Symmetry in Network Coding
Formalization, Graph-theoretic Characterization, and Computation

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ISIT, 2015
Outline

1. Motivation, Context, & Bigger Picture

2. Formalization via Groups
   - Group Action
   - Network Symmetry Group
   - Symmetries of Network Codes and Rate Regions

3. Graph Theoretic Characterization
   - Two Special Network Coding Problem Classes
   - Automorphism Groups
   - Characterization for k-Unicast Networks
   - Characterization for MSNC Networks
   - Computation

4. Applications
   - Computation of Polyhedral Bounds on Rate Regions
   - A generalization of partition-symmetrical entropy functions
General Network Coding Problem

\[ A[R_1, R_2, \ldots]^T \geq B[H(Y_1), H(Y_2), \ldots]^T \]
Rate Region

- Rate region: all possible rate and source entropy vectors satisfying all network constraints.
- Collect the $N$ network random variables and their joint entropies.
- Define $\Gamma^*_N$: 2$^N$ − 1-dim. cone, region of valid entropy vectors. (revisit later)
- Constraints from network $A$:
  \[ L_1 = \{ \mathbf{h} \in \Gamma^*_N : h_{Y_S} = \sum_{s \in S} h_{Y_s} \} \]  
  \[ L_2 = \{ \mathbf{h} \in \Gamma^*_N : h_{X_{Out(k)}|Y_s} = 0 \} \]  
  \[ L_3 = \{ \mathbf{h} \in \Gamma^*_N : h_{X_{Out(i)}|X_{In(i)}} = 0 \} \]  
  \[ L_4 = \{ (\mathbf{h}^T, \mathbf{R}^T)^T \in \mathbb{R}_+^{2N-1+|\mathcal{E}|} : R_e \geq h_{U_e}, e \in \mathcal{E} \} \]  
  \[ L_5 = \{ \mathbf{h} \in \Gamma^*_N : h_{Y_{\beta(t)}|U_{In(t)}} = 0 \} \].
- Rate region (cone) in terms of edge rates and source entropies (derived from [Yan, Yeung, Zhang TranIT 2012]):
  \[ \mathcal{R}_*(A) = \text{proj}_{R_{\mathcal{E}}, H(Y_S)} \left( \text{con}(\Gamma^*_N \cap L_{123}) \cap L_{45} \right) \]
A (3, 3) network and its rate region $R_*(A)$

Rate region: a cone with dimensions of all variables in the network

$$R_1 \geq H(Y_3)$$
$$R_2 \geq H(Y_1)$$
$$R_1 + R_2 \geq H(Y_1) + H(Y_2) + H(Y_3)$$
$$R_2 + R_3 \geq H(Y_1) + H(Y_3)$$
$$R_1 + R_2 + 2R_3 \geq H(Y_1) + H(Y_2) + 2H(Y_3)$$
Rate Regions through Algorithms & Software

MAJOR RECENT FOCUS OF ASPITRG:

\[
\begin{bmatrix}
H(X_1)

\vdots

H(X_K)

R_1

\vdots

R_{|E|}
\end{bmatrix} \geq 0
\]

\[
R_1 + R_2 + 2R_3 \geq 2H(X) + H(Y)
\]

\[
R_1 + 2R_2 + 2R_3 \geq 2H(X) + H(Y)
\]

\[
2R_1 + R_2 + R_3 \geq 2H(X) + H(Y)
\]

\[
3R_1 + 2R_2 + 2R_3 \geq 3H(X) + 2H(Y)
\]

EXAMPLE:


Listing Networks & Calculating Their Rate Regions

List of All Networks
up to a Certain Size

Our Software

All Their Rate Regions
Listing Networks & Calculating Their Rate Regions

List of All Networks up to a Certain Size

Our Software

All Their Rate Regions

group networks into equivalence classes (only differing in labels)
Listing Networks & Calculating Their Rate Regions

List of All Networks up to a Certain Size

group networks into equivalence classes (only differing in labels)

list only canonical representative of each equivalence class, (Li NetCod '15 & Trans IT sub)

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All Their Rate Regions

network symmetry group (today's talk) determines size of the equivalence class, & is constructed in enumeration
Listing Networks & Calculating Their Rate Regions

