Overhead Performance Tradeoffs in Distributed Wireless Networks

John Walsh, S. Weber & L. Cimini, J. Garcia-Frias
Contributing Students: Congduan Li, Yunshu Liu, Jayant Apte
Department of Electrical and Computer Engineering
Drexel University & University of Delaware
contact: jwalsh@ece.drexel.edu

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Outline

1. Project Overview
2. Computing Rate Regions via Entropy Geometry Bounds
   (a) Shannon & Non-Shannon Outer Bounds
   (b) Inner Bounds from Representable Matroids & Subspace Relations
   (c) Computational Geometry for Computing Rate Regions
   (d) Codes & Extreme Entropies
3. Achieving Limits: Characterizing Extreme Entropies with Information Geometry
   (a) Information Geometric Characterization of Distributions on Shannon Facets
   (b) Some Thoughts on Information Geometry of Non-Shannon Exposed Faces

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Project Overview – Motivation & Goals

Motivation: PHY/MAC Control & Reference Information becoming a large $\%$ of Wireless Network Traffic

- eg. LTE - $\approx 25\%$ of downlink transmission or more (paid $20 B for spectrum!)
- OFDMA/MIMO & turbo $\rightarrow$ PHY limits, but requires $\uparrow$ reference & control
- *wireless control information (overhead) itself is largely not efficiently encoded*

Main Project Goals:

- Determine Fundamental Tradeoff between Overhead and Performance in Wireless Resource Allocation
- Assess Overhead & Performance of Well Known Resource Controllers
- Design New Controllers to Achieve/Approach the Fundamental Tradeoff

Project Overview – Meta Problem

Problem: Design a distributed controller for a wireless network that maximizes performance while satisfying QoS demands (resource allocation, rate control, scheduling, routing, …)

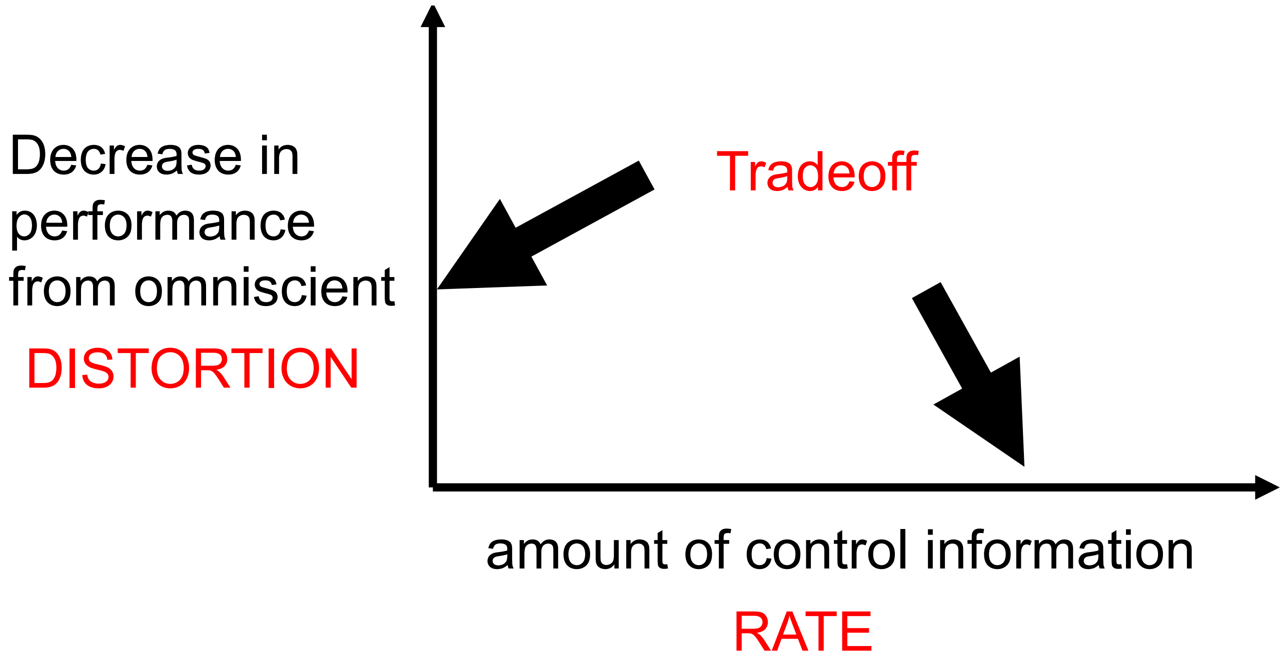
Assumption: Observations (e.g., channel gains, queue lengths) are distributed at different nodes throughout the network

For a given amount of overhead, what is the performance degradation from what an omniscient centralized controller could achieve?

How much information do nodes need to exchange to achieve a target performance and QoS?

How can a distributed, low-complexity, scheduler be built to approach the performance of the centralized controller's design?

