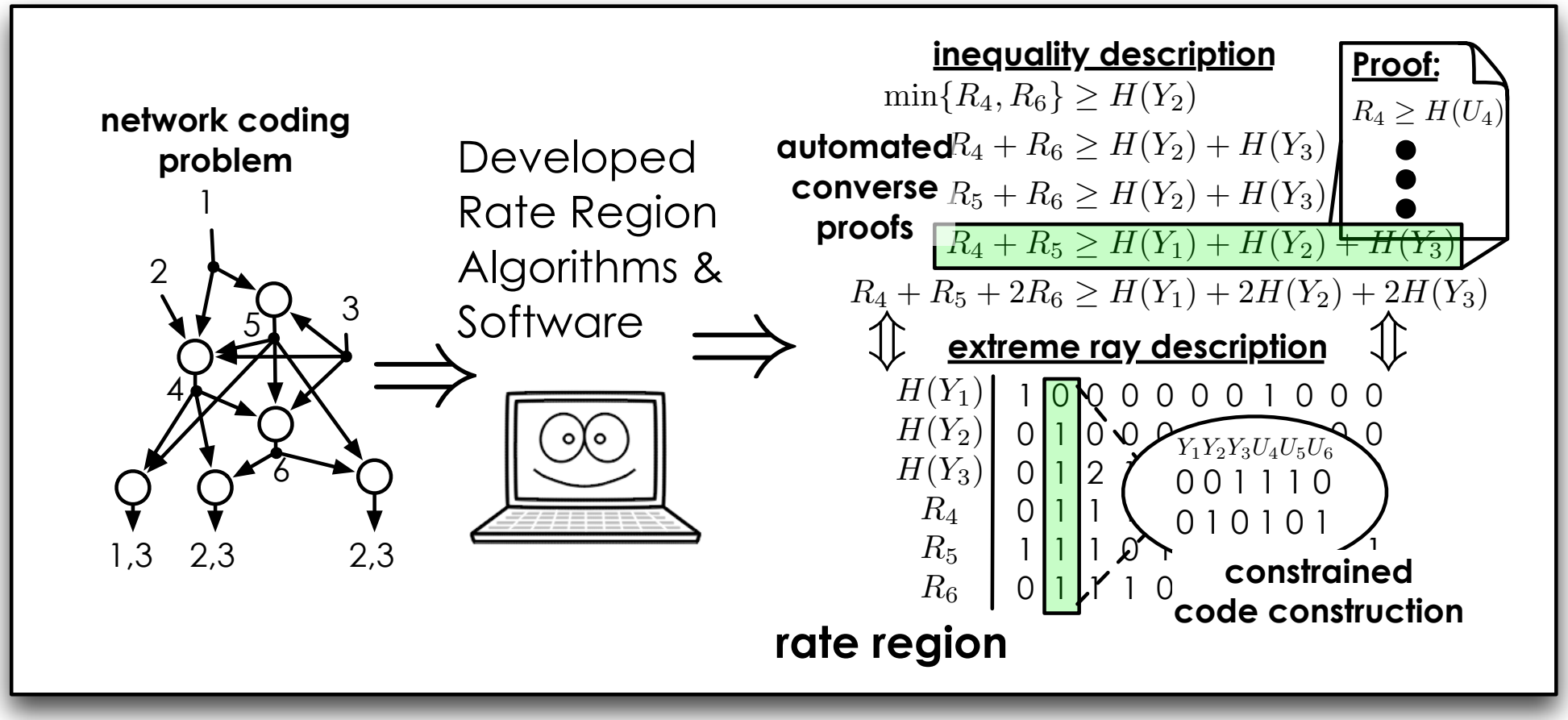
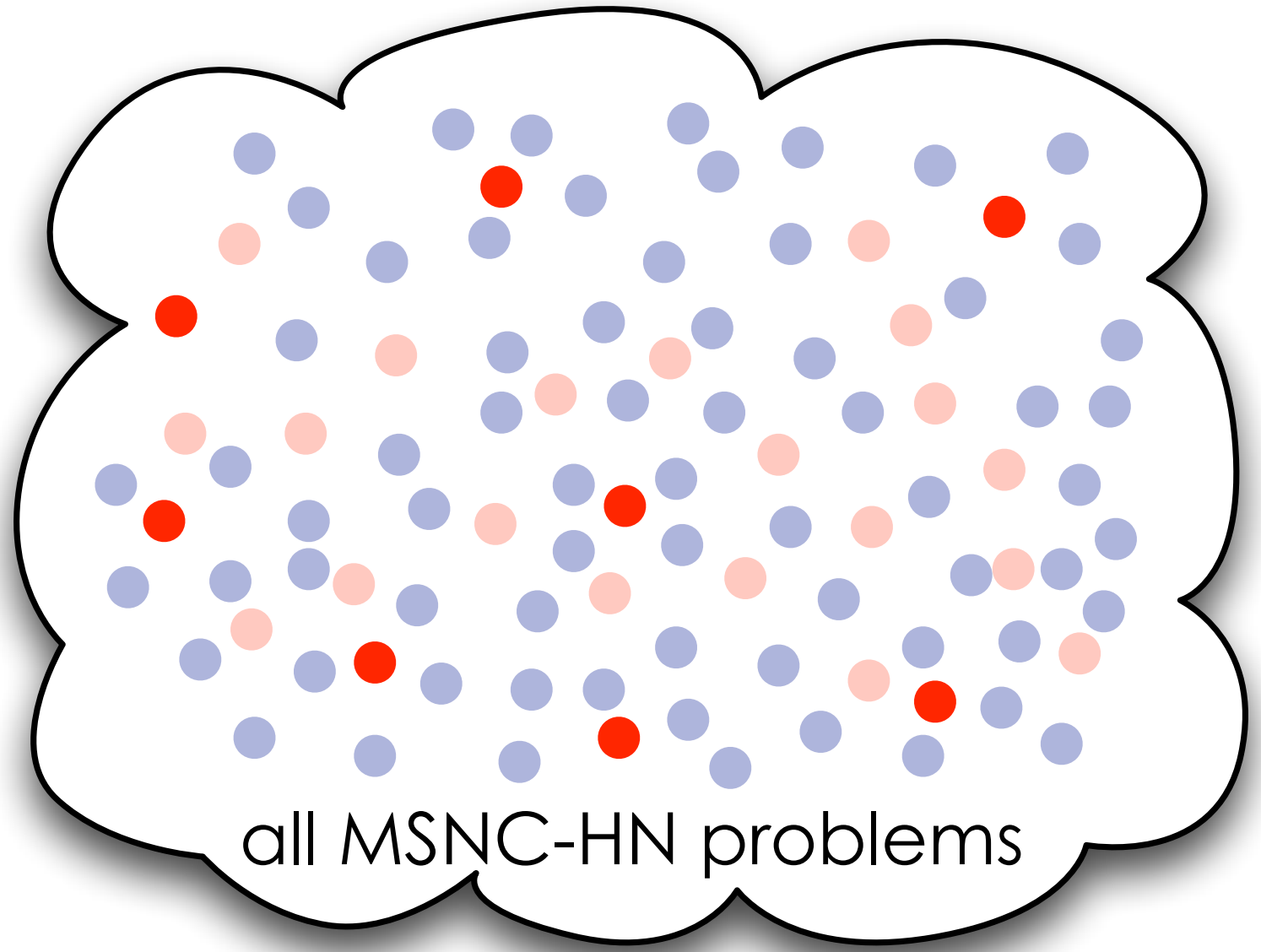


- Want software to reach as large networks as possible. $\Longrightarrow$ Define & Exploit Problem Symmetry
- Want the database of solved problems to be as large as possible. $\Longrightarrow$ Define Problem Equivalence Classes via Canonical & Minimal Representatives

Computationally Enabled Research Agenda – Minimality & Symmetry



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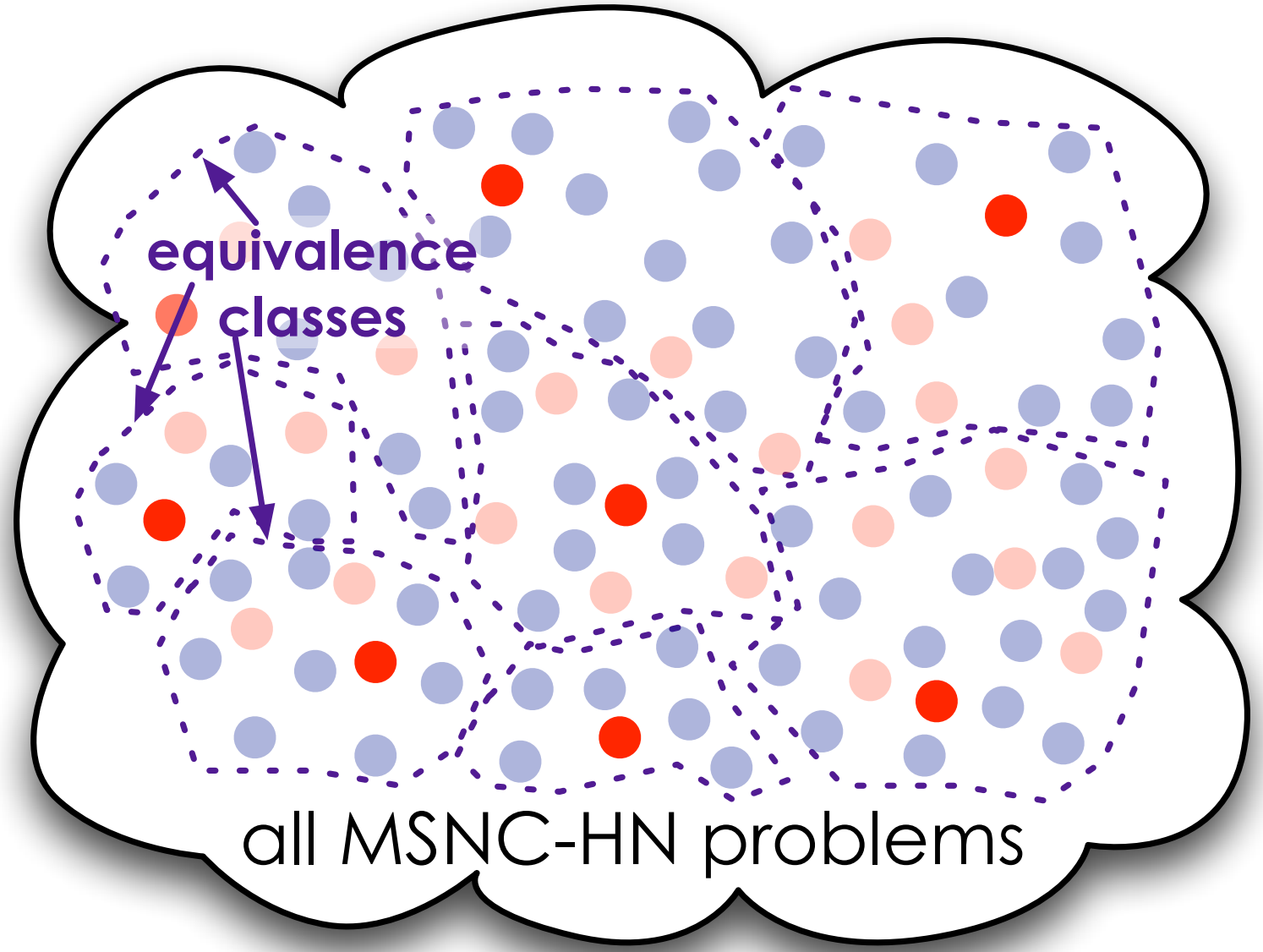