List of All Networks up to a Certain Size

network symmetry group can be exploited to reduce complexity of rate region calculation (Apte,Walsh NetCod'15)

Our Software

network symmetry group (today’s talk) determines size of the equivalence class, & is constructed in enumeration

All Their Rate Regions
Rate Region Database

Network Coding Rate Region Database

Database of ~7000 Rate Regions of >100k Networks (Li Trans IT sub 2014)
Database of ~635k Rate Regions of > 7M or 545B and bounds for ~1.8T more (Li Trans IT sub 2015)
Network Coding Rate Region Database

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What now?
Rate Region Database

Network Coding Rate Region Database

Database of ~7000 Rate Regions of >100k Networks (Li Trans IT sub 2014)
Database of ~635k Rate Regions of > 7M or 545B and bounds for ~1.8T more (Li Trans IT sub 2015)

What now? Submit 635,000 transactions papers?
Rate Region Database

Network Coding Rate Region Database

- Database of ~7000 Rate Regions of >100k Networks (Li Trans IT sub 2014)
- Database of ~635k Rate Regions of > 7M or 545B and bounds for ~1.8T more (Li Trans IT sub 2015)

What now? Analyze, learn, and explain something! Can't read and remember 635,000 network proofs.
Rate Region Database

Network Coding Rate Region Database

Database of ~7000 Rate Regions of >100k Networks (Li Trans IT sub 2014)
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(Li Trans IT sub 2015)

Investigate Structure through Network Hierarchy
- Minor-closed graphs: finite number of forbidden minors [RobertsonSeymour1983-2004]
- Rota’s conjecture in matroid theory: finite number of forbidden minors for $\mathbb{F}_q$ representability [Oxley2011]
- 3 Embedding Operators w/ Big $\rightarrow$ Small rate region expression [Li Allerton 2014, Li Trans IT sub 2014]
- Forbidden embedded networks for $\mathbb{F}_q$ codes sufficiency.
Example: Forbidden embedded networks

- Goal: minimal forbidden networks for sufficiency
- Scalar binary codes considered
- \( k = 1, 2, 3; |\mathcal{E}| = 2, 3, 4, 7360 \) non-isomorphic MDCS
- 1922 sufficient, 5438 insufficient
- 12 minimal forbidden minors [Li, et. al TransIT sub 2014]

5438 / 7360 insufficient

1922 / 7360 sufficient

12 forbidden embedded networks
Hierarchy: Combination Operators

- Combination operators [Li ITW sub 2015, Li Trans IT sub 2015]
- Rules for combining smaller networks into large ones.
- Rate region of large expressed in terms of those of small.
Hierarchy: Combination Operators: Ex. [Li ITW sub 2015]
Use operators together to get RR for big networks. Partial Network Closure.
Hierarchy: Both Embedding & Combination Ops [Li Trans IT sub 2015]

Start with the single (1, 1), single (2, 1), and the four (1, 2) networks; These 6 tiny networks can generate new 11635 networks w/ small cap!

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With the increase of cap size, number of new networks increases!

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Embedding operations are important in the process!

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Listing Networks & Calculating Their Rate Regions

List of All Networks up to a Certain Size

group networks into equivalence classes (only differing in labels)
list only canonical representative of each equivalence class
(Li NetCod '15 & Trans IT sub)

Our Software

All Their Rate Regions

network symmetry group
(today's talk) determines size of the equivalence class, & is constructed in enumeration
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   - Characterization for MSNC Networks
   - Computation

4 Applications
   - Computation of Polyhedral Bounds on Rate Regions
   - A generalization of partition-symmetrical entropy functions
Group Action

Action of a finite group on a set

Definition

The (left) group action of a group $G \leq S_n$ on set $S$ is the map

$$\phi : G \times S \rightarrow S : (g, x) \mapsto \phi(g, x)$$

that satisfies:

1. **G1** $\phi((g \circ h), x) = \phi(g, \phi(h, x)), \forall g, h \in G, x \in S$
2. **G2** $\phi(e, x) = x, \forall x \in S$

$a\leq$ denotes subgroups, $S_n$ denotes symmetric group

- Denote $\phi(g, x)$ as $x^g$ and, $\phi(g, X)$ as $X^g$ for $x \in S$, and $X \subseteq S$
- For $X \subseteq S$, an element $g \in G$ is said to stabilize $X$ setwise if $X^g = X$
- The collection of all group elements $g \in G$ that setwise stabilize a subset $X \subseteq S$ forms a group called stabilizer subgroup, denoted as $G_X$. 

