Network Coding: Computation, Symmetry, and Hierarchy

John MacLaren Walsh
Department of Electrical and Computer Engineering
Drexel University
Philadelphia, PA
jwalsh@ece.drexel.edu

Thanks to NSF CCF-1421828 & NSF CCF-1053702.
Collaborators & Co-Authors

Congduan Li, Ph.D.
Postdoctoral Fellow,
Institute for Network Coding
Chinese Univ. of Hong Kong
rate region database
network operators
forbidden net. minors

Steven Weber, Ph.D.
Professor, Dept. of ECE
Drexel University
co-advisor for
Congduan Li

Jayant Apte,
Ph.D. Candidate,
ASPIRG,
Drexel University
network symmetry
software: ITAP & CHM

Yunshu Liu,
Ph.D. Candidate
ASPIRG,
Drexel University
nonlinear entropic
vectors & codes
Outline

1. Problem Statement. Minimality & Symmetry

2. Symmetry Exploiting Rate Region Computation
   (a) Converse: Symmetry Exploiting CHM
   (b) Achievability:
      i. Linear Code Enumeration
      ii. Nonlinear Code (Partition) Enumeration

3. Rate Region Database

4. Hierarchy: Network Operators
   (a) Big $\rightarrow$ Small: Embedding Operators
   (b) Small $\rightarrow$ Big: Combinations Operators
   (c) Operator Concatenation
   (d) Coverage
Computationally Enabled Research Agenda – Motivation

Developed
Rate Region
Algorithms &
Software

network coding
problem
1
2
3
4
5
6

1,3
2,3
2,3
Developed Rate Region Algorithms & Software

rate region

**network coding problem**

**inequality description**

\[
\begin{align*}
\min \{R_4, R_6\} & \geq H(Y_2) \\
R_4 + R_6 & \geq H(Y_2) + H(Y_3) \\
R_5 + R_6 & \geq H(Y_2) + H(Y_3) \\
R_4 + R_5 & \geq H(Y_1) + H(Y_2) + H(Y_3) \\
R_4 + R_5 + 2R_6 & \geq H(Y_1) + 2H(Y_2) + 2H(Y_3)
\end{align*}
\]
Developed Rate Region Algorithms & Software

**network coding problem**

**inequality description**
\[
\begin{align*}
\min\{R_4, R_6\} & \geq H(Y_2) \\
R_4 + R_6 & \geq H(Y_2) + H(Y_3) \\
R_5 + R_6 & \geq H(Y_2) + H(Y_3) \\
R_4 + R_5 & \geq H(Y_1) + H(Y_2) + H(Y_3) \\
R_4 + R_5 + 2R_6 & \geq H(Y_1) + 2H(Y_2) + 2H(Y_3)
\end{align*}
\]

**extreme ray description**
\[
\begin{array}{cccccccccccc}
H(Y_1) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
H(Y_2) & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
H(Y_3) & 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
R_4 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
R_5 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
R_6 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0
\end{array}
\]
network coding problem

Developed Rate Region Algorithms & Software

inequality description
\[
\begin{align*}
\min\{R_4, R_6\} & \geq H(Y_2) \\
R_4 + R_6 & \geq H(Y_2) + H(Y_3) \\
R_5 + R_6 & \geq H(Y_2) + H(Y_3) \\
R_4 + R_5 & \geq H(Y_1) + H(Y_2) + H(Y_3) \\
R_4 + R_5 + 2R_6 & \geq H(Y_1) + 2H(Y_2) + 2H(Y_3)
\end{align*}
\]

\[\iff\]

extreme ray description
\[
\begin{array}{cccccccccccc}
H(Y_1) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
H(Y_2) & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
H(Y_3) & 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
R_4 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
R_5 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
R_6 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0
\end{array}
\]
Computationally Enabled Research Agenda – Motivation

Developed Rate Region Algorithms & Software

network coding problem

inequality description
\[ \min \{R_4, R_6\} \geq H(Y_2) \]
\[ R_4 + R_6 \geq H(Y_2) + H(Y_3) \]
\[ R_5 + R_6 \geq H(Y_2) + H(Y_3) \]
\[ R_4 + R_5 \geq H(Y_1) + H(Y_2) + H(Y_3) \]
\[ R_4 + R_5 + 2R_6 \geq H(Y_1) + 2H(Y_2) + 2H(Y_3) \]

extreme ray description
\[ \begin{bmatrix}
H(Y_1) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
H(Y_2) & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
H(Y_3) & 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
R_4 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
R_5 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
R_6 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 
\end{bmatrix} \]