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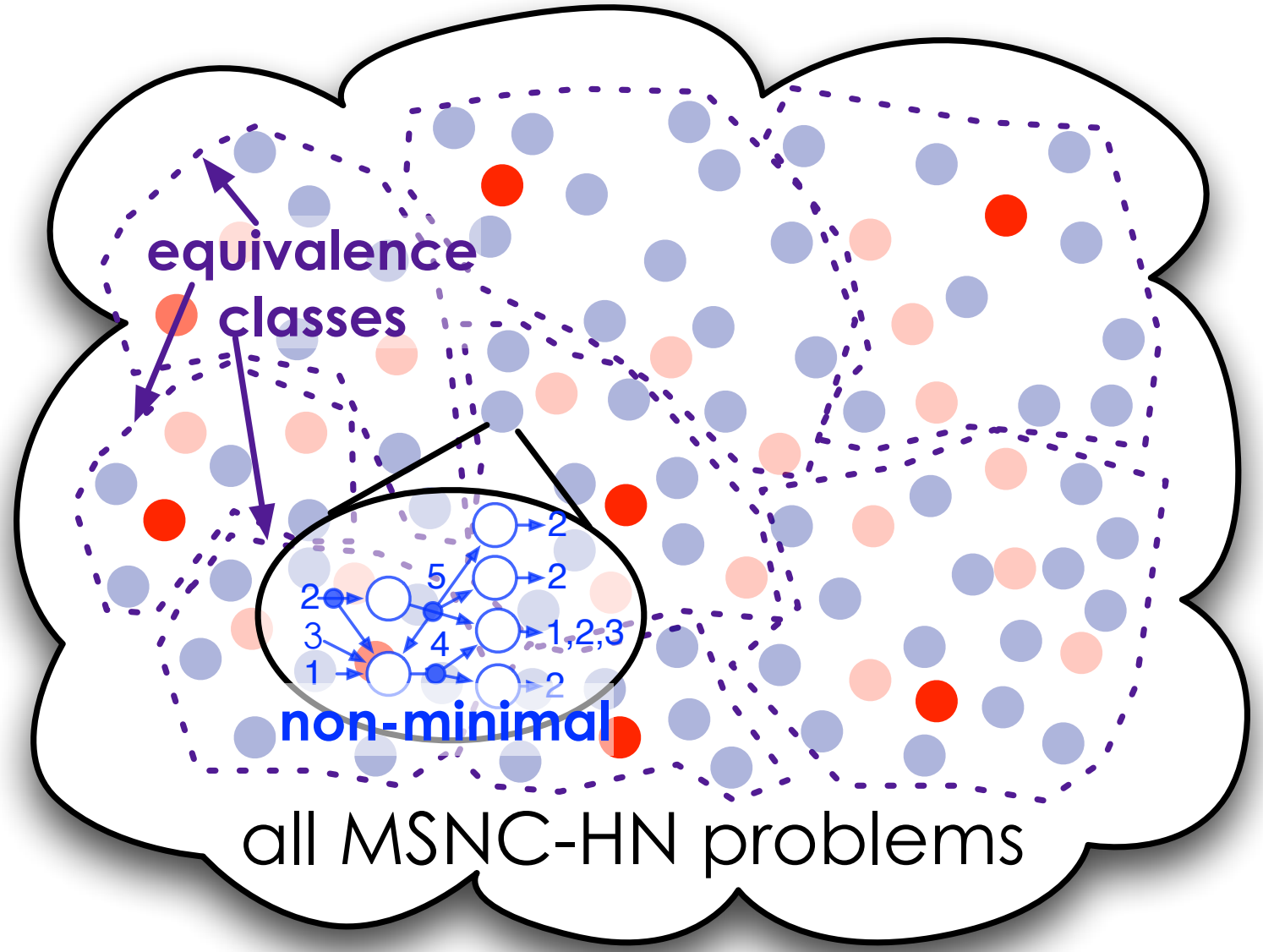

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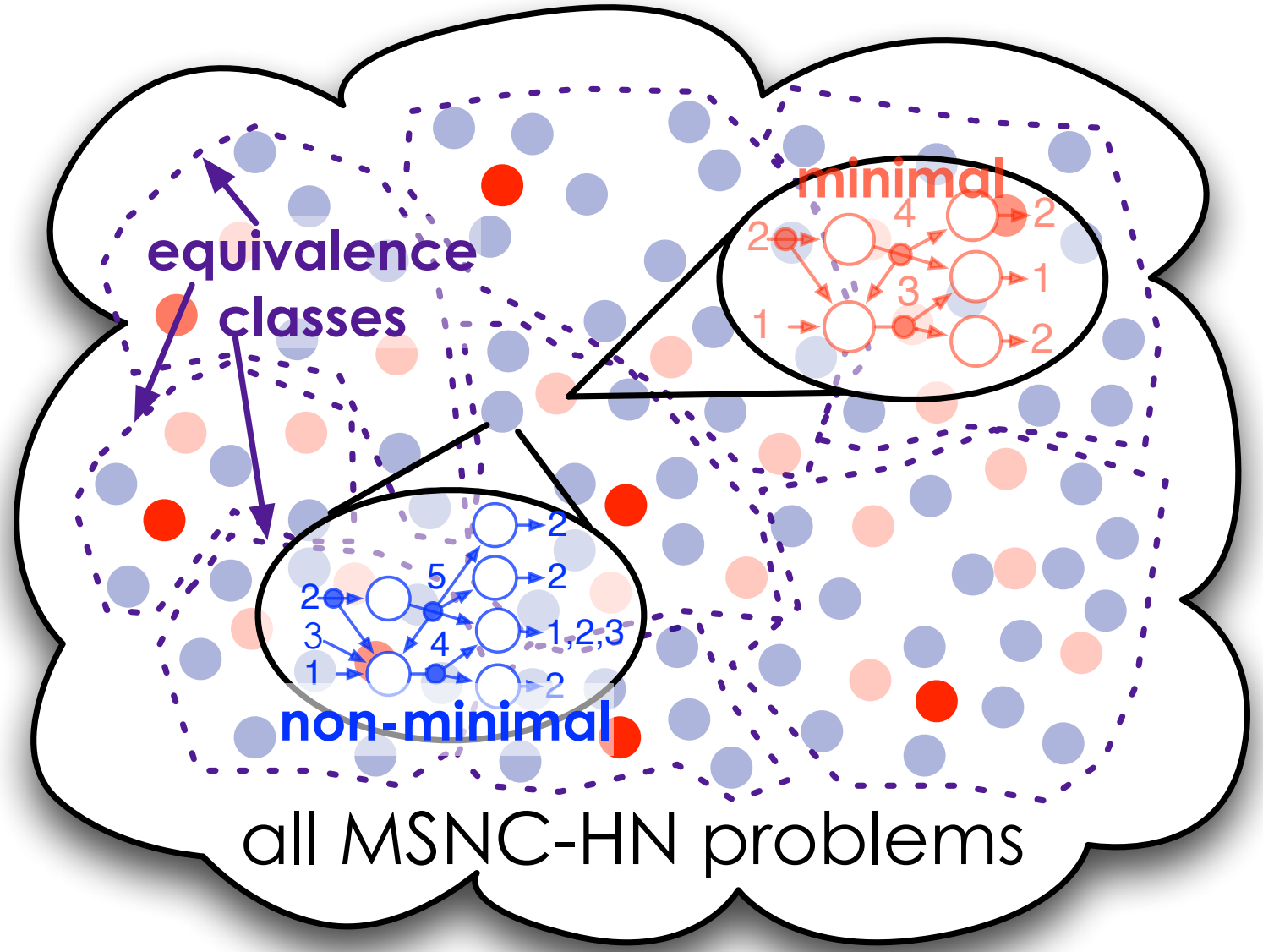
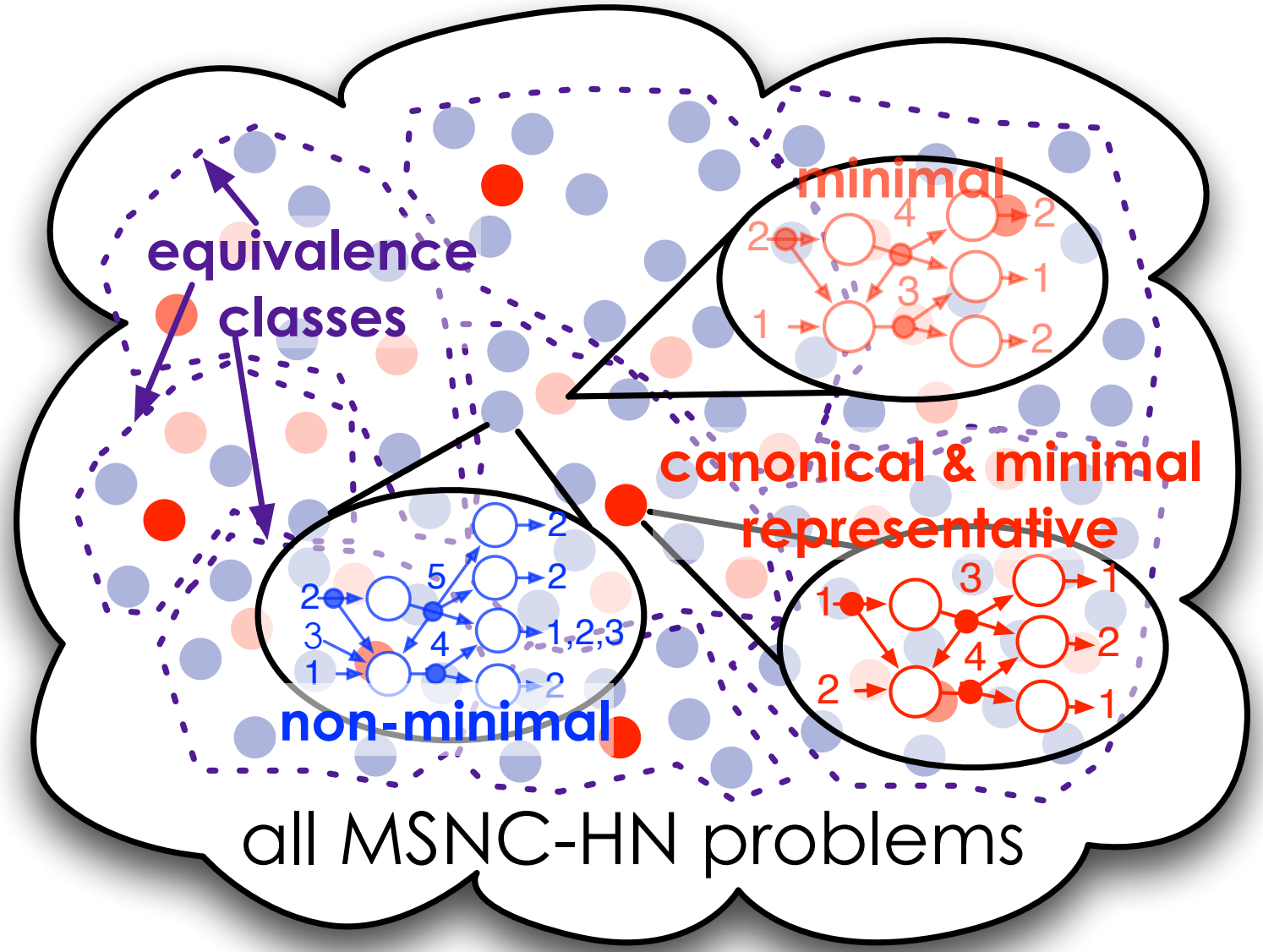
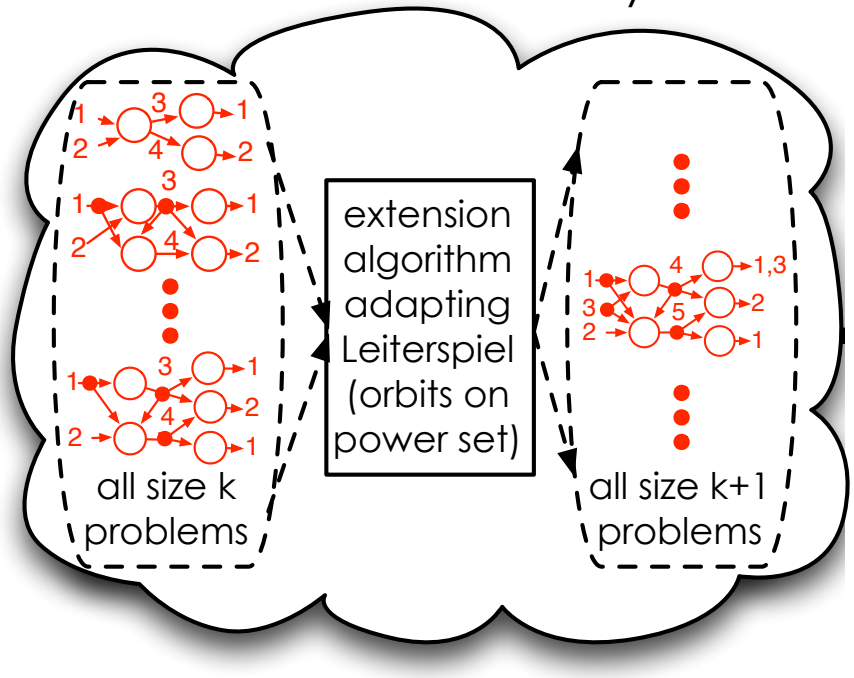