Jayant Apte, John Walsh (Drexel University) Symmetry in Network Coding

ISIT, 2015 27 / 55
Group Action on Network Constraints

- Recall network constraints $\mathcal{L}_{12345} = \mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_3 \cap \mathcal{L}_4 \cap \mathcal{L}_5$
- Let $[n]$ be the set of subscripts of RVs in $\mathcal{X}_n$ and $\hat{\mathcal{L}}$ be the set of sets of constraints arising from different network coding problems
- The natural action of $G \leq S_n$ on $[n]$ induces following sequence of actions: $\operatorname{Act}([n]) \xrightarrow{\text{induced}} \operatorname{Act}(\mathcal{X}_n) \xrightarrow{\text{induced}} \operatorname{Act}(2^{\mathcal{X}_n}) \xrightarrow{\text{induced}} \operatorname{Act}(\hat{\mathcal{L}})$

**Example**

Let $g = (1, 2)(3, 4)(7, 8) \in G \leq S_8$ (natural action on $[8]$ and $\mathcal{X}_8$)

$\{1, 3\} \mapsto \{2, 4\}, \{1, 3, 4, 8\} \mapsto \{2, 4, 3, 7\}$ (induced action on $2^{\mathcal{X}_8}$)

$\left\{ \begin{array}{l}
\{h_{\{1,3\}} = h_{\{1,3,4,8\}}\} \mapsto \{h_{\{2,4\}} = h_{\{2,4,3,7\}}\} \\
\{h_8 \leq R_8\} \mapsto \{h_7 \leq R_7\}
\end{array} \right\}$ (induced action on $\hat{\mathcal{L}}$)
Rate Region

- Rate region: all possible rate and source entropy vectors satisfying all network constraints.
- Collect the $N$ network random variables and their joint entropies.
- Define $\Gamma_N^*$: $2^N - 1$-dim. cone, region of valid entropy vectors. (revisit later)
- Constraints from network $A$:
  \begin{align*}
  \mathcal{L}_1 &= \{ \mathbf{h} \in \Gamma_N^* : h_{Y_S} = \sum_{s \in \mathcal{S}} h_{Y_s} \} \\
  \mathcal{L}_2 &= \{ \mathbf{h} \in \Gamma_N^* : h_{X_{\text{Out}(k)|Y_s}} = 0 \} \\
  \mathcal{L}_3 &= \{ \mathbf{h} \in \Gamma_N^* : h_{X_{\text{Out}(i)|X_{\text{In}(i)}} = 0 \}
  \\
  \mathcal{L}_4 &= \{ (\mathbf{h}^T, \mathbf{R}^T)^T \in \mathbb{R}^{2^N-1+|\mathcal{E}|} : R_e \geq h_{U_e}, e \in \mathcal{E} \} \\
  \mathcal{L}_5 &= \{ \mathbf{h} \in \Gamma_N^* : h_{Y_{\beta(t)|U_{\text{In}(t)}}} = 0 \}.
  \end{align*}

- Rate region (cone) in terms of edge rates and source entropies (derived from [Yan, Yeung, Zhang TranIT 2012]):
  \[
  \mathcal{R}_*(A) = \text{proj}_{\mathcal{R}^N_{\mathcal{E},H(Y_S)}}(\text{con}(\Gamma_N^* \cap \mathcal{L}_{123}) \cap \mathcal{L}_{45})
  \]
Recall network constraints $\mathcal{L}_{12345} = \mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_3 \cap \mathcal{L}_4 \cap \mathcal{L}_5$

Let $[n]$ be the set of subscripts of RVs in $\mathcal{X}_n$ and $\hat{\mathcal{L}}$ be the set of sets of constraints arising from different network coding problems.