Proof:
\[ R_4 \geq H(U_4) \]
\[ \vdots \]

automated converse proofs

rate region
Computedally Enabled Research Agenda – Motivation

Developed Rate Region Algorithms & Software

Inequality description
\[ \min\{R_4, R_6\} \geq H(Y_2) \]

Automated converse proofs
\[ R_4 + R_6 \geq H(Y_2) + H(Y_3) \]
\[ R_5 + R_6 \geq H(Y_2) + H(Y_3) \]
\[ R_4 + R_5 \geq H(Y_1) + H(Y_2) + H(Y_3) \]
\[ R_4 + R_5 + 2R_6 \geq H(Y_1) + 2H(Y_2) + 2H(Y_3) \]

Extreme ray description
\[
\begin{array}{cccccccccccc}
H(Y_1) & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
H(Y_2) & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
H(Y_3) & 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
R_4 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
R_5 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
R_6 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
\end{array}
\]

Rate region

Network coding problem 1 2 3 4 5 6

1,3 2,3 2,3
Developed Rate Region Algorithms & Software

Developed Rate Region Algorithms & Software

rate region

Proof:

\[
R_4 \geq H(U_4)
\]

\[
\begin{align*}
R_4 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_5 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_4 + R_5 &\geq H(Y_1) + H(Y_2) + H(Y_3) \\
R_4 + R_5 + 2R_6 &\geq H(Y_1) + 2H(Y_2) + 2H(Y_3)
\end{align*}
\]

inequality description

\[
\begin{align*}
\min\{R_4, R_6\} &\geq H(Y_2) \\
R_4 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_5 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_4 + R_5 &\geq H(Y_1) + H(Y_2) + H(Y_3) \\
R_4 + R_5 + 2R_6 &\geq H(Y_1) + 2H(Y_2) + 2H(Y_3)
\end{align*}
\]

Proof:

\[
R_4 \geq H(U_4)
\]

\[
\begin{align*}
R_4 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_5 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_4 + R_5 &\geq H(Y_1) + H(Y_2) + H(Y_3) \\
R_4 + R_5 + 2R_6 &\geq H(Y_1) + 2H(Y_2) + 2H(Y_3)
\end{align*}
\]

inequality description

\[
\begin{align*}
\min\{R_4, R_6\} &\geq H(Y_2) \\
R_4 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_5 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_4 + R_5 &\geq H(Y_1) + H(Y_2) + H(Y_3) \\
R_4 + R_5 + 2R_6 &\geq H(Y_1) + 2H(Y_2) + 2H(Y_3)
\end{align*}
\]

Proof:

\[
R_4 \geq H(U_4)
\]

\[
\begin{align*}
R_4 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_5 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_4 + R_5 &\geq H(Y_1) + H(Y_2) + H(Y_3) \\
R_4 + R_5 + 2R_6 &\geq H(Y_1) + 2H(Y_2) + 2H(Y_3)
\end{align*}
\]

inequality description

\[
\begin{align*}
\min\{R_4, R_6\} &\geq H(Y_2) \\
R_4 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_5 + R_6 &\geq H(Y_2) + H(Y_3) \\ 
R_4 + R_5 &\geq H(Y_1) + H(Y_2) + H(Y_3) \\
R_4 + R_5 + 2R_6 &\geq H(Y_1) + 2H(Y_2) + 2H(Y_3)
\end{align*}
\]

Proof:

\[
R_4 \geq H(U_4)
\]

\[
\begin{align*}
R_4 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_5 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_4 + R_5 &\geq H(Y_1) + H(Y_2) + H(Y_3) \\
R_4 + R_5 + 2R_6 &\geq H(Y_1) + 2H(Y_2) + 2H(Y_3)
\end{align*}
\]

inequality description

\[
\begin{align*}
\min\{R_4, R_6\} &\geq H(Y_2) \\
R_4 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_5 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_4 + R_5 &\geq H(Y_1) + H(Y_2) + H(Y_3) \\
R_4 + R_5 + 2R_6 &\geq H(Y_1) + 2H(Y_2) + 2H(Y_3)
\end{align*}
\]