Computationally Enabled Research Agenda – Minimality & Symmetry

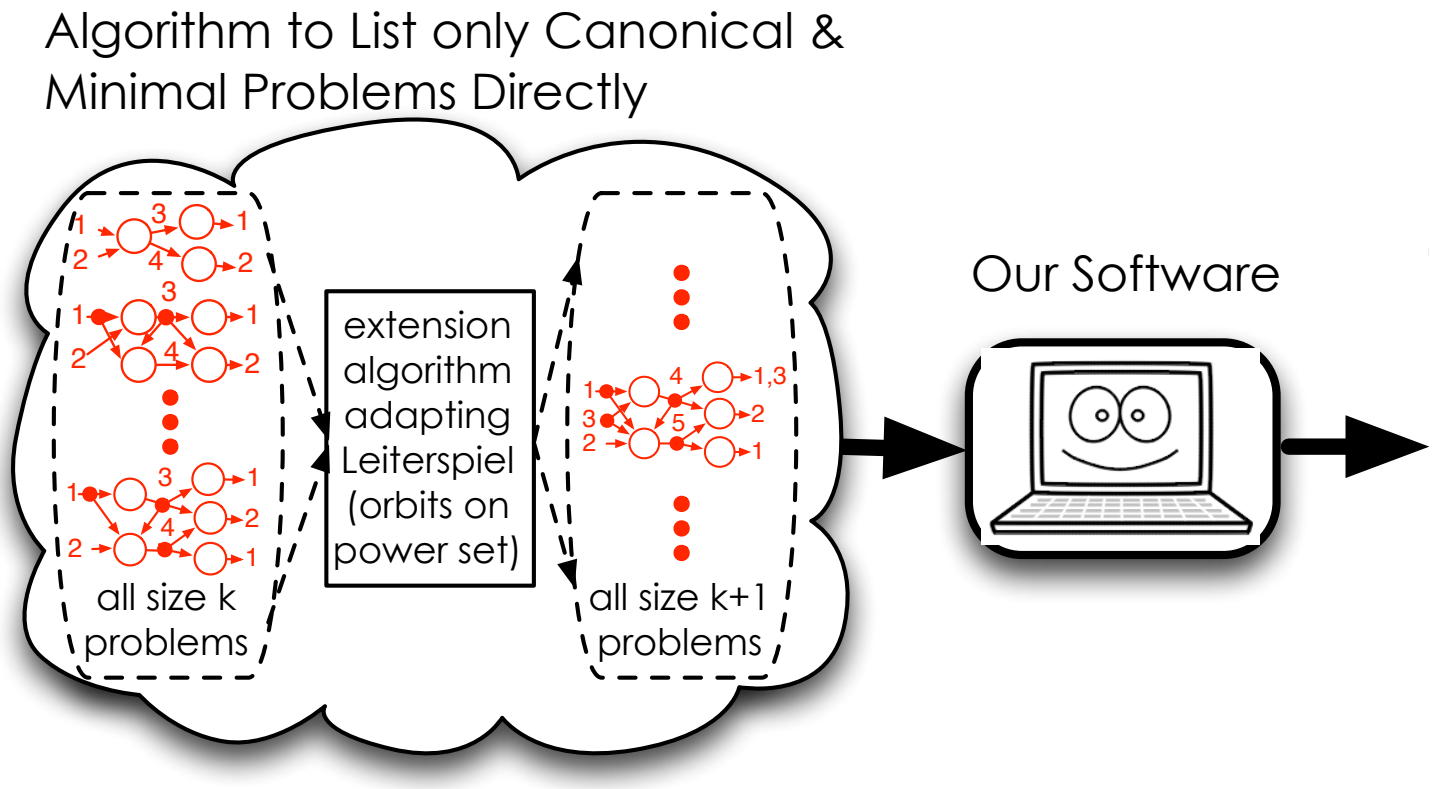


Computationally Enabled Research Agenda – Rate Region Database

Algorithm to List only Canonical & Minimal Problems Directly

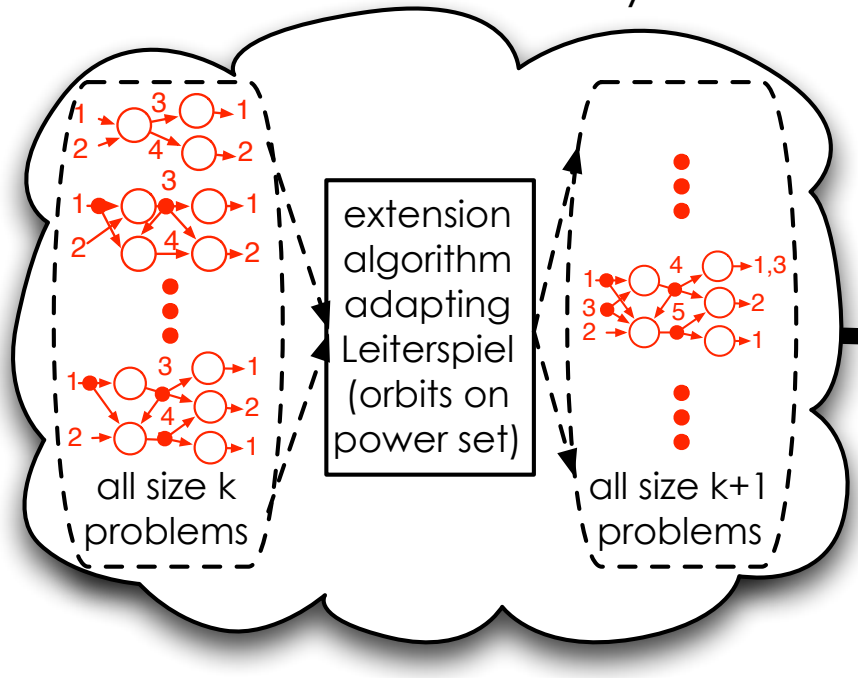


Computationally Enabled Research Agenda – Rate Region Database

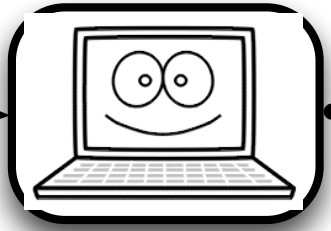


Computationally Enabled Research Agenda – Rate Region Database

Algorithm to List only Canonical & Minimal Problems Directly

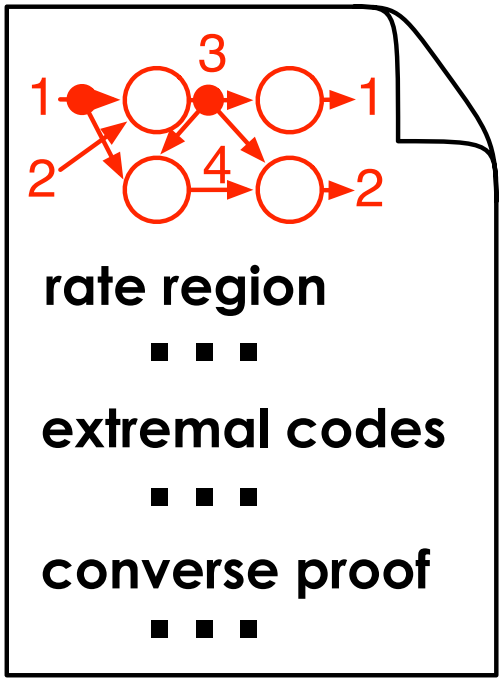


Our Software



Rate Region Database:
rate region,
converse proof,
efficient codes
for every canonical & minimal problem