The natural action of $G \leq S_n$ on $[n]$ induces following sequence of actions: $\text{Act } ([n]) \xrightarrow{\text{induced}} \text{Act}(\mathcal{X}_n) \xrightarrow{\text{induced}} \text{Act}(2^{\mathcal{X}_n}) \xrightarrow{\text{induced}} \text{Act}(\hat{\mathcal{L}})$

Example

Let $g = (1, 2)(3, 4)(7, 8) \in G \leq S_8$ (natural action on $[8]$ and $\mathcal{X}_8$)

$\{1, 3\} \mapsto \{2, 4\}$, $\{1, 3, 4, 8\} \mapsto \{2, 4, 3, 7\}$ (induced action on $2^{\mathcal{X}_8}$)

\[
\begin{align*}
\{h_{\{1,3\}} = h_{\{1,3,4,8\}}\} &\mapsto \{h_{\{2,4\}} = h_{\{2,4,3,7\}}\} \\
\{h_8 \leq R_8\} &\mapsto \{h_7 \leq R_7\}
\end{align*}
\]

(induced action on $\hat{\mathcal{L}}$)
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The network symmetry group $G^\mathcal{I}$ of a MSNC instance $\mathcal{I} = (\mathcal{G} = (V, E), S, T, \beta)$ is the subgroup of $S_n$, $n = |S| + |E|$, that stabilizes $\mathcal{L}_{12345}$ setwise under its induced action on $\hat{\mathcal{L}}$. 
Network Symmetry Group

Example

$\omega_2$ $x_2$ $2$ $R_5$ $R_4$ $1$ $x_1$ $\omega_1$

$R_3$ $x_3$ $3$ $x_4$

$R_6$ $x_6$ $R_7$

$x_7$

$R_9$ $x_9$ $4$ $x_8$ $R_8$

$R_9$

$x_1$ $x_2$

$L_1$ source independence

$h_1 + h_2 = h_{\{1,2\}} \quad \Rightarrow \quad h_1 + h_2 = h_{\{1,2\}}$

$h_1 = h_{\{1,4,6\}}$

$h_2 = h_{\{2,5,7\}}$

$h_{\{4,5\}} = h_{\{3,4,5\}}$

$h_3 = h_{\{3,8,9\}}$

$h_{\{6,8\}} = h_{\{2,6,8\}}$

$h_{\{7,9\}} = h_{\{1,7,9\}}$

$h_{\{6,8\}} = h_{\{2,6,8\}}$

$L_2$ node constraints

$h_1 \geq \omega_1$

$h_2 \geq \omega_2$

$h_3 \leq R_3$

$h_4 \leq R_4$

$h_5 \leq R_5$

$h_6 \leq R_6$

$h_7 \leq R_7$

$h_8 \leq R_8$

$h_9 \leq R_9$

$L_3$ rate constraints

$h_{\{1,2\}}(3)(4,5)(6,7)(8,9)$
Network Symmetry Group

Example

Network Symmetry Group of Butterfly Network is of order 2 with \((1,2)(3)(4,5)(6,7)(8,9)\) being the only non-trivial symmetry.
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Symmetries of Network Codes

Definition

A network code \((\{f_e\}, \{g_t\})\) for a MSNC instance \(I\) is an assignment of a function \(f_e\) to each edge \(e \in E\) and a function \(g_t\) to each sink \(t \in T\).
Definition

A network code \( (\{f_e\}, \{g_t\}) \) for a MSNC instance \( \mathcal{I} \) is an assignment of a function \( f_e \) to each edge \( e \in E \) and a function \( g_t \) to each sink \( t \in T \).

Theorem

Let \( G^\mathcal{I} \) be the NSG associated with MSNC instance \( \mathcal{I} \). Then, for any \( g \in G^\mathcal{I} \), if a network code \( (\{f_i\}, \{g_t\}) \) satisfies \( \mathcal{L}_{12345} \), so does \( (\{f_{ig}\}, \{g_{tg}\}) \) for every \( g \in G^\mathcal{I} \).
Theorem

If $\omega, r$ is an achievable source information rates and edge rates vector pair for a MSNC instance $I$, so are $[\omega_s \mid s \in S]$, $[R_e \mid e \in E]$ for every $g \in G^I$. 