Proof:

\[
R_4 \geq H(U_4)
\]

\[
\begin{align*}
R_4 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_5 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_4 + R_5 &\geq H(Y_1) + H(Y_2) + H(Y_3) \\
R_4 + R_5 + 2R_6 &\geq H(Y_1) + 2H(Y_2) + 2H(Y_3)
\end{align*}
\]

Proof:

\[
R_4 \geq H(U_4)
\]

\[
\begin{align*}
R_4 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_5 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_4 + R_5 &\geq H(Y_1) + H(Y_2) + H(Y_3) \\
R_4 + R_5 + 2R_6 &\geq H(Y_1) + 2H(Y_2) + 2H(Y_3)
\end{align*}
\]

Proof:

\[
R_4 \geq H(U_4)
\]

\[
\begin{align*}
R_4 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_5 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_4 + R_5 &\geq H(Y_1) + H(Y_2) + H(Y_3) \\
R_4 + R_5 + 2R_6 &\geq H(Y_1) + 2H(Y_2) + 2H(Y_3)
\end{align*}
\]

Proof:

\[
R_4 \geq H(U_4)
\]

\[
\begin{align*}
R_4 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_5 + R_6 &\geq H(Y_2) + H(Y_3) \\
R_4 + R_5 &\geq H(Y_1) + H(Y_2) + H(Y_3) \\
R_4 + R_5 + 2R_6 &\geq H(Y_1) + 2H(Y_2) + 2H(Y_3)
\end{align*}
\]
Substitutes outer/inner bounds to $\Gamma_N^*$ into Yan, Yeung, Zhang ’12

$$\mathcal{R}_* = \text{proj}_{R_e, [H(Y_s)|s \in S]} \left( \text{con}(\Gamma_N^* \cap \mathcal{L}_{123}) \cap \mathcal{L}_{45} \right)$$

where $\mathcal{L}_{123} := \left\{ \mathbf{h} \mid h_{Y_S} = \sum_{s \in S} h_{Y_s}, h_{X_{\text{Out}(k)}|Y_s} = 0, h_{X_{\text{Out}(i)}|X_{\text{In}(i)}} = 0 \right\}$ and $\mathcal{L}_{45} := \left\{ (\mathbf{h}^T, \mathbf{R}^T)^T \in \mathbb{R}_{++}^{2N-1+|\mathcal{E}|} : R_e \geq h_{U_e}, e \in \mathcal{E}, h_{Y_{\beta(t)}|U_{\text{In}(t)}} = 0 \right\}$
Computationally Enabled Research Agenda – Motivation

- Want software to reach as large networks as possible.  \(\implies\) Define & Exploit Problem Symmetry
Exploiting Symmetry in Rate Region Computation – Definition [Li Arxiv '15]

- Represent a network as $A := (Q, W)$:
  - sources: $1 \ldots K$
  - edges: $K + 1, \ldots, K + |E_U|$
  - Edge definitions $W$ a set of $(i, A)$, $i \in \{K + 1, \ldots, K + |E_U|\}$, $A \subseteq \{1, \ldots, K + |E_U|\} \setminus \{i\}$;
  - Sink definitions $Q$ a set of $(i, A)$, $i \in \{1, \ldots, K\}$, $A \subseteq \{1, \ldots, K + |E_U|\} \setminus \{i\}$;
  - Same $i$ can appear in $W$ but not in $Q$

- Group $G := S_{1,2,\ldots,K} \times S_{K+1,\ldots,K+|E_U|}$ acts naturally on $(Q, W)$.

- Network Symmetry Group (NSG): stabilizer subgroup $G_A = \{g \in G | g((Q, W)) = (Q, W)\}$.

- NSG Stabilizes constraints $L_{123}$ and $L_{45}$ in rate region expression.
  \[ \pi(R_*(A)) = R_*(A) \text{ for all } \pi \in G_A \]

- any $g \in G$ w/ $g \notin G_A$ gives an equivalent network $A' = gA$ s.t. $g(R(A)) = R_*(A')$.