Network Coding Rate Region Database



rate region

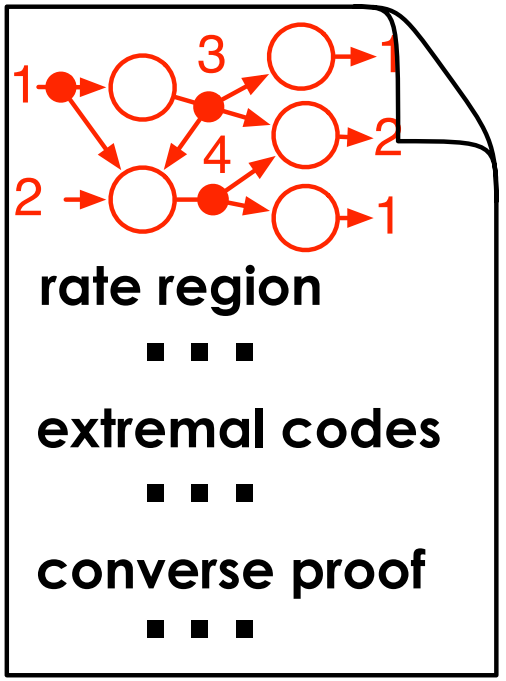
▪ ▪ ▪

extremal codes

▪ ▪ ▪

converse proof

▪ ▪ ▪



rate region

▪ ▪ ▪

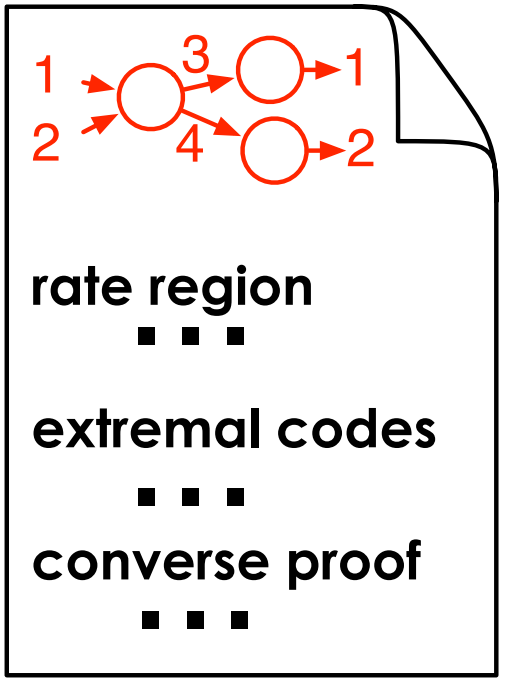
extremal codes

▪ ▪ ▪

converse proof

▪ ▪ ▪

▪ ▪ ▪



rate region

▪ ▪ ▪

extremal codes

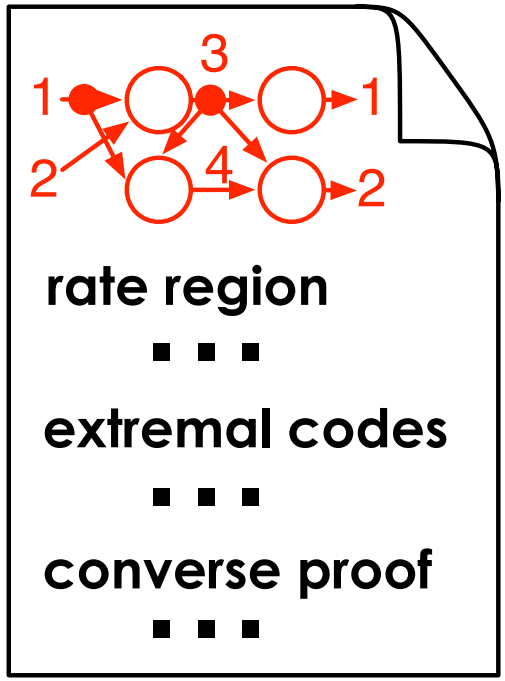
▪ ▪ ▪

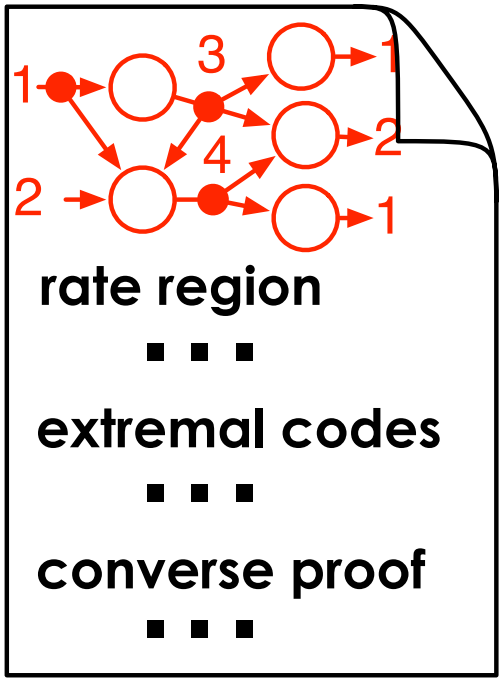
converse proof

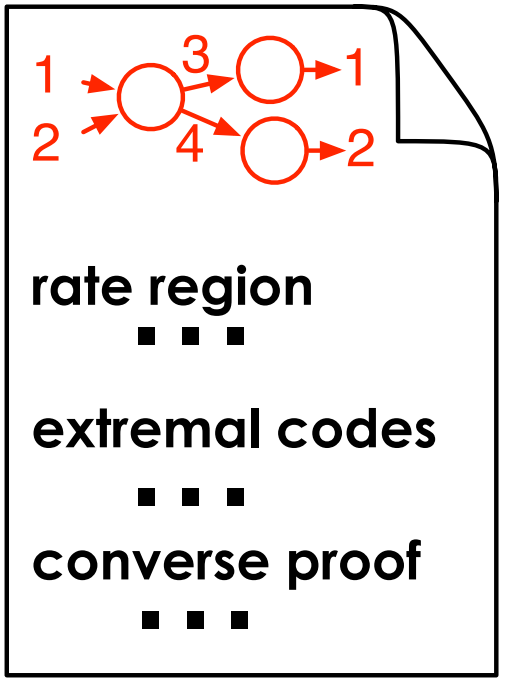
▪ ▪ ▪

Database of ~**7000** Rate Regions of >**100k** Networks (Li Trans IT sub 2014)
Database of ~**744k** Rate Regions of > 7M or ~**2.3T** Minimal Networks
(Li Trans IT sub 2015)

Network Coding Rate Region Database



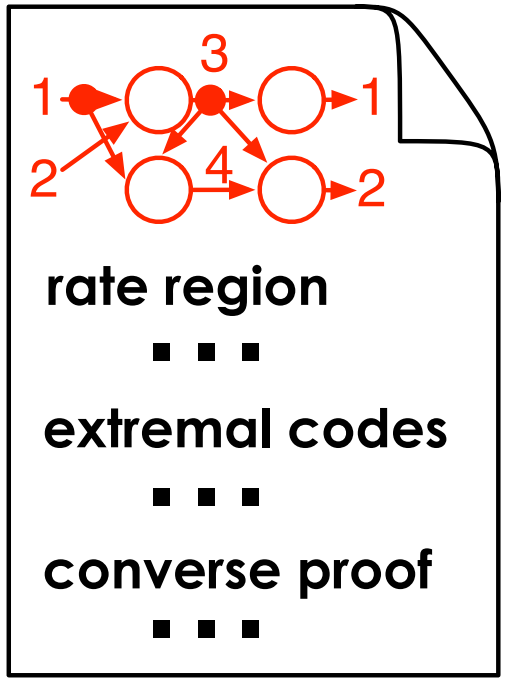


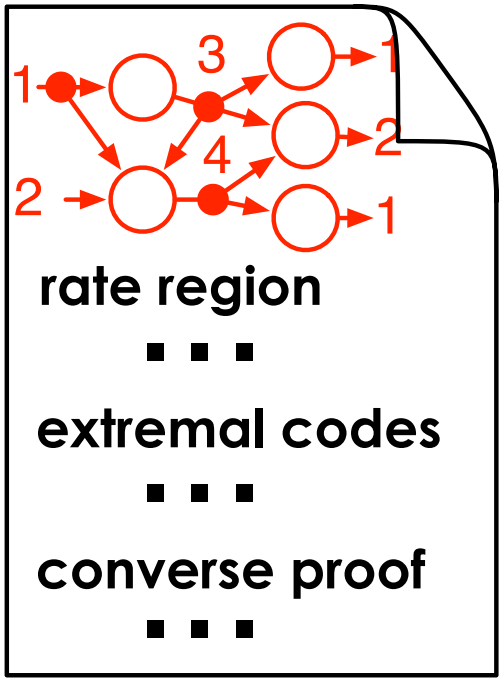


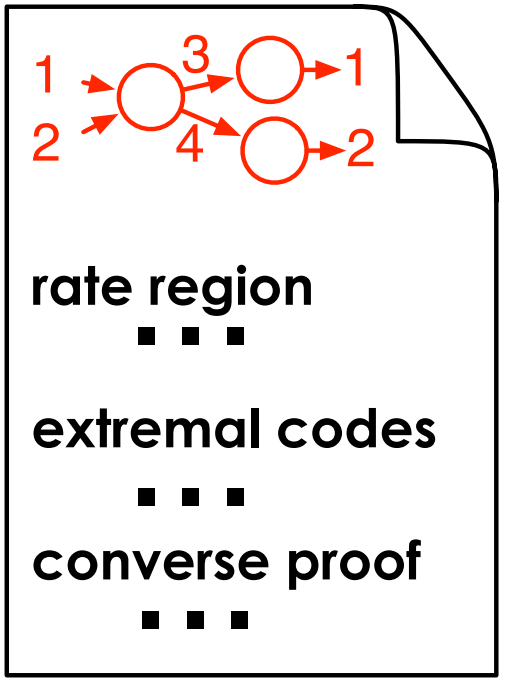
Database of ~**7000** Rate Regions of >**100k** Networks (Li Trans IT sub 2014)
Database of ~**744k** Rate Regions of > 7M or ~**2.3T** Minimal Networks (Li Trans IT sub 2015)

What now?