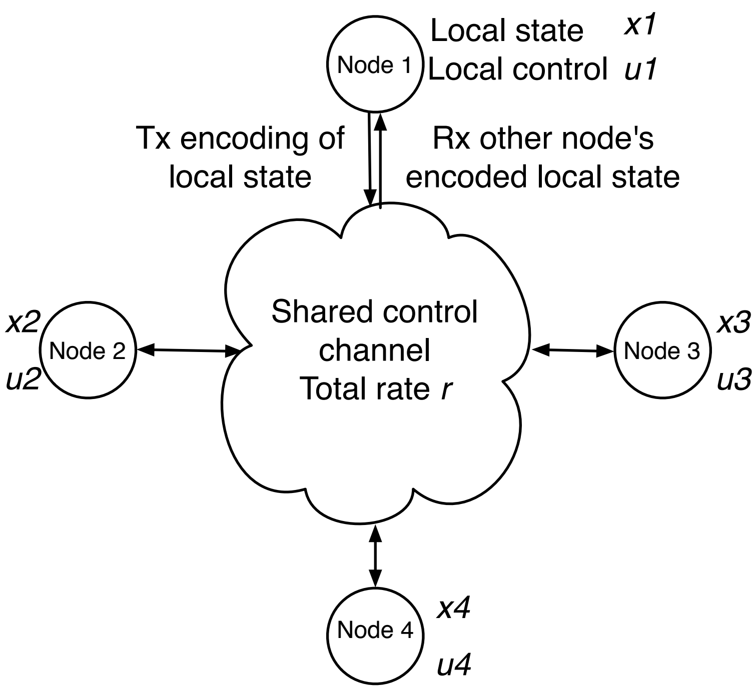
Project Overview – Meta Idea

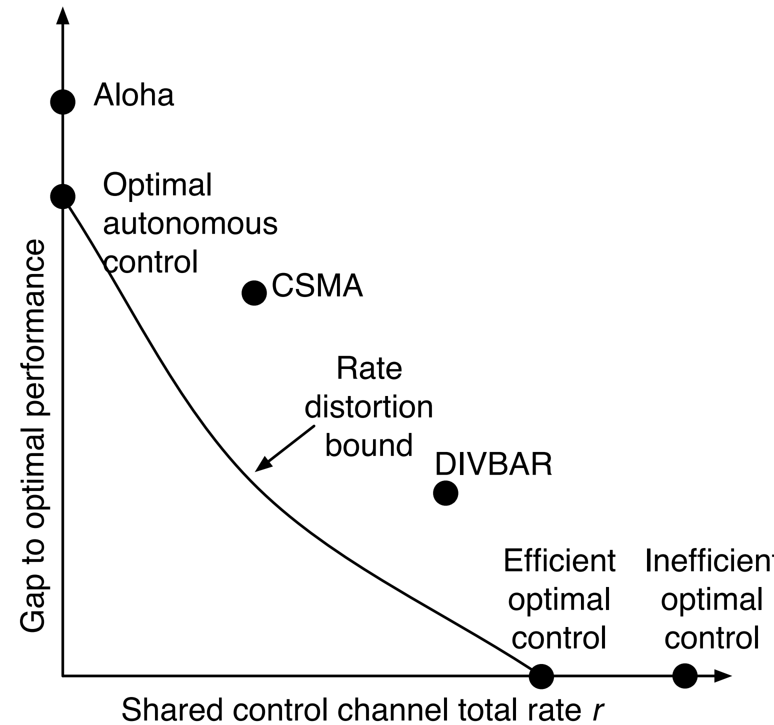


What is the *minimal* amount of control information to achieve a *given* performance? ⇒ distributed lossy source coding limit problem (*multiterminal rate distortion*)

What algorithms approach this minimum amount of overhead while meeting a target perf.? ⇒ lossy source codec design problem

Project Overview – Overall Technique

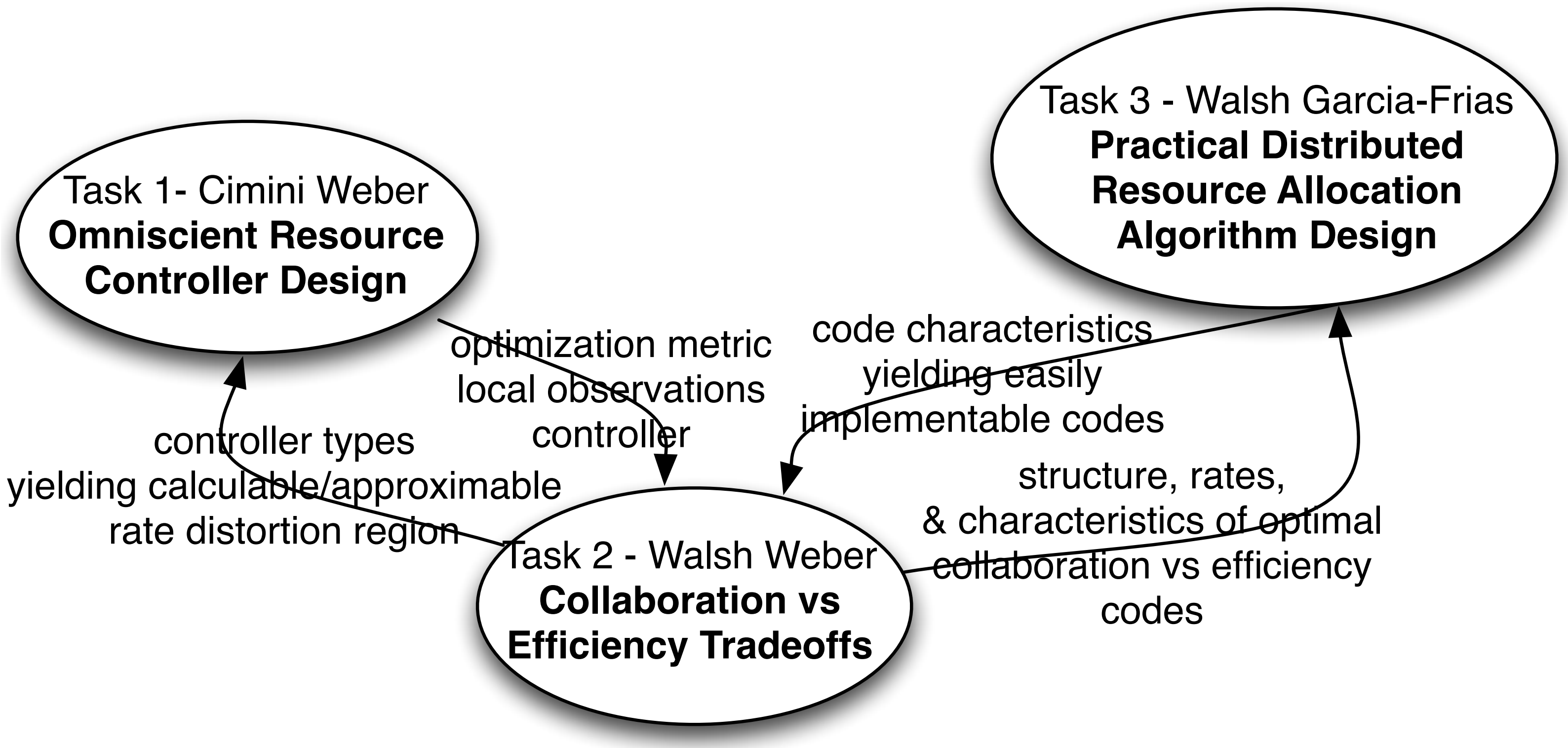




Key Idea: View Optimal Overhead vs. Performance tradeoff as a Multiterminal Rate Distortion Problem

- Nodes 1) observe (eg. CQIs, demands), 2) send encoded observation, 3) decode into local control
- Distortion: difference between performance of *omniscient* optimal controller and performance of inferred control decisions