Symmetries of Rate Regions
Symmetries of Rate Regions

Theorem
If \( \omega, r \) is an achievable source information rates and edge rates vector pair for a MSNC instance \( I \), so are \([\omega_{sg} | s \in S], [R_{eg} | e \in E]\) for every \( g \in G^I \).

Example
\((\omega_1, \omega_2)\) is achievable rate tuple iff 
\((\omega_2, \omega_1)\) is achievable rate tuple

(bpnu: bits per network use)
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Two Special Problems

MSNC

- Multisource Network Coding
- Described by tuple \((\mathcal{G}, S, \mathcal{T}, \beta)\)

\[ \beta : \mathcal{T} \rightarrow 2^S \setminus \emptyset \]

- \( \beta : \mathcal{T} \rightarrow 2^S \setminus \emptyset \) allows arbitrary subset demands
Two Special Problems

MSNC

- Multisource Network Coding
- Described by tuple \((G, S, T, \beta)\)

\[ G = (V, E) \]
\[ \beta : T \rightarrow 2^S \setminus \emptyset \]
\[ \beta(t_1) \subseteq S \]
\[ \beta(t_2) \subseteq S \]
\[ \beta(t_{|T|}) \subseteq S \]

- \( \beta : T \rightarrow 2^S \setminus \emptyset \) allows arbitrary subset demands

k-UNC

- k-Unicast Network Coding
- Described by tuple \((G, S, T, \beta)\)

\[ G = (V, E) \]
\[ \beta : T \rightarrow S \]
\[ \beta(t_1) \in S \]
\[ \beta(t_2) \in S \]
\[ \ldots \]
\[ \beta(t_k) \in S \]

- \( \beta : T \rightarrow S \) is a bijection
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   - Two Special Network Coding Problem Classes
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   - Computation of Polyhedral Bounds on Rate Regions
   - A generalization of partition-symmetrical entropy functions
Several combinatorial objects can be described as collections of subsets of some set eg. graphs, digraphs, linear codes and so on...

Automorphism group of a combinatorial object is the group containing all permutations that keep the collection of subsets invariant
Several combinatorial objects can be described as collections of subsets of some set eg. graphs, digraphs, linear codes and so on...

Automorphism group of a combinatorial object is the group containing all permutations that keep the collection of subsets invariant.

Example

V=S, E=\{\{1,2\},\{2,3\},\{1,3\}\}

S=\{1,2,3\}
V is S,
E is a collection of 2 subsets of V

Aut(G) contains all 3! permutations of \{1,2,3\}

V=S, E=\{(1,2),(1,3)\}

S=\{1,2,3\}
V is S,
E is a subset of V \times V

Aut(G) contains all identity and (2,3)
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Construction

1. Start with the underlying DAG $\mathcal{G} = (V, E)$
2. Add a feedback edges $F = \{(t, \beta(t)) \mid t \in T\}$ to create circulation graph $\mathcal{G}_c = (V, E \cup F_c)$
3. Take line graph: vertices are edges of $\mathcal{G}_c$, edges are length 2 directed paths in $\mathcal{G}_c$ to create dual circulation graph $\mathcal{G}_c^*$
4. Associate random variables with vertices of $\mathcal{G}_c^*$
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Theorem

The network symmetry group of a k-UNC instance is $G^\mathcal{T} = G_{F_c} \cap \text{Aut}(\mathcal{G}_c^*)$, the group of permutations of $\chi_n$ induced by the subgroup of automorphism group of its dual circulation graph that setwise stabilizes the feedback edge set.
symmetries are the subgroup of that setwise stabilizes the feedback edges \{(6,1),(5,2)\}

\[\text{Problem Symmetries} \]

symmetries are the subgroup of \(\text{Aut}(G_c^*)\) that setwise stabilizes the feedback edges \{(6,1),(5,2)\}
NSG via Graphs for k-Unicast

Example

A k-UNC instance

\[ \begin{align*}
X_2 & \rightarrow X_3 \\
X_1 & \rightarrow X_4 \\
X_7 & \rightarrow X_8 \\
X_9 & \rightarrow X_6
\end{align*} \]

line graph of circulation graph \( G_c^* \)

\[ \begin{align*}
X_2 & \rightarrow X_3 \\
X_1 & \rightarrow X_4 \\
X_7 & \rightarrow X_8 \\
X_9 & \rightarrow X_6
\end{align*} \]