Network Coding Rate Region Database



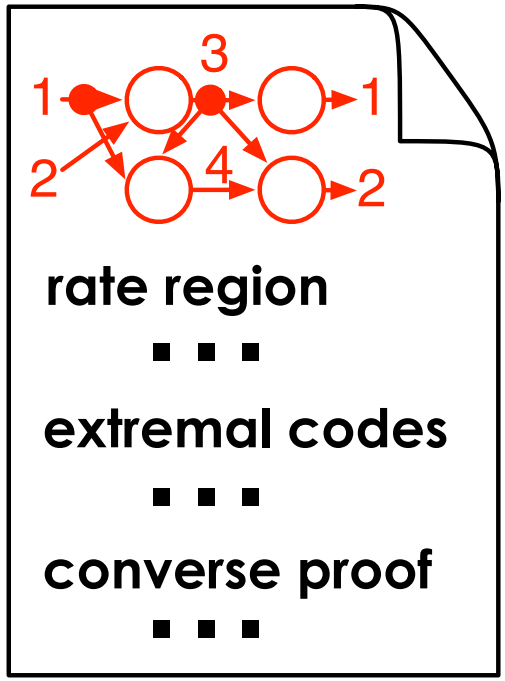




Database of ~**7000** Rate Regions of >**100k** Networks (Li Trans IT sub 2014)
Database of ~**744k** Rate Regions of > 7M or ~**2.3T** Minimal Networks (Li Trans IT sub 2015)

What now? Submit 744,000 transactions papers?

Network Coding Rate Region Database



rate region

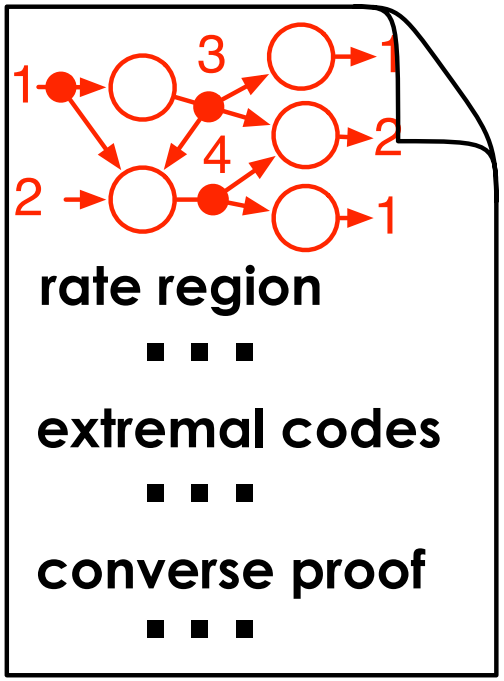
▪ ▪ ▪

extremal codes

▪ ▪ ▪

converse proof

▪ ▪ ▪



rate region

▪ ▪ ▪

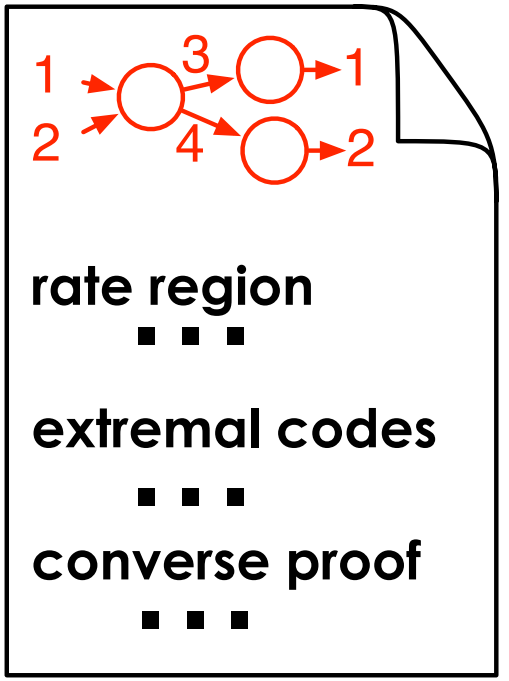
extremal codes

▪ ▪ ▪

converse proof

▪ ▪ ▪

▪ ▪ ▪



rate region

▪ ▪ ▪

extremal codes

▪ ▪ ▪

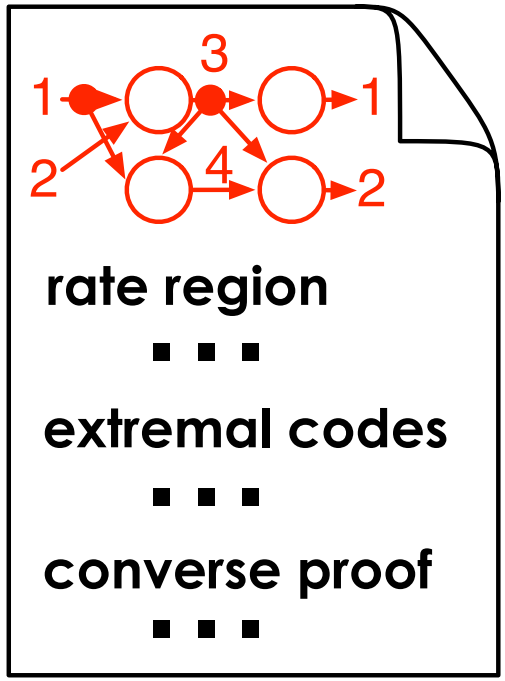
converse proof

▪ ▪ ▪

Database of ~**7000** Rate Regions of >**100k** Networks (Li Trans IT sub 2014)
Database of ~**744k** Rate Regions of > 7M or ~**2.3T** Minimal Networks
(Li Trans IT sub 2015)

What now? Analyze, learn, and explain something!
Can't read and remember 744,000 network proofs.

Network Coding Rate Region Database



rate region

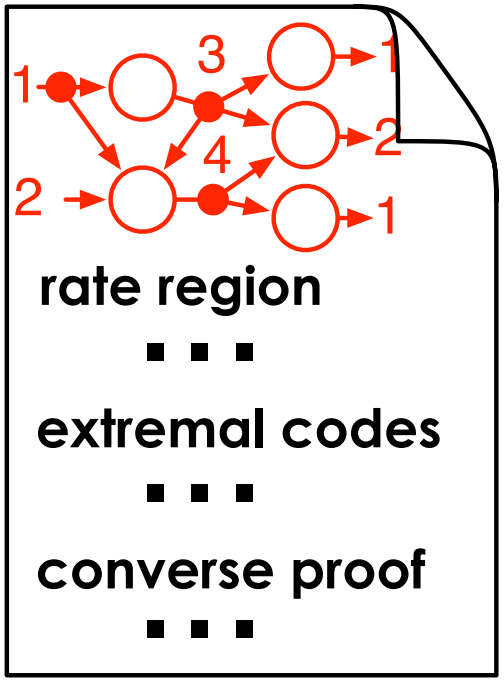
▪ ▪ ▪

extremal codes

▪ ▪ ▪

converse proof

▪ ▪ ▪



rate region

▪ ▪ ▪

extremal codes

▪ ▪ ▪

converse proof

▪ ▪ ▪

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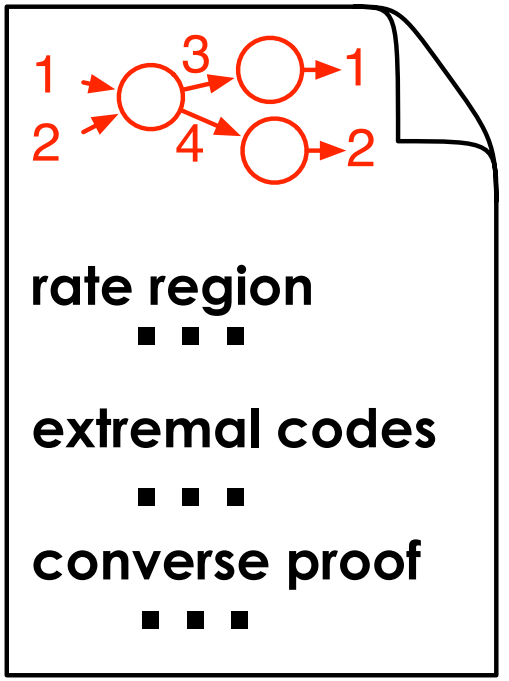


rate region

▪ ▪ ▪

extremal codes

▪ ▪ ▪

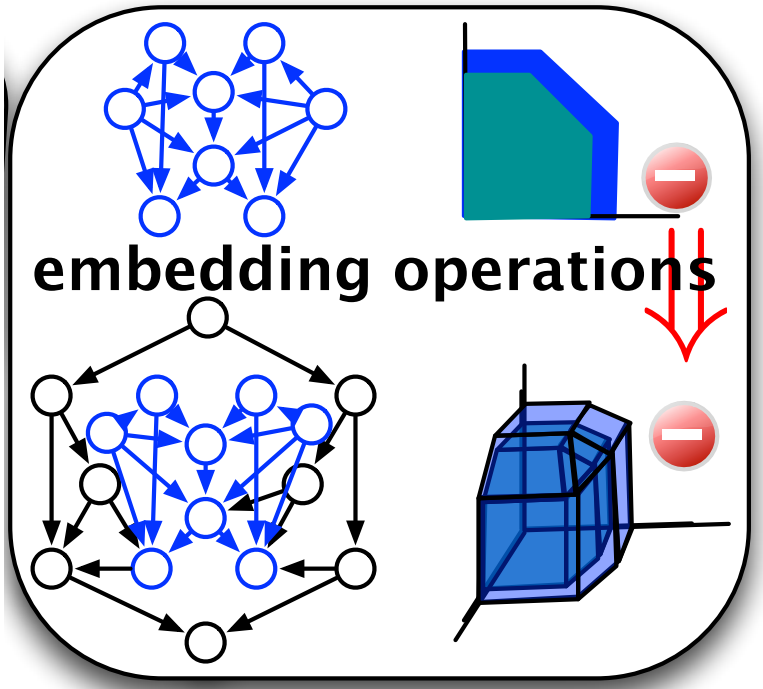
converse proof

▪ ▪ ▪

Database of ~**7000** Rate Regions of >**100k** Networks (Li Trans IT sub 2014)
Database of ~**744k** Rate Regions of > 7M or ~**2.3T** Minimal Networks
(Li Trans IT sub 2015)