Project Overview – Agenda & Personnel

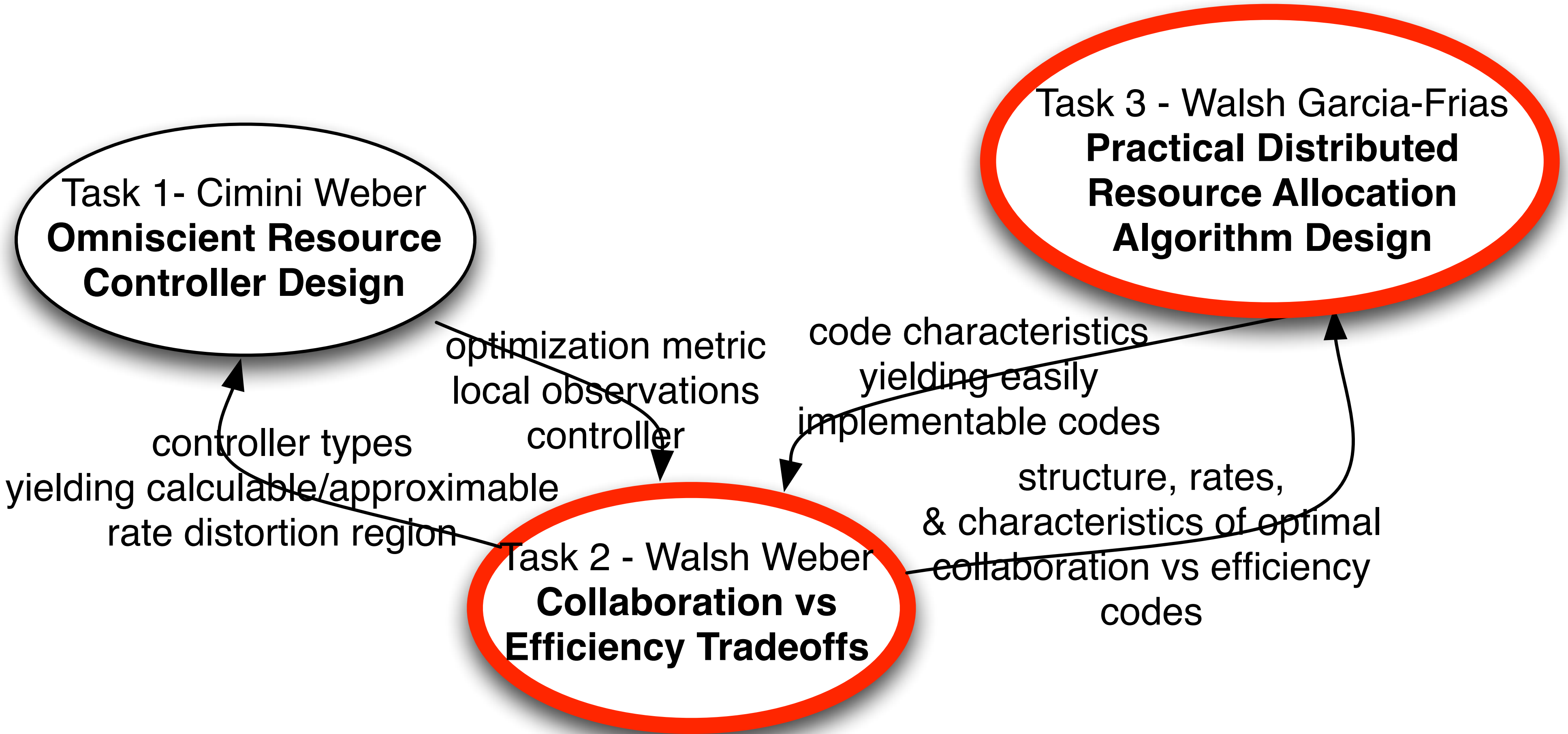


1. Characterize Omniscient Controllers & Joint Distributions (Weber & Cimini)
2. Assess Overhead & Performance of well known Controllers (Weber & Cimini)
3. Determine/Bound Rate Distortion Function (Walsh & Weber)
4. Develop New Resource Controllers as Practical Distributed Source Codes (Walsh & Garcia-Frias)

each one of these is *very* hard.

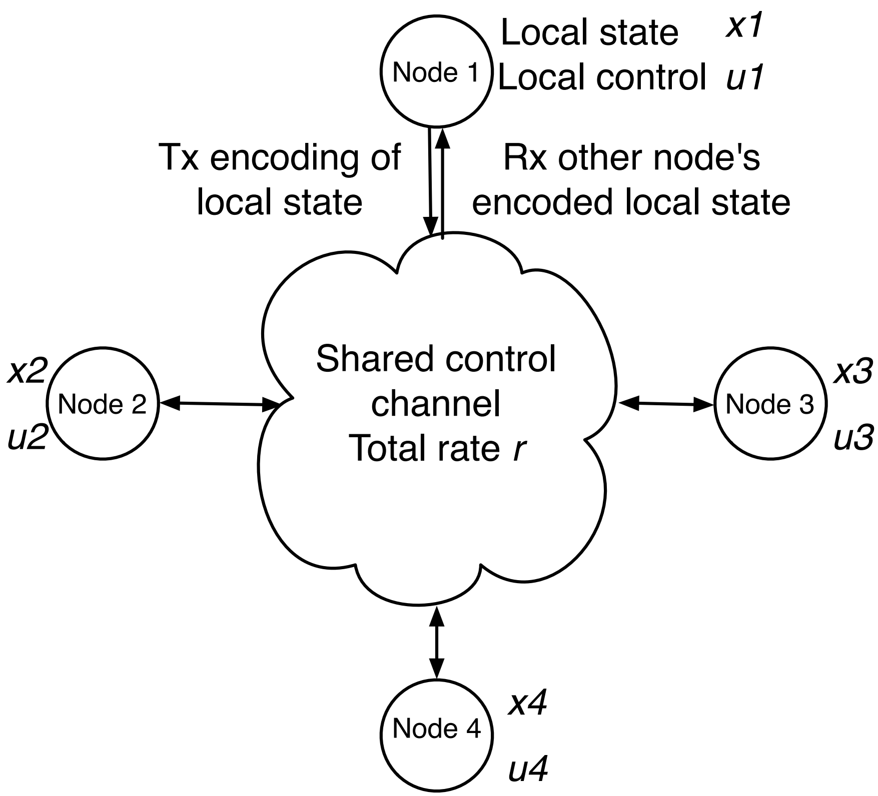
Today's Talk - Relation to Project Overview