- Orbit Stabilizer Theorem $\implies$ there are $\frac{|G|}{|G_A|}$ such equivalent networks
**Exploiting Symmetry in Rate Region Computation – Converse**

Converse: calculate

\[
\mathcal{R}_o = \text{proj}_{\mathcal{R}_E,[H(Y_s)|s \in S]}(\Gamma_N^o \cap \mathcal{L}_{12345})
\]  

(2)

through polyhedral projection.

---

**Extreme Ray/Point Projection**

- Project Extreme Rays & Points
- Redundancy Removal (convex hull)
- Extremal repr. of projected polyhedron
- Representation conversion
- Inequality repr. of projected polyhedron

**Fourier Motzkin Elimination**

- Successively Eliminates Variables (slow)
- Inequalities of N dimen. polyhedron
- N-1 dim. proj. (elim. 1 var)
- N-2 dim. proj. (elim. 1 var)

**Convex Hull Method**

- Works directly in projected space
- Inequality repr. of projected polyhedron

**Benson’s Outer Approximation Algorithm**

- Only deals with Pareto frontier in projected space
- Inequality repr. of projected polyhedron
Exploiting Symmetry in Rate Region Computation – Converse

Apte’s CHM w/ rational LP solver implementation available from ASPITRG website.
Exploiting Symmetry in Rate Region Computation – Converse [Apte Netcod 2015]

If we know one of these then we know all of these!

projection of a hypercube to 3-d
Exploiting Symmetry in Rate Region Computation – Converse [Apte Netcod 2015]
Exploiting Symmetry in Rate Region Computation – Converse [Apte Netcod 2015]

Symmetry exploitation

- Symmetric improvements
- Symmetric linear programs
- Symmetric updates

Permutation of standard bases is a restricted symmetry of this polytope. It also leaves cost invariant.

There exists a solution of this LP in fixed space: the subspace of points mapping to themselves under the basis vector permutation.

\[
\begin{align*}
  x_1 &\leq 2.5 \\
  x_1 + x_2 &\leq 3.7 \\
  x_2 &\leq 2.5
\end{align*}
\]

\[
\max c^T x = x_1 + x_2
\]
Exploiting Symmetry in Rate Region Computation – Converse [Apte Netcod 2015]
Exploiting Symmetry in Rate Region Computation – Achievability

- Key Idea: linear network code = (\mathcal{V}, \phi)
  - \mathcal{V} a set of \ell \leq K + |E_U| subspaces of GF(q)^r
  - a map \phi : \mathcal{V} \rightarrow \{1, \ldots, K + |E_U|\} assigning these subspaces to network sources & edges.

- Note: rate region only depends on subset entropies=dimensions/ranks of these subspaces and their sums! (associated polymatroid).

- Two sets \mathcal{V} and \mathcal{V}' will give the same ranks if \mathcal{V}' = \mathcal{V}g for some g \in PGL_q^r!

- Linear Code Enumeration Problem:
  - finding only canonical representatives of orbits in the sets of subspaces \mathcal{V} under \ PGL_q^r which can be grouped with a \phi to obey network constraints (& optionally have specified \ Re, H(Y_s) dimension vectors)
  - Algorithm Leiterspiel [Schmalz, Betten Braun & Fripertinger] for finding orbits in the power-set obeying an inherited property recursively in subset-size can be applied to solve this problem.
- Inherited property: ability to map (partial $\phi$ or p-map) to a (thus far defined) subset of network variables and obey the constraints they are involved in.
- NSG (together with stabilizer subgroup of the subspaces $\mathcal{V}$) makes some $\phi$ equivalent with one another, removing the need to test all of them.

• Method above implemented in our ITAP (github) GAP package

• C++ version building on Orbiter [Betten] forthcoming.

• Although enumeration oriented, when used as a verification algorithm (w/ specified rate vector) it can still be faster than the Groebner basis (w/ Singular) based path-gain verification of Subramanian & Thangaraj!