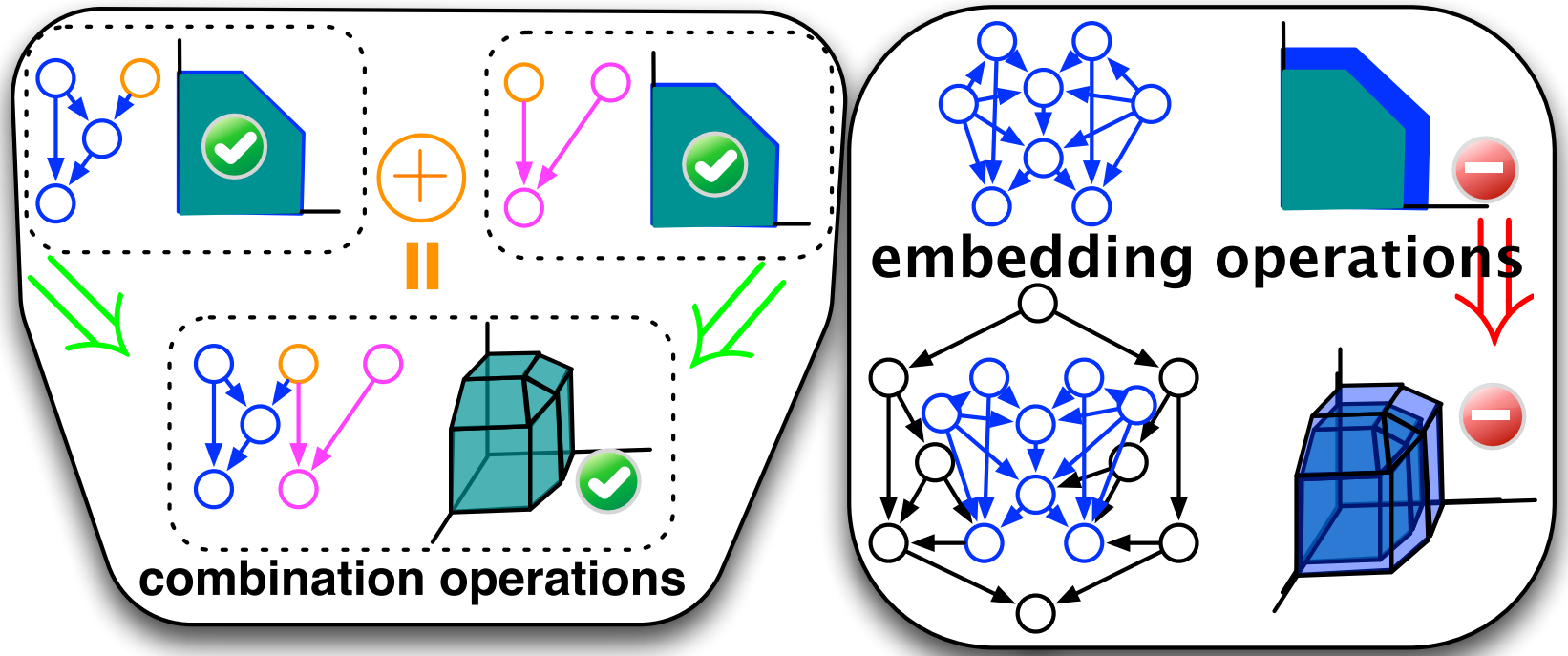
Investigate Structure through Network Hierarchy

Computationally Enabled Research Agenda – Hierarchy



- Understand existing large rate region database through structure: embedding operators & forbidden network minors

Computationally Enabled Research Agenda – Hierarchy

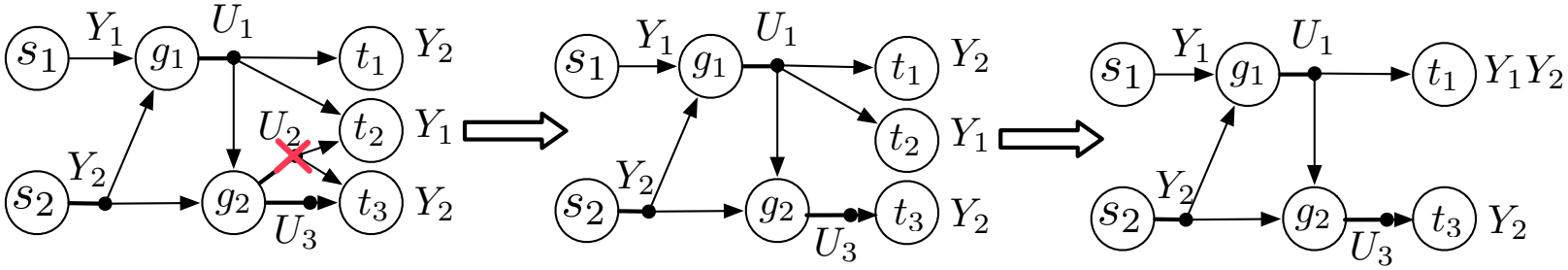


Network well-quasi-ordering Forbidden network Minors

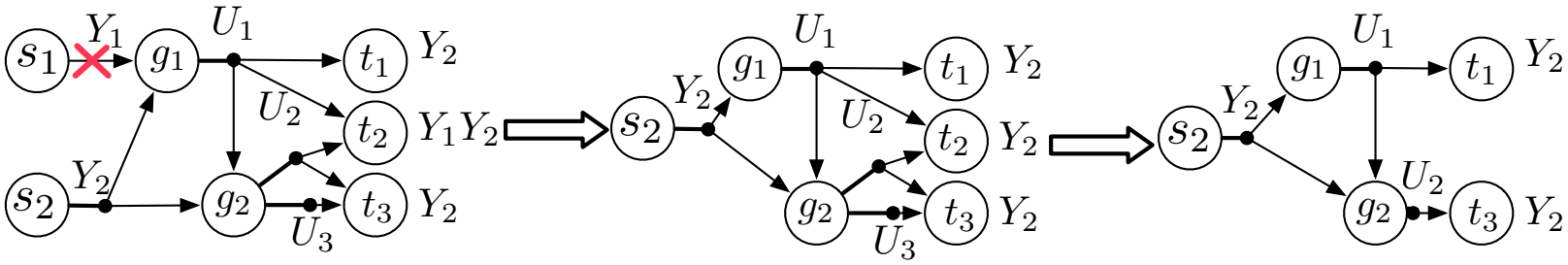
- Understand existing large rate region database through structure: embedding operators & forbidden network minors
- Build solutions to arbitrarily sized networks: combinations operators, operator concatenation

Computationally Enabled Research Agenda – Hierarchy: Embedding Operators

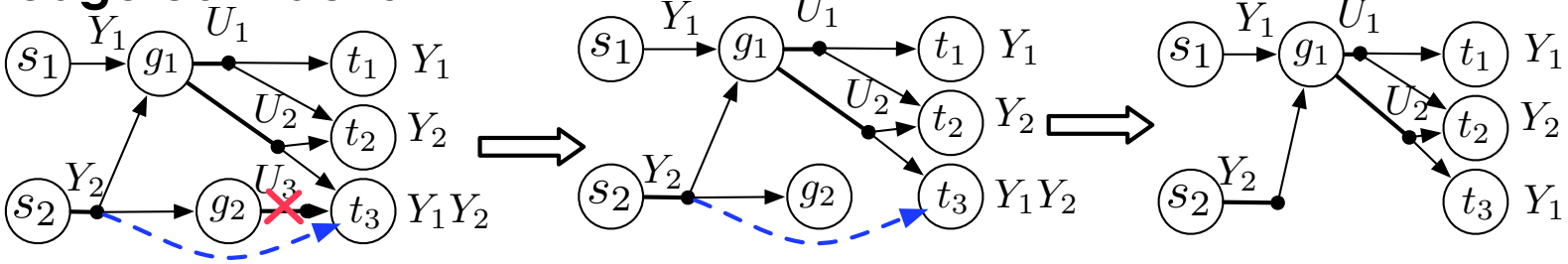
edge deletion



source deletion

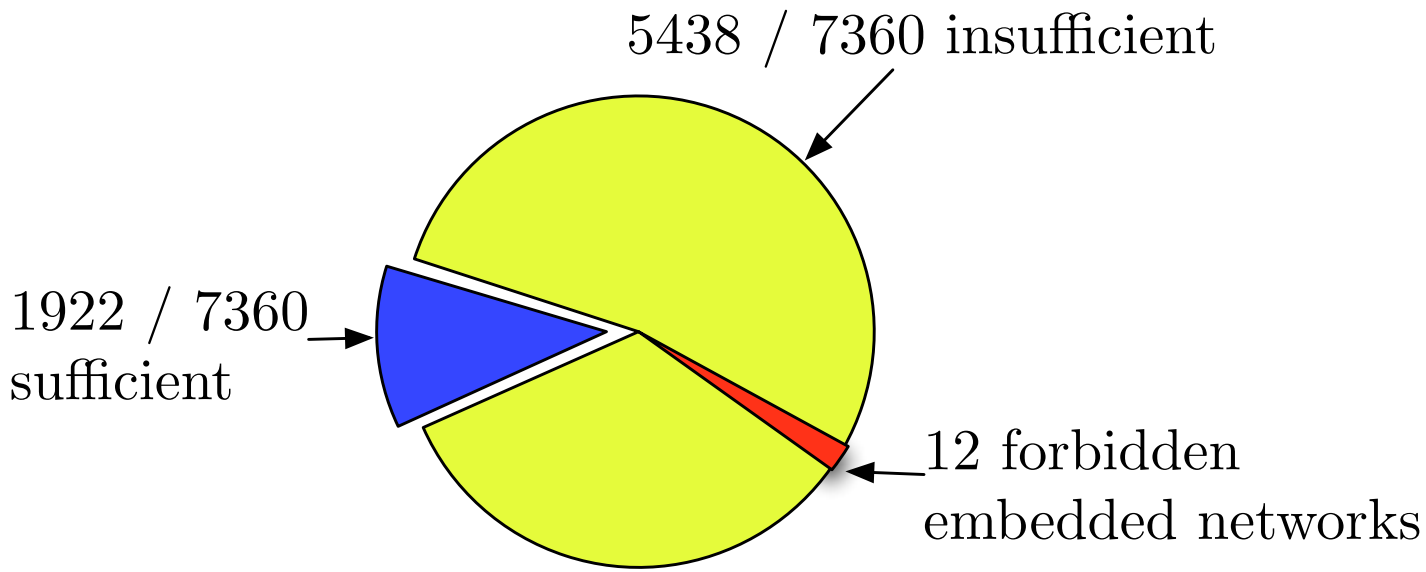


edge contraction



- Rate region (bound) of embedded network can be directly obtained from rate region (bound) of parent network.
- Insufficiency of class of codes of small $\Longrightarrow$ insufficiency of class of codes of big. (*forbidden network minor*)