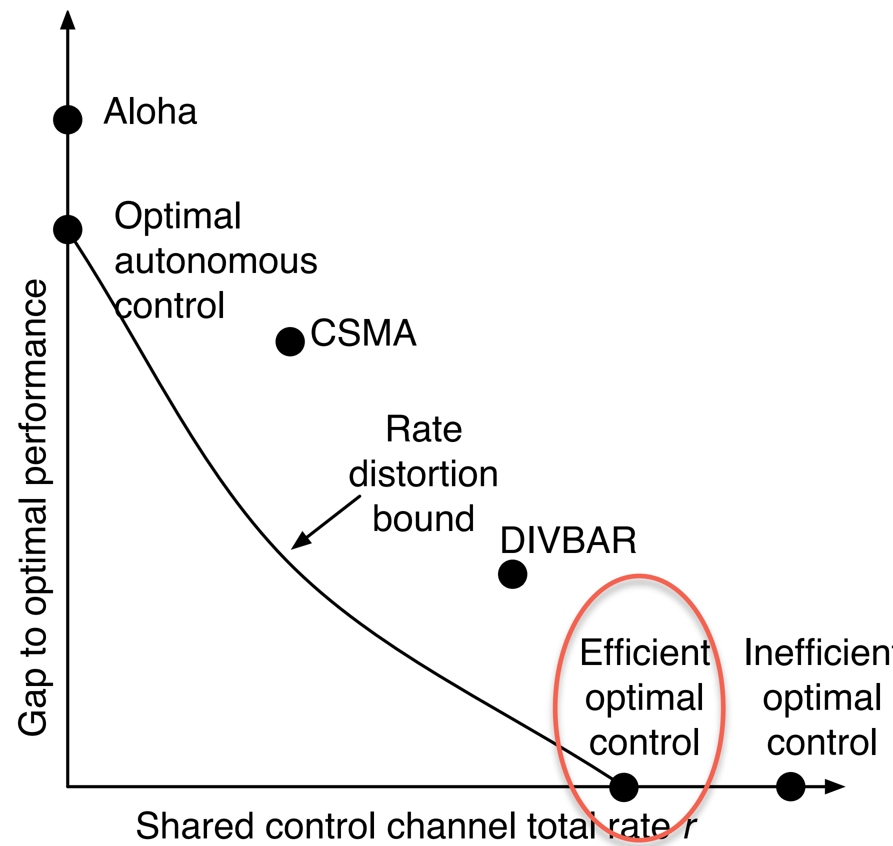
- Focus on two parts of Agenda



Today's Talk - Relation to Project Overview

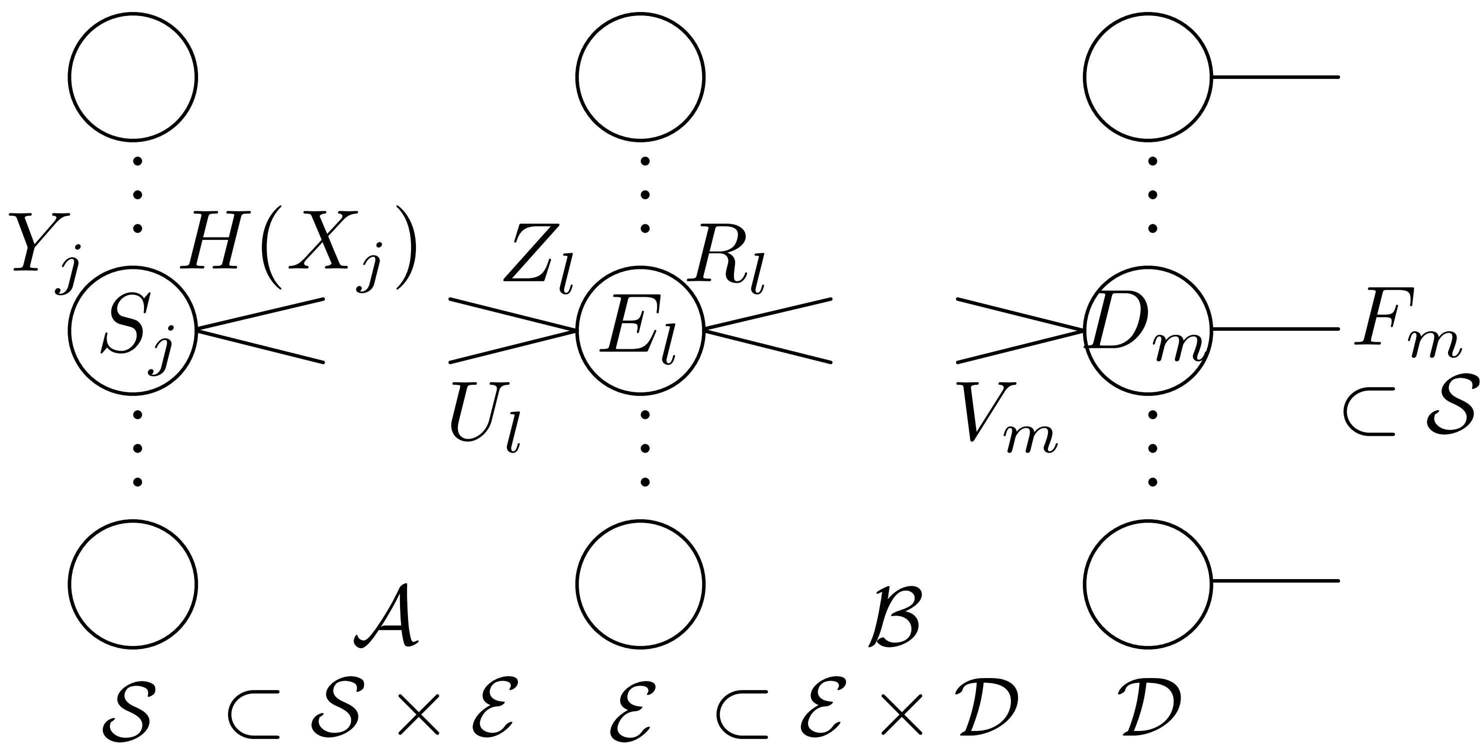
- Focus on two parts of Agenda (Rate distortion function determination & efficient code characterization)
- Zoom in to 1 Point on Tradeoff





Today's Talk - Relation to Project Overview

- Focus on two parts of Agenda (Rate distortion function determination & efficient code characterization)
- Zoom in to 1 Point on Tradeoff
  - lossless reproduction of salient part of omniscient control information (min rate for 0 distortion)
- Further simplify the model:



Today's Talk - Relation to Project Overview

- Focus on two parts of Agenda (Rate distortion function determination & efficient code characterization)
- Zoom in to 1 Point on Tradeoff
  - lossless reproduction of salient part of omniscient control information (min rate for 0 distortion)
- Further simplify the model: (distributed source coding for satellite communications – Zhang & Yeung 1998)
  - independent local states available to subsets of nodes and are themselves part of control
  - encoded messages received by subsets of nodes
  - each node wishes to losslessly reproduce some subset of the local states
- Problem: *What rates allow this?*

Today's Talk – Rate Region for Simplified Problem

Zhang & Yeung [1] introduced bounds for this type of rate region, and Yeung et. al. [2, 3] calculated it explicitly as

$$\mathcal{R} = \mathsf{Ex}(\mathsf{proj}_{U_e}(\overline{\mathsf{con}(\Gamma^*_{\mathfrak{n}} \cap \mathcal{L}_{123})} \cap \mathcal{L}_4)) \tag{1}$$

i.e. expressed as a projection of a lin $\cap$ w/ the *region of entropic vectors* (REV)

Obtained as an instance of a capacity region for a network code.