• Nonlinear codes: view discrete random variables (sources & edges) each as partitions of a finite $\Omega$ from $(\Omega, \mathcal{F}, \mathcal{P})$. Group is $S_{|\Omega|}$. Have done this for E.V.s [Liu arxiv 2016] but could be coupled with p-maps $\phi$ to yield nonlinear codes.
• Want software to reach as large networks as possible.  \(\implies\) Define & Exploit Problem Symmetry
Want software to reach as large networks as possible. \(\implies\) Define & Exploit Problem Symmetry

Want the database of solved problems to be as large as possible. \(\implies\) Define Problem Equivalence Classes via Canonical & Minimal Representatives
Computationally Enabled Research Agenda – Minimality & Symmetry

all MSNC-HN problems
all MSNC-HN problems

equivalence classes
Computationally Enabled Research Agenda – Minimality & Symmetry
Computationally Enabled Research Agenda – Minimality & Symmetry

all MSNC-HN problems

equivalence classes

minimal

non-minimal

all MSNC-HN problems
Computationally Enabled Research Agenda – Minimality & Symmetry

all MSNC-HN problems
equivalence
1,2,3
non-minimal
canonical & minimal
representative
minimal
Algorithm to List only Canonical & Minimal Problems Directly

extension algorithm adapting Leiterspiel (orbits on power set)

all size k problems

all size k+1 problems
Computationally Enabled Research Agenda – Rate Region Database

Algorithm to List only Canonical & Minimal Problems Directly

Our Software

extension algorithm adapting Leiterspiel (orbits on power set)
Computationally Enabled Research Agenda – Rate Region Database

Algorithm to List only Canonical & Minimal Problems Directly

Our Software

Rate Region Database:
rate region, converse proof, efficient codes for every canonical & minimal problem
Network Coding Rate Region Database

Database of ~7000 Rate Regions of >100k Networks (Li Trans IT sub 2014)
Database of ~744k Rate Regions of >7M or ~2.3T Minimal Networks (Li Trans IT sub 2015)
Network Coding Rate Region Database

Database of ~7000 Rate Regions of >100k Networks (Li Trans IT sub 2014)
Database of ~744k Rate Regions of > 7M or ~2.3T Minimal Networks (Li Trans IT sub 2015)

What now?
Network Coding Rate Region Database

Database of ~7000 Rate Regions of >100k Networks (Li Trans IT sub 2014)
Database of ~744k Rate Regions of > 7M or ~2.3T Minimal Networks (Li Trans IT sub 2015)

What now? Submit 744,000 transactions papers?
Network Coding Rate Region Database

Database of ~7000 Rate Regions of >100k Networks (Li Trans IT sub 2014)
Database of ~744k Rate Regions of >7M or ~2.3T Minimal Networks (Li Trans IT sub 2015)

What now? Analyze, learn, and explain something! Can't read and remember 744,000 network proofs.
Network Coding Rate Region Database

Database of ~7000 Rate Regions of >100k Networks (Li Trans IT sub 2014)
Database of ~744k Rate Regions of > 7M or ~2.3T Minimal Networks (Li Trans IT sub 2015)

Investigate Structure through Network Hierarchy
• Understand existing large rate region database through structure: embedding operators & forbidden network minors
Computationally Enabled Research Agenda – Hierarchy

- Understand existing large rate region database through structure: embedding operators & forbidden network minors
- Build solutions to arbitrarily sized networks: combinations operators, operator concatenation
Rate region (bound) of embedded network can be directly obtained from rate region (bound) of parent network.

- Insufficiency of class of codes of small $\Rightarrow$ insufficiency of class of codes of big. (forbidden network minor)
First database: 5438 canonical MDCS for which scalar binary codes are insufficient can be boiled down to 12 forbidden minor networks.
• rate region of big directly expressible from rate regions of smalls
Computationally Enabled Research Agenda – Hierarchy: Combination Operators

\[ R_i, \quad R_1 + R_2, \quad R_i, \quad R_3 + R_i, \quad R_3 + R_i + R_5, \quad R_6, \quad R_i + R_8, \quad R_6 + R_i, \quad R_6 + R_i + 2R_8, \quad R_9, \quad R_{10}, \quad R_{11}, \quad R_{12}, \quad R_{10}' + R_{11} + R_{12} \geq H(X_4) + 2H(X_5) + H(X_6) \]