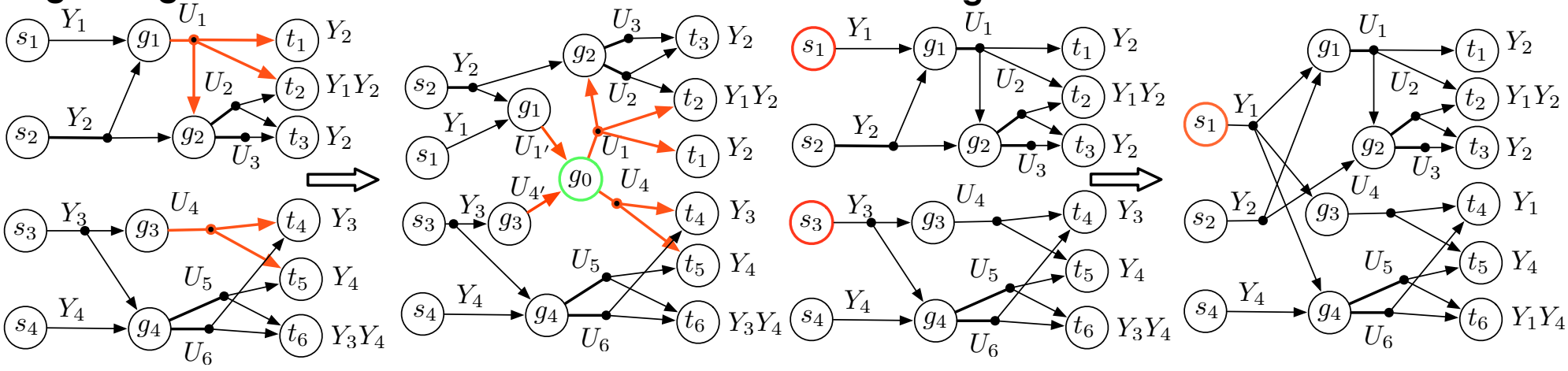
Computationally Enabled Research Agenda – Hierarchy: Embedding Operators



- First database: 5438 canonical MDCS for which scalar binary codes are insufficient can be boiled down to 12 forbidden minor networks.

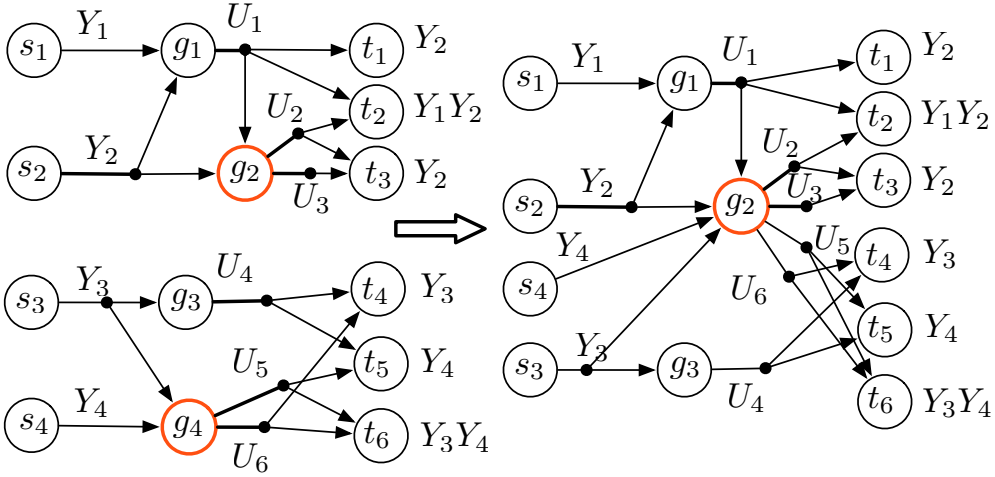
Computationally Enabled Research Agenda – Hierarchy: Combination Operators

Edge Merge

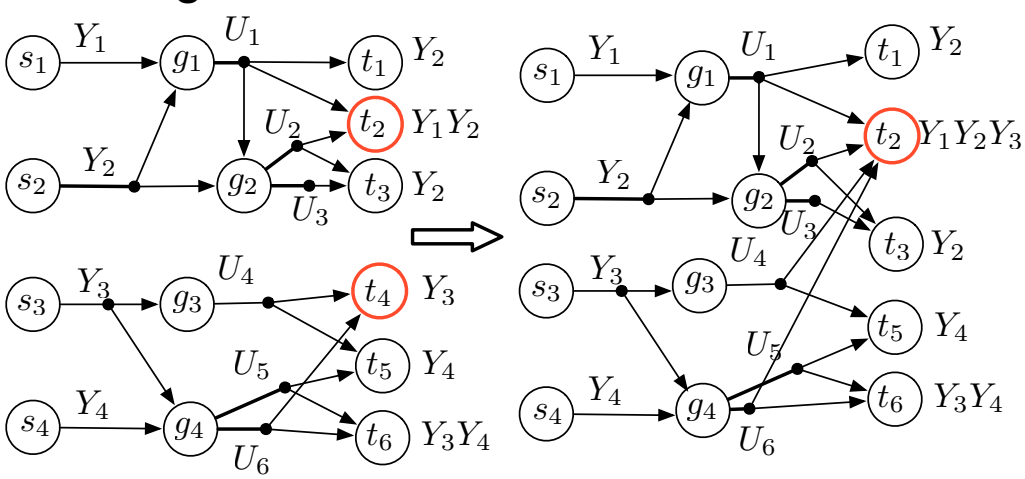


Source Merge

Node Merge

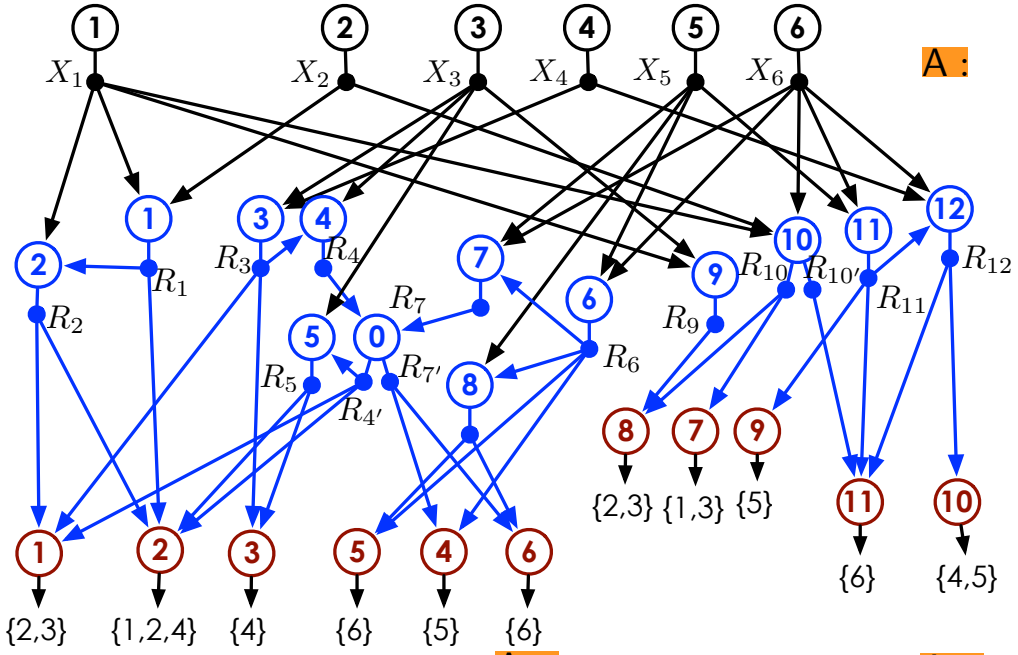


Sink Merge

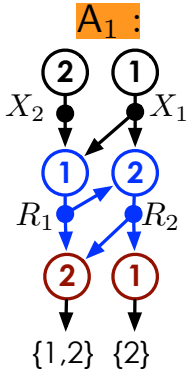


- rate region of big directly expressible from rate regions of smalls

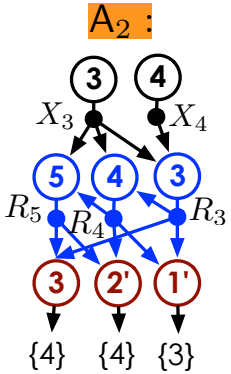
Computationally Enabled Research Agenda – Hierarchy: Combination Operators



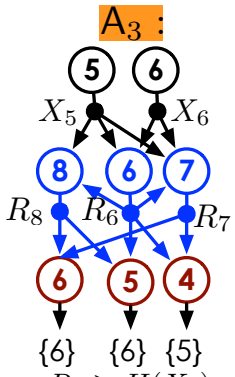
$$
\begin{array}{rcl}
R_i & \geq & H(X_2), i = 1, 2 \\
R_1 + R_2 & \geq & H(X_1) + H(X_2) \\
R_i & \geq & H(X_4), i = 3, 4, 4' \\
R_3 + R_i & \geq & H(X_3) + H(X_4), i = 4, 4' \\
R_3 + R_i + R_5 & \geq & H(X_3) + 2H(X_4), i = 4, 4' \\
R_6 & \geq & H(X_6) \\
R_i + R_8 & \geq & H(X_6), i = 7, 7' \\
R_6 + R_i & \geq & H(X_5) + H(X_6), i = 7, 7' \\
R_6 + R_i + 2R_8 & \geq & H(X_5) + 2H(X_6), i = 7, 7' \\
R_9 & \geq & H(X_3) \\
R_{10} & \geq & \sum_{i=1}^{3} H(X_i) \\
R_{11} & \geq & H(X_5) \\
R_{12} & \geq & H(X_4) + H(X_5) \\
R_{10'} + R_{11} + R_{12} & \geq & H(X_4) + 2H(X_5) + H(X_6)
\end{array}
$$



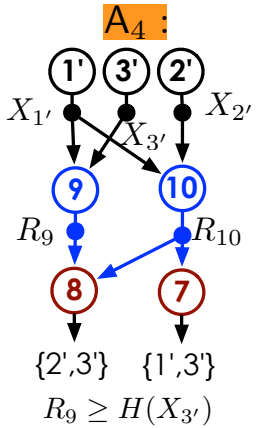
$$R_1 \geq H(X_2)$$
$$R_2 \geq H(X_2)$$
$$R_1 + R_2 \geq H(X_1) + H(X_2)$$