Today:

1. explain *computation techniques* for rate regions using bounds for REV.
2. characterize the codes which achieve capacity
   (a) linear codes – representable matroids & subspace relations
   (b) other codes – information geometry of distributions associated with entropy facets

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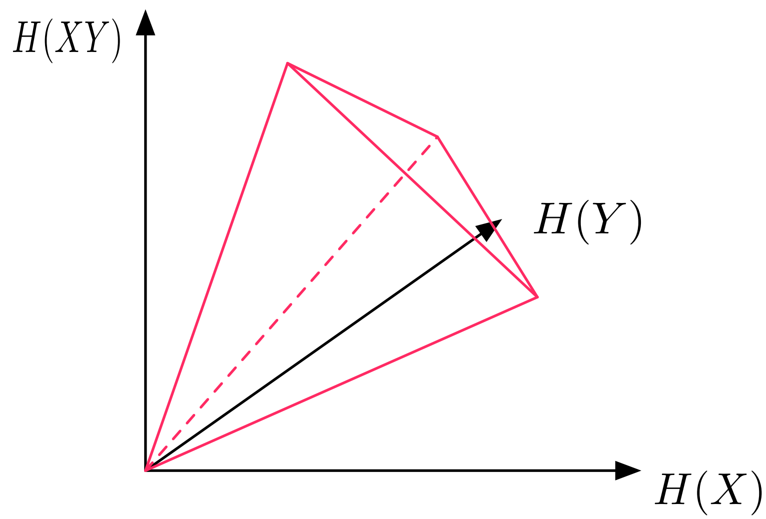
Region of Entropic Vectors $\bar{\Gamma}^*_N$ – What is it?

1. $\boldsymbol{X} = (X_1, \ldots, X_N)$ N discrete RVs
2. $h(\boldsymbol{X}_{\mathcal{A}})$ entropy of subset $\boldsymbol{X}_{\mathcal{A}} = (X_i, i \in \mathcal{A})$ $\mathcal{A} \subseteq \{1, \ldots, N\}$.
3. Let $\boldsymbol{h} = (h(\boldsymbol{X}_{\mathcal{A}}),\ \mathcal{A} \subseteq [N])$ be the vector of entropies of each non-empty subset $\mathcal{A} \subseteq [N]$. Note $\boldsymbol{h}$ has $2^N - 1$ entries.
   - Example: for $N = 3$, $\boldsymbol{h} = (h_1, h_2, h_3, h_{12}, h_{13}, h_{23}, h_{123})$.
4. A vector $\boldsymbol{h} \in \mathbb{R}^{2^N-1}$ is called entropic if its elements are the entropies for some joint distribution $\boldsymbol{p}$ on the N rvs $\boldsymbol{X}$.
5. REV $\bar{\Gamma}^*_N$ is the closure of the set of all entropic vectors. [3]

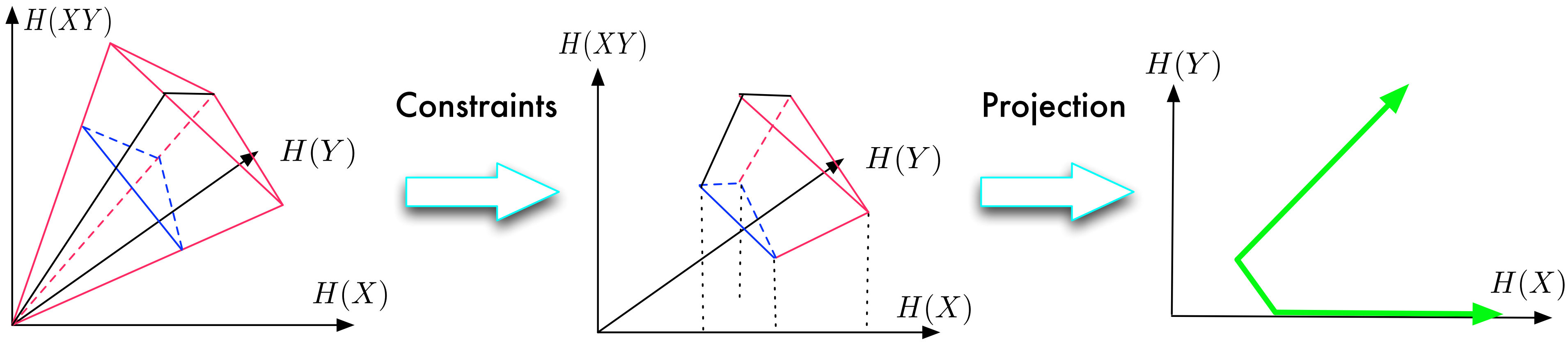


REV for $N = 2$:

$$H(X) \leq H(XY)$$
$$H(Y) \leq H(XY)$$
$$H(XY) \leq H(X) + H(Y)$$

$\bar{\Gamma}^*_N$ *is an* **unknown** *non-polyhedral convex cone for* $N \geq 4$.

Rate Regions from REV Bounds – Yeung et. al.



Constraints:

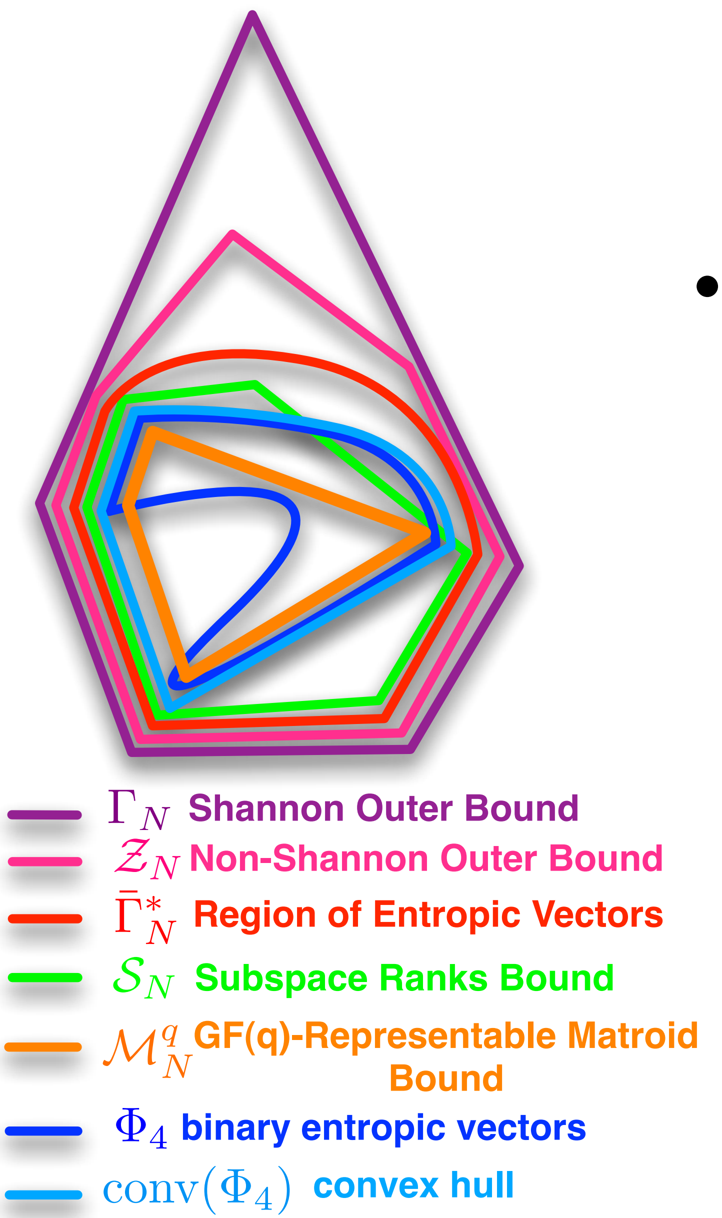
- Rate Constr.
- Source Constr.
- Encoding Constr.
- Decoding Constr.