\( G^I = \{(1, (1, 2)(3)(4, 5)(6, 7)(8, 9)\} \)

network symmetry group \( G^I \)

contains identity and

\[ \begin{align*}
(1, 2)(3)(4, 5)(6, 7)(8, 9)
\end{align*} \]
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A slightly more complicated characterization

1. Start with underlying DAG $\mathcal{G} = (V, E)$
2. Add new vertices $S'$, edges $E_S$ and $F_c$ to create circulation graph $\mathcal{G}_c$
3. Take line graph to create dual circulation graph $\mathcal{G}_c^*$
4. Compute $\text{AUT}(\mathcal{G}_c^*)$
5. Obtain network symmetry group as an induced group
A slightly more complicated characterization

1. Start with underlying DAG $G = (V, E)$
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4. Compute $\text{AUT}(G_c^*)$
5. Obtain network symmetry group as an induced group

Theorem

The network symmetry group of a MSNC instance is the group of permutations induced onto $E \cup E_S$ by the subgroup of $\text{Aut}(G_c^*)$ that stabilizes $E_S$ setwise.
NSG via Graphs for MSNC Networks

Example

An MSNC instance

Circulation Graph $G_c$

$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4$

$1 \rightarrow 2 \rightarrow 3 \rightarrow 4$

$5 \rightarrow 6$

$7 \rightarrow 8 \rightarrow 9$

$X_1X_3 \rightarrow X_1X_4 \rightarrow X_2X_3 \rightarrow X_2X_4$

$1 \rightarrow 2 \rightarrow 3 \rightarrow 4$

$1' \rightarrow 2' \rightarrow 3' \rightarrow 4'$

$5 \rightarrow 6$

$7 \rightarrow 8 \rightarrow 9$

$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4$

$X_5 \rightarrow X_6 \rightarrow X_7 \rightarrow X_8$

$X_9 \rightarrow X_{10} \rightarrow X_{11} \rightarrow X_{12}$

$X_{13} \rightarrow X_{14} \rightarrow X_{15} \rightarrow X_{16}$

$X_{17} \rightarrow X_{18} \rightarrow X_{19} \rightarrow X_{20}$

$X_{21} \rightarrow X_{22} \rightarrow X_{23} \rightarrow X_{24}$

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The Network Symmetry group is generated by:

\[ g_1 = (1, 3)(2, 4)(5, 7)(6, 8)(9, 12)(10, 11)(13, 21)(14, 23)(15, 22)(16, 24)(18, 19) \]

\[ g_2 = (3, 4)(7, 8)(13, 14)(15, 16)(17, 18)(19, 20)(21, 22)(23, 24) \]

\[ g_3 = (1, 2)(5, 6)(13, 15)(14, 16)(17, 19)(18, 20)(21, 23)(22, 24) \]

A group of order 8
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Computing NSGs

- One can use off-the-shelf (di)graph automorphism group computation software for computing NSGs
- **nauty**: No Automorphisms, Yes? implements Brendan McKay’s canonical labeling algorithm
- We used SageMath’s implementation of the same algorithm
- Can also be enumerated along with canonical networks [Li, Walsh, Weber, NetCod 2015, Trans IT sub 2015]
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Let $\mathcal{H}'_n \triangleq \mathbb{R}^{2^n-1+n}$

$\text{Act } ([n]) \xrightarrow{\text{induced}} \text{Act}(\mathcal{X}_n) \xrightarrow{\text{induced}} \text{Act}(2\mathcal{X}_n) \xrightarrow{\text{induced}} \text{Act}(\mathcal{H}'_n)$
Let $\mathcal{H}'_n \triangleq \mathbb{R}^{2^n - 1 + n}$

Act $([n]) \xrightarrow{\text{induced}} \text{Act}(\mathcal{X}_n) \xrightarrow{\text{induced}} \text{Act}(2^{\mathcal{X}_n}) \xrightarrow{\text{induced}} \text{Act}(\mathcal{H}'_n)$