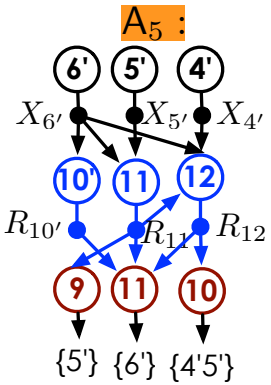
$$R_i \geq H(X_4), i = 3, 4$$
$$R_3 + R_4 \geq H(X_3) + H(X_4)$$
$$R_3 + R_4 + 2R_5 \geq H(X_3) + 2H(X_4)$$



$$R_6 \geq H(X_6)$$
$$R_7 + R_8 \geq H(X_6)$$
$$R_6 + R_7 \geq H(X_5) + H(X_6)$$
$$R_6 + R_7 + 2R_8 \geq H(X_5) + 2H(X_6)$$

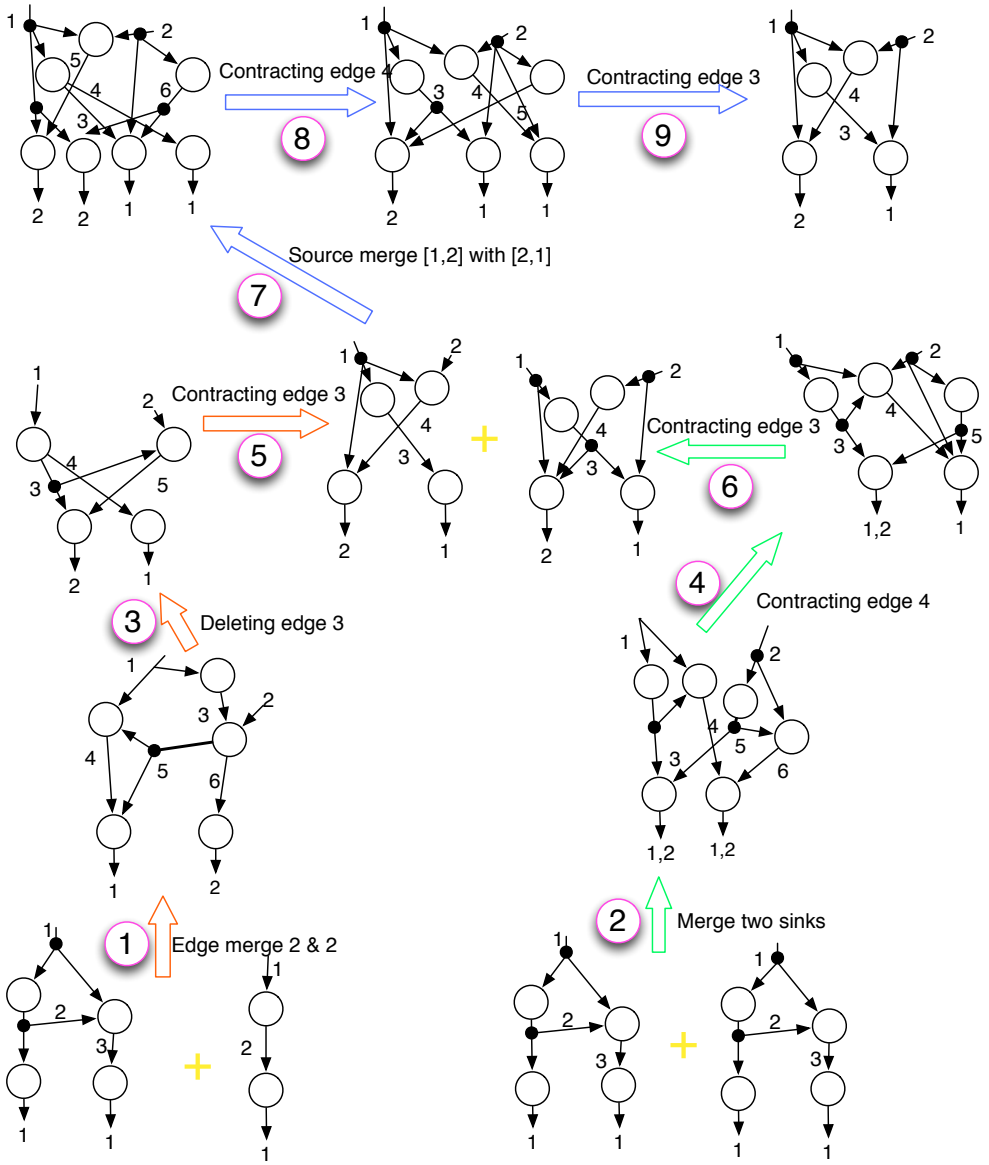


$$R_9 \geq H(X_{3'})$$
$$R_{10} \geq H(X_{1'}) + H(X_{2'}) + H(X_{3'})$$

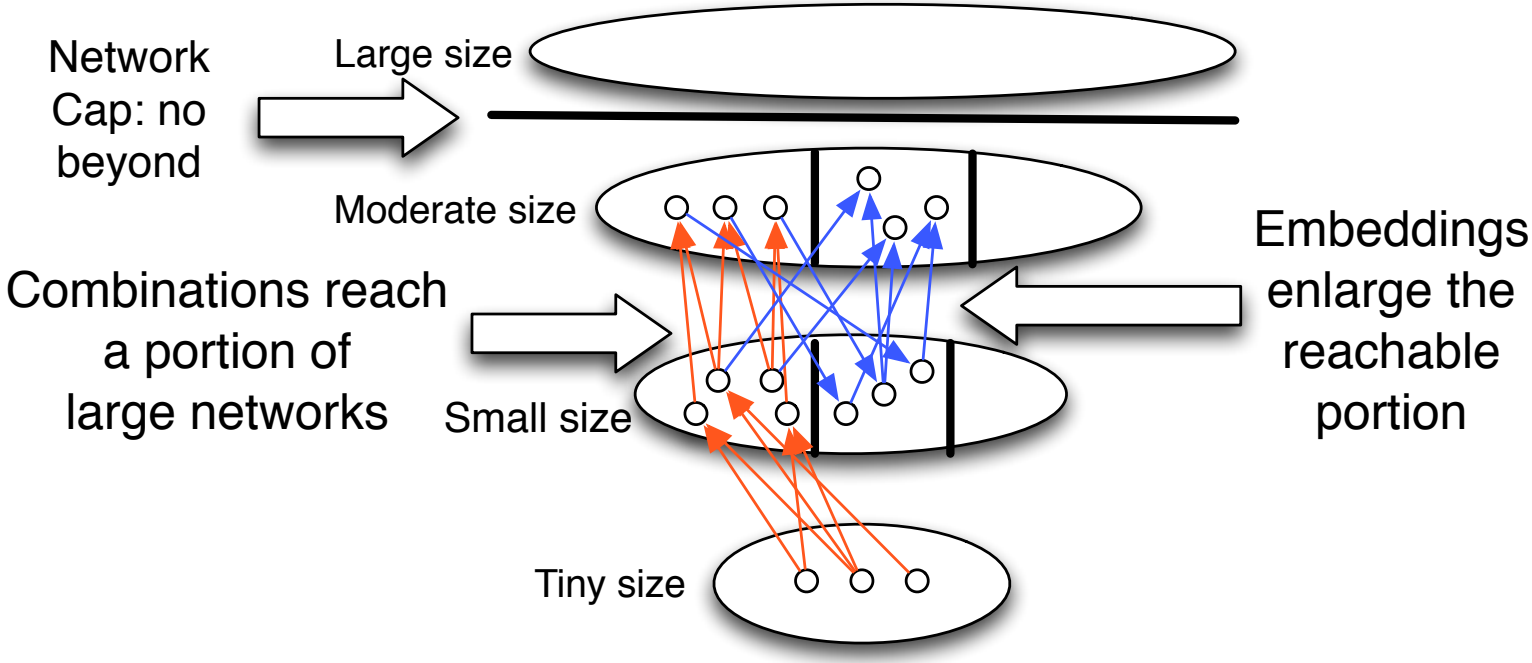


$$R_{11} \geq H(X_{5'})$$
$$R_{12} \geq H(X_{4'}) + H(X_{5'})$$
$$R_{10'} + R_{11} + R_{12} \geq H(X_{4'}) + 2H(X_{5'}) + H(X_{6'})$$

Computationally Enabled Research Agenda – Hierarchy: Operator Concatenation



Computationally Enabled Research Agenda – Hierarchy: Operator Concatenation



Use operators *together* to get RR for big networks. Partial Network Closure.

Computationally Enabled Research Agenda – Hierarchy: Operator Concatenation

Start with the single $(1,1)$, single $(2,1)$, and the four $(1,2)$ networks; These 6 tiny networks can generate new 11635 networks w/ small cap!

	combination operators only			embedding and combinations		
size\cap	(3,3)	(3,4)	(4,4)	(3,3)	(3,4)	(4,4)
(1,3)	4	4	4	4	4	4
(1,4)	0	10	10	0	10	10
(2,2)	3	3	3	8	15	16
(2,3)	13	16	16	30	131	155
(2,4)	0	97	101	0	516	648
(3,2)	2	3	2	4	10	11
(3,3)	24	24	24	42	353	833
(3,4)	0	135	135	0	2361	5481
(4,2)	0	0	3	0	0	3
(4,3)	0	0	17	0	0	44
(4,4)	0	0	253	0	0	4430
all	46	292	568	88	3400	11635

Computationally Enabled Research Agenda – Hierarchy: Operator Concatenation

With the increase of cap size, number of new networks increases!

	combination operators only			embedding and combinations		
size\cap	(3,3)	(3,4)	(4,4)	(3,3)	(3,4)	(4,4)
(1,3)	4	4	4	4	4	4
(1,4)	0	10	10	0	10	10
(2,2)	3	3	3	8	15	16
(2,3)	13	16	16	30	131	155
(2,4)	0	97	101	0	516	648
(3,2)	2	3	2	4	10	11
(3,3)	24	24	24	42	353	833
(3,4)	0	135	135	0	2361	5481
(4,2)	0	0	3	0	0	3
(4,3)	0	0	17	0	0	44
(4,4)	0	0	253	0	0	4430
all	46	292	568	88	3400	11635

Computationally Enabled Research Agenda – Hierarchy: Operator Concatenation

Embedding operations are important in the process!

	combination operators only			embedding and combinations		
size\cap	(3,3)	(3,4)	(4,4)	(3,3)	(3,4)	(4,4)
(1,3)	4	4	4	4	4	4
(1,4)	0	10	10	0	10	10
(2,2)	3	3	3	8	15	16
(2,3)	13	16	16	30	131	155
(2,4)	0	97	101	0	516	648
(3,2)	2	3	2	4	10	11
(3,3)	24	24	24	42	353	833
(3,4)	0	135	135	0	2361	5481
(4,2)	0	0	3	0	0	3
(4,3)	0	0	17	0	0	44
(4,4)	0	0	253	0	0	4430
all	46	292	568	88	3400	11635