Project:

- Keep only rates & source entropies:
- Rates Above Work Too:

Since $\bar{\Gamma}_N^*$ *unknown for* $N \geq 4$ *have to substitute INNER and OUTER BOUNDs.*
Determining $\bar{\Gamma}_N^*$ *is* **equivalent** *to finding MSNC cap. reg. for all networks [4, 5, 6].*

REV Bounds – Outer Bounds



- **Shannon Outer Bound**: Γ_N

$$I(\boldsymbol{X}_{\mathcal{A}}; \boldsymbol{X}_{\mathcal{B}} | \boldsymbol{X}_{\mathcal{C}}) \geq 0 \quad \forall \mathcal{A}, \mathcal{B}, \mathcal{C}$$

$\Gamma_2 = \bar{\Gamma}^*_2, \Gamma_3 = \bar{\Gamma}^*_3$.

$\Gamma_N \neq \bar{\Gamma}^*_N, N \geq 4$ $\bar{\Gamma}^*_N$ non-polyhedral convex cone

- **Non-Shannon Outer Bounds**: [7, 8, 9, 10, 11, 12, 13]

Yeung & Zhang, Dougherty & Freiling & Zeger, Matus

Start with 4 unconstr. r.v.s

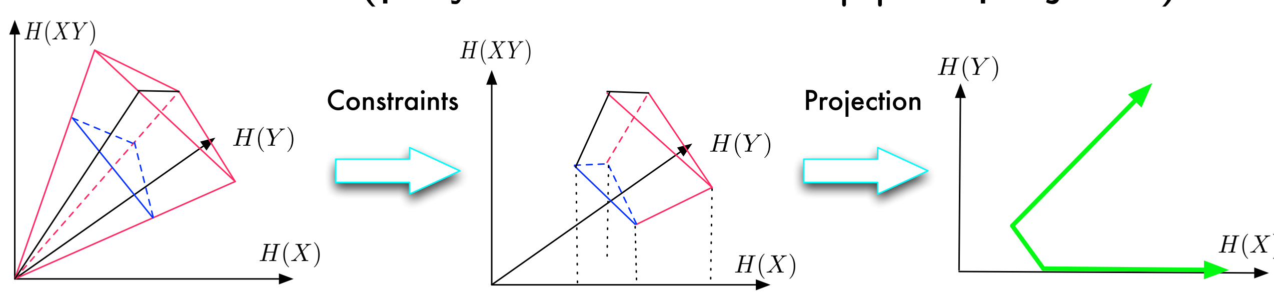
add rv. obeying distr. match & Markov. cond.

Intersect Γ_N for $N \geq 5$ w/ Markov & distr. match

Project back to orig. 4 unconstr. vars.

obtain new information inequalities this way!

Look familiar? (poly. bound $\rightarrow$ lin. $\bigcap$ $\rightarrow$ project?)



REV Bounds – Inner Bounds: Representable Matroids

Matroids: $\boldsymbol{r} \in \Gamma_N \cap \mathbb{Z}^{2^N-1}$, $r(\mathcal{A}) \leq |\mathcal{A}|$

- All non-isomorphic matroids for $N \leq 9$ [14] '08
- enumerating non-iso. matroids is difficult

$GF(q)$-**Representable Matroid**: $\boldsymbol{r} \in \Gamma_N \cap \mathbb{Z}^{2^N-1}$ s.t. $\exists \mathbf{A} \in GF(q)^{M \times N}$ s.t. $r(\mathcal{A}) = \mathrm{rank}(\mathbf{A}_{:,\mathcal{A}})$

- repr. matroid = scaled EV!: $\boldsymbol{u} \sim \mathcal{U}(GF(q)^M)$

$$\boldsymbol{X} = \boldsymbol{u}\mathbf{A} \;\Rightarrow\; h_{\mathcal{A}} = r(\mathcal{A}) \log_2 q$$

- Key: representability $\Leftrightarrow$ no forbidden minors:
  - *complete small list known for* $q \in \{2,3,4\}$ [15, 16, 17, 18, 19, 20] eg.:$GF(2)$ repr. $\Leftrightarrow$ no $U(2,4)$ minor
- $\bar{\Gamma}^*_N$ bound: $\mathcal{T}^q_N$ conic hull of $GF(q)$-repr. matroids.

Build inner bound $\mathcal{T}^q_N$ [15]

Forbid. minor size k

1. list non-isomorphic matroids size k
2. remove Forbidden minor
3. convex hull, extreme rep. $\rightarrow$ ineq. rep
4. for every size k subset of N, substitute into inequalities rank func. of minor
5. append to Γ_N

save comp. vs. checking in N *directly*

REV Bounds – Inner Bounds: Subspaces

Subspace Bounds: $\boldsymbol{r} \in \Gamma_N \cap \mathbb{Z}^{2^N-1}$ projections of representable matroids, $N' \geq N$, partition $\{1,\ldots,N'\} = \bigcup_{n=1}^{N} \mathcal{G}_n$, $\mathcal{G}_n \cap \mathcal{G}_k = \emptyset$ $n \neq k$

$$r(\mathcal{A}) = \text{rank}([\mathbf{A}_{:,\mathcal{G}_n} | n \in \mathcal{A}]) \qquad (2)$$

subspace ranks $=$ scaled EV!:

$$\boldsymbol{X}_n = \boldsymbol{u}\mathbf{A}_{:,\mathcal{G}_n} \Rightarrow h_{\mathcal{A}} = r(\mathcal{A}) \log_2 q$$

$\mathcal{S}_N$: conic hull of all subspace ranks

$\mathcal{S}_4$: $\Gamma_4 \cap$ Ingleton's [21, 22, 23]

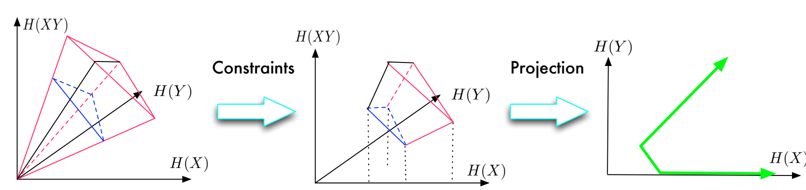
$$I(X_1;X_2) + I(X_3;X_4|X_1) + I(X_3;X_4|X_2) - I(X_3;X_4) \geq 0$$