Let $\mathcal{L}_{12345} \bigcap$ be the intersection of hyperplanes and halfspaces in $\mathcal{H}'_n$ associated with network constraints

$\Gamma_n \cap \mathcal{L}_{12345}$ is stabilized setwise under action of NSG on $\mathcal{H}'_n$

NSG gives polyhedral symmetry group of $\Gamma_n \cap \mathcal{L}_{12345}$

NSG gives polyhedral symmetry group of $\text{proj}_{\omega, r}(\Gamma_n \cap \mathcal{L}_{12345})$
Computation of Polyhedral Bounds on Rate Regions

- Let $\mathcal{H}_n' \triangleq \mathbb{R}^{2^n−1+n}$
- $\text{Act} \left([n]\right) \xrightarrow{\text{induced}} \text{Act}(\mathcal{X}_n) \xrightarrow{\text{induced}} \text{Act}(2^{\mathcal{X}_n}) \xrightarrow{\text{induced}} \text{Act}(\mathcal{H}_n')$
- Let $\mathcal{L}_{12345}$ be the intersection of hyperplanes and halfspaces in $\mathcal{H}_n'$ associated with network constraints
- $\Gamma_n \cap \mathcal{L}_{12345}$ is stabilized setwise under action of NSG on $\mathcal{H}_n'$
- NSG gives polyhedral symmetry group of $\Gamma_n \cap \mathcal{L}_{12345}$
- NSG gives polyhedral symmetry group of $\text{proj}_{\omega,r}(\Gamma_n \cap \mathcal{L}_{12345})$
- Extend to any outer/inner polyhedral bound $\Gamma_n^k$, $k \in \{\text{in},\text{out}\}$ provided that $\Gamma_n^k$ is stabilised setwise under induced action of $S_n$ on $\mathcal{H}_n'$
- e.g. ZY, DFZ non-Shannon outer bounds, matroid inner bounds, subspace inner bounds
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A generalization of partition-symmetrical entropy functions

- Let $\text{Fix}_G(S)$ for $S \subseteq \mathcal{H}'_n$ be the subset of $S$ fixed under the induced action of $G$ on $\mathcal{H}'_n$
- Let $p$ be the partition of $\mathcal{X}_n$ arising from orbits of $G^I$, $\Sigma_p$ be the partition respecting subgroup of $S_n$
A generalization of partition-symmetrical entropy functions

- Let \( \text{Fix}_G(S) \) for \( S \subseteq \mathcal{H}'_n \) be the subset of \( S \) fixed under the induced action of \( G \) on \( \mathcal{H}'_n \)
- Let \( p \) be the partition of \( \mathcal{X}_n \) arising from orbits of \( G^\mathcal{I} \), \( \Sigma_p \) be the partition respecting subgroup of \( S_n \)
- Qi Chen and R. W. Yeung [arXiv] defined following regions:

\[
\Psi_p = \text{Fix}_{\Sigma_p}(\Gamma_n), \quad \Psi_p^* = \text{Fix}_{\Sigma_p}(\Gamma_n^*)
\]

(13)

- For a MSNC instance \( \mathcal{I} \), one can also define:

\[
\Psi_{\mathcal{I}} = \text{Fix}_{G^\mathcal{I}}(\Gamma_n), \quad \Psi_{\mathcal{I}}^* = \text{Fix}_{G^\mathcal{I}}(\Gamma_n^*)
\]

(14)

Theorem

For a MSNC instance \( \mathcal{I} \) with NSG \( G^\mathcal{I} \) and associated orbits in \( \mathcal{X}_n \) denoted as \( p \), \( \Psi_p \subseteq \Psi_{\mathcal{I}} \) and \( \Psi_p^* \subseteq \Psi_{\mathcal{I}}^* \)
Conclusions

- Defined a network symmetry group
- Showed that it gives symmetries of rate regions & codes
- Showed how to calculate it with graph automorphism based tools
- Mentioned applications in
  - enumeration [Li Walsh Weber Trans IT Sub 2015],
  - rate region calculation [Apte Walsh NetCod’15], and
  - partition-symmetrical entropy functions
- Provided context in computational network coding research