\[ H(X_2), \quad i = 1, 2 \]
\[ H(X_1) + H(X_2) \]
\[ H(X_4), \quad i = 3, 4, 4' \]
\[ H(X_3) + H(X_4), \quad i = 4, 4' \]
\[ H(X_3) + 2H(X_4), \quad i = 4, 4' \]
\[ H(X_6) \]
\[ H(X_6), \quad i = 7, 7' \]
\[ H(X_5) + H(X_6), \quad i = 7, 7' \]
\[ H(X_3) \]
\[ \sum_{i=1}^{3} H(X_i) \]
\[ H(X_5) \]
\[ H(X_4) + H(X_5) \]

\[ R_7 + R_8 \geq H(X_6) \]
\[ R_6 \geq H(X_6) \]
\[ R_7 + R_8 \geq H(X_6) \]
\[ R_6 + R_i \geq H(X_4) + 2H(X_4) \]
\[ R_6 + R_7 \geq H(X_5) + H(X_6) \]
\[ R_6 + R_7 + 2R_8 \geq H(X_5) + 2H(X_6) \]
Computationally Enabled Research Agenda – Hierarchy: Operator Concatenation

- Contracting edge 3
- Contracting edge 4
- Source merge $[1,2]$ with $[2,1]$
- Deleting edge 3
- Edge merge 2 & 2
- Merge two sinks
**Computationally Enabled Research Agenda** – Hierarchy: Operator Concatenation

Use operators *together* to get RR for big networks. Partial Network Closure.
**Computationally Enabled Research Agenda** – Hierarchy: Operator Concatenation

Start with the single $(1,1)$, single $(2,1)$, and the four $(1,2)$ networks; These 6 tiny networks can generate new 11635 networks w/ small cap!

<table>
<thead>
<tr>
<th>size\cap</th>
<th>combination operators only</th>
<th>embedding and combinations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(3,3)</td>
<td>(3,4)</td>
</tr>
<tr>
<td>(1,3)</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>(1,4)</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>(2,2)</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>(2,3)</td>
<td>13</td>
<td>16</td>
</tr>
<tr>
<td>(2,4)</td>
<td>0</td>
<td>97</td>
</tr>
<tr>
<td>(3,2)</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>(3,3)</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>(3,4)</td>
<td>0</td>
<td>135</td>
</tr>
<tr>
<td>(4,2)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(4,3)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(4,4)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>all</td>
<td>46</td>
<td>292</td>
</tr>
</tbody>
</table>
**Computationally Enabled Research Agenda** – Hierarchy: Operator Concatenation

With the increase of cap size, number of new networks increases!

<table>
<thead>
<tr>
<th>size\cap</th>
<th>combination operators only</th>
<th>embedding and combinations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(3,3)</td>
<td>(3,4)</td>
</tr>
<tr>
<td>(1,3)</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>(1,4)</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>(2,2)</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>(2,3)</td>
<td>13</td>
<td>16</td>
</tr>
<tr>
<td>(2,4)</td>
<td>0</td>
<td>97</td>
</tr>
<tr>
<td>(3,2)</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>(3,3)</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>(3,4)</td>
<td>0</td>
<td>135</td>
</tr>
<tr>
<td>(4,2)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(4,3)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(4,4)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>all</td>
<td>46</td>
<td>292</td>
</tr>
</tbody>
</table>
**Computationally Enabled Research Agenda** – Hierarchy: Operator Concatenation

Embedding operations are important in the process!

<table>
<thead>
<tr>
<th>size\cap</th>
<th>combination operators only</th>
<th>embedding and combinations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(3,3)</td>
<td>(3,4)</td>
</tr>
<tr>
<td>(1,3)</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>(1,4)</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>(2,2)</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>(2,3)</td>
<td>13</td>
<td>16</td>
</tr>
<tr>
<td>(2,4)</td>
<td>0</td>
<td>97</td>
</tr>
<tr>
<td>(3,2)</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>(3,3)</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>(3,4)</td>
<td>0</td>
<td>135</td>
</tr>
<tr>
<td>(4,2)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(4,3)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(4,4)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>all</td>
<td>46</td>
<td>292</td>
</tr>
</tbody>
</table>