$\mathcal{S}_5$ recently characterized by DFZ [24, 25] using Γ_5

$\mathcal{S}_N$ unknown for $N \geq 6$, but can inner bounded by projecting $\mathcal{T}_N^q$

Build inner bound for $\mathcal{S}_N \subsetneq \bar{\Gamma}_N^*$:

1. Obtain $\mathcal{M}_{N'}^q$ using method from previous slide, i.e. *intersect Γ_N with inequalities from forbidden minors*
2. project (remove all but entropies where each element in $\boldsymbol{X}_n$ appears together)



sound familiar??? (poly. bound $\rightarrow$ lin. $\cap$ $\rightarrow$ project)

Computational Geometry for Computing Rate Regions [26]

In each case (region computation, inner bound, and outer bound) have *same 3 steps*:

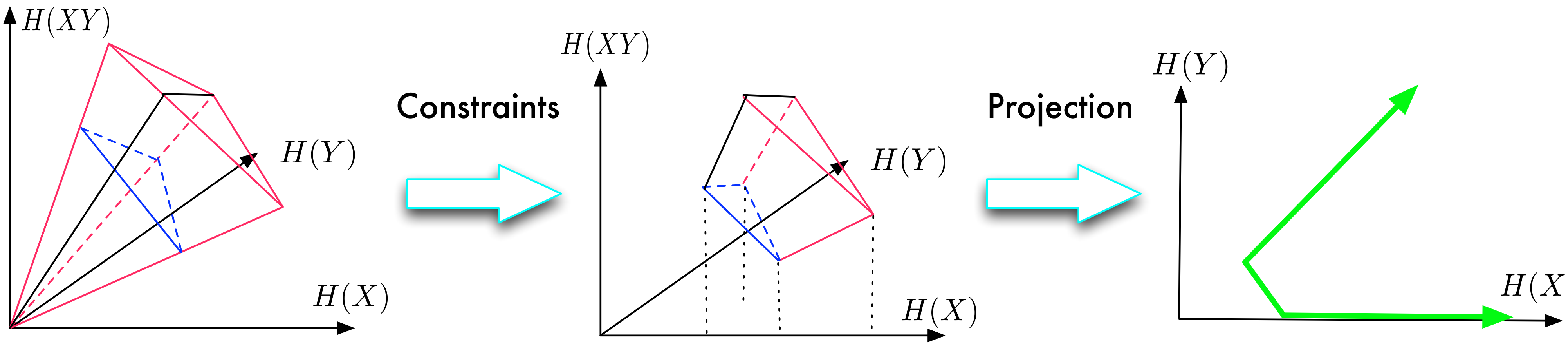
$$\text{(poly. bound} \rightarrow \text{lin.} \cap \rightarrow \text{project)}$$

Sounds simple, but, *there are a number of different ways of calculating each of these steps* and *they have drastically differing complexity!* Some of the issues:

- Which representation of the polyhedral cone? (Inequality or extreme ray) one can be gigantic while the other is tiny? Matroid list may not be available.
- How to add the constraints:
  - simply concatenate the inequalities? (then convert to extremes?)
  - if using extreme representation: double descriptions step
- Which method to use to project:
  - Simply remove the elements in extreme rays, then take a convex hull, method?
  - Fourier Motzkin, Convex Hull Method (CHM) [27]
- Which method for representation conversion? (DD [28], LRS [29]), parallelization?

area of active software development. future: exploiting symmetries [30]

Codes & Extreme Entropies



- efficient points in rate region are projections of extreme points/rays in constrained bound (extreme entropies)
- Using representable matroid based inner bounds and subspace based inner bounds allows for optimal points in rate region to be associated with their *efficient code*

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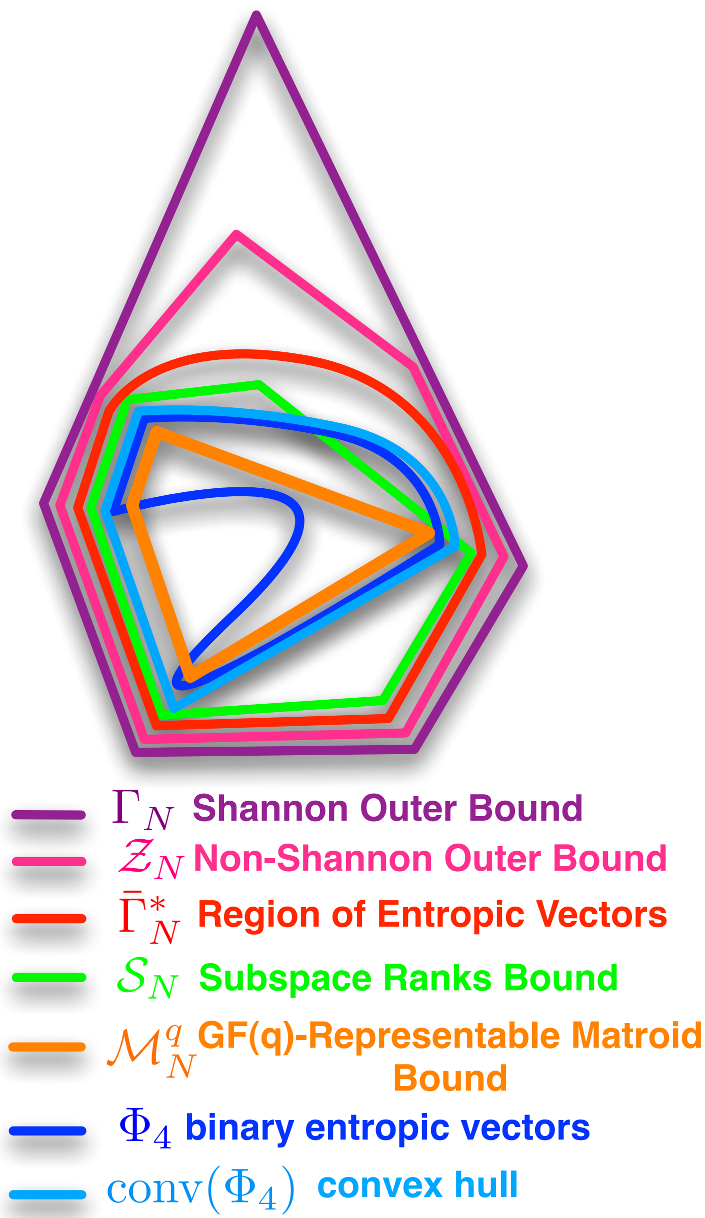
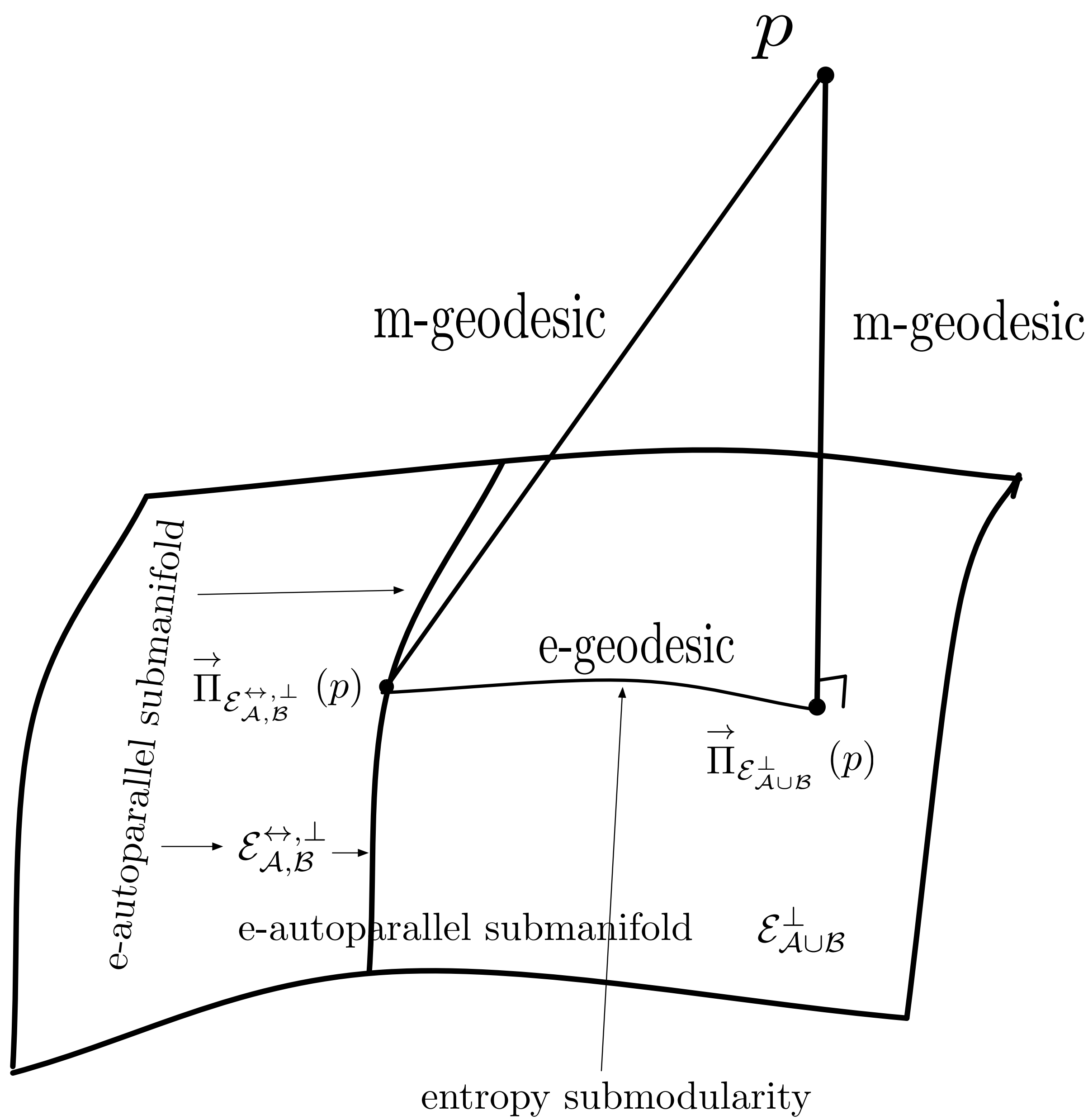
Extreme Entropy Characterization via Information Geometry[31]: Shannon Facets

- Intersection of $\bar{\Gamma}^*_N$ with a facet of Γ_N = those entropies for which a basic Shannon type inequality is tight.
- As basic inequalities are of the form $H(X_{\mathcal{A}}|X_{\mathcal{B}}) \geq 0$, $I(X_{\mathcal{A}};X_{\mathcal{B}}) \geq 0$, or $I(X_{\mathcal{A}};X_{\mathcal{B}}|X_{\mathcal{C}}) \geq 0$, tightness implies conditionally deterministic, independence, or conditional independence.
- Set of distributions satisfying one of these w/ equality has a nice information geometric structure (mutually dual foliation):
  - marginal distributions on implicated variables $(X_{\mathcal{A}}, X_{\mathcal{A}\cup\mathcal{B}}, X_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}})$ satisfying independence/conditional indep. $==$ *an e-affine submanifold*
  - set of distributions having these marginals *an m-affine submanifold*
- $\exists$ coordinate system for probabilities where Shannon facet (affine set intersection with *entropies*) is associated with an *affine sets of distributions*!
- *every Shannon type inequality can be viewed as positivity of divergence to a projection to an e-affine submanifold*!

Extreme Entropy Characterization via Information Geometry: Shannon Facets [32]





Extreme Entropy Characterization via Information Geometry: Non-Shannon Exposed Faces [32]

- recall $\mathcal{S}_4 = \Gamma_4 \cap \mathrm{Ingleton}$
- $\mathcal{S}_4 \subseteq \bar{\Gamma}_4^*$ and polars reverse containment, hence *every non-Shannon inequality can be expressed as*

$$\mathrm{Ingleton} + \sum_i \alpha_i I(X_{\mathcal{A}_i}; X_{\mathcal{B}_i} | X_{\mathcal{C}_i}) \geq 0 \tag{3}$$

($\leq$ 8 conditional entropies used for all known non-Shannon ineq.s) where, recall

$$\mathrm{Ingleton} = I(X_1; X_2) + I(X_3; X_4 | X_1) + I(X_3; X_4 | X_2) - I(X_3; X_4) \geq 0. \tag{4}$$

- hence every non-Shannon exposed face can be written as

$$I(X_3; X_4) = I(X_1; X_2) + I(X_3; X_4 | X_1) + I(X_3; X_4 | X_2) + \sum_i \alpha_i I(X_{\mathcal{A}_i}; X_{\mathcal{B}_i} | X_{\mathcal{C}_i})$$

this is equality of a divergence to a projection to a weighted sum of divergences to other projections

Summary

1. Big Project Idea: overhead vs. performance as rate distortion. resource controllers as distributed source codes.

2. One point on tradeoff (omniscient performance) and particular independence model assumptions to simplify

3. Fundamental limits expressed in terms of region of entropic vectors.

4. *Computation* of rate regions, inner and outer bounds for REV

5. information geometric properties of Shannon and non-Shannon exposed faces

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