Symmetry in Network Coding

Formalization, Graph-theoretic Characterization, and Computation

Jayant Apte John Walsh

Department of Electrical and Computer Engineering
Drexel University, Philadelphia

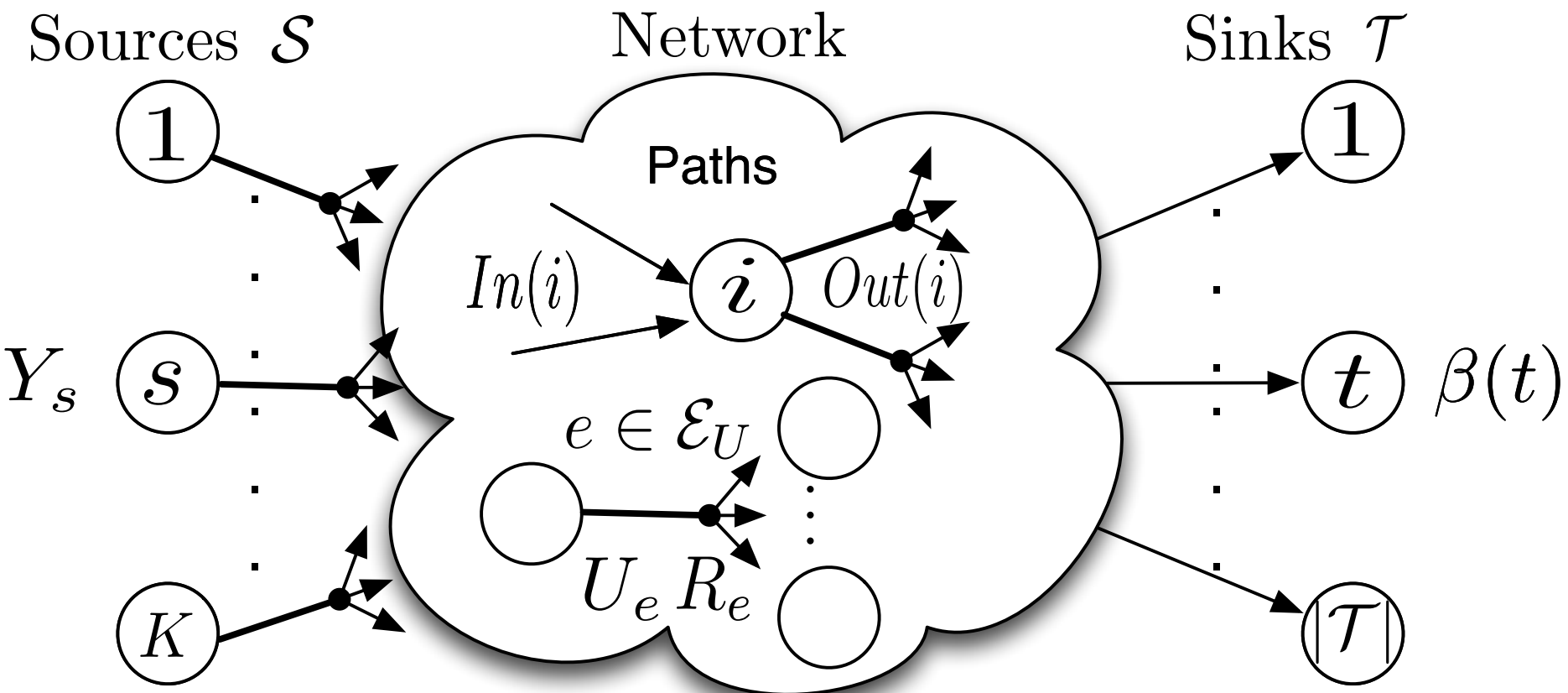
ISIT, 2015

Outline

1. Motivation, Context, & Bigger Picture
2. Formalization via Groups
   - Group Action
   - Network Symmetry Group
   - Symmetries of Network Codes and Rate Regions
3. Graph Theoretic Characterization
   - Two Special Network Coding Problem Classes
   - Automorphism Groups
   - Characterization for k-Unicast Networks
   - Characterization for MSNC Networks
   - Computation
4. Applications
   - Computation of Polyhedral Bounds on Rate Regions
   - A generalization of partition-symmetrical entropy functions

Drexel UNIVERSITY

General Network Coding Problem



$$\mathbb{A}[R_1, R_2, \ldots]^T \geq \mathbb{B}[H(Y_1), H(Y_2), \ldots]^T$$

Rate Region

- Rate region: all possible rate and source entropy vectors satisfying all network constraints.
- Collect the N network random variables and their joint entropies.
- Define Γ_N^*: $2^N - 1$-dim. cone, region of valid entropy vectors. (revisit later)
- Constraints from network A:

$$\mathcal{L}_1 = \{\mathbf{h} \in \Gamma_N^* : h_{Y_S} = \Sigma_{s \in \mathcal{S}} h_{Y_s}\} \quad (1)$$

$$\mathcal{L}_2 = \{\mathbf{h} \in \Gamma_N^* : h_{X_{\mathrm{Out}(k)}|Y_s} = 0\} \quad (2)$$

$$\mathcal{L}_3 = \{\mathbf{h} \in \Gamma_N^* : h_{X_{\mathrm{Out}(i)}|X_{\mathrm{In}(i)}} = 0\} \quad (3)$$

$$\mathcal{L}_4 = \{(\mathbf{h}^T, \mathbf{R}^T)^T \in \mathbb{R}_+^{2^N-1+|\mathcal{E}|} : R_e \geq h_{U_e}, e \in \mathcal{E}\} \quad (4)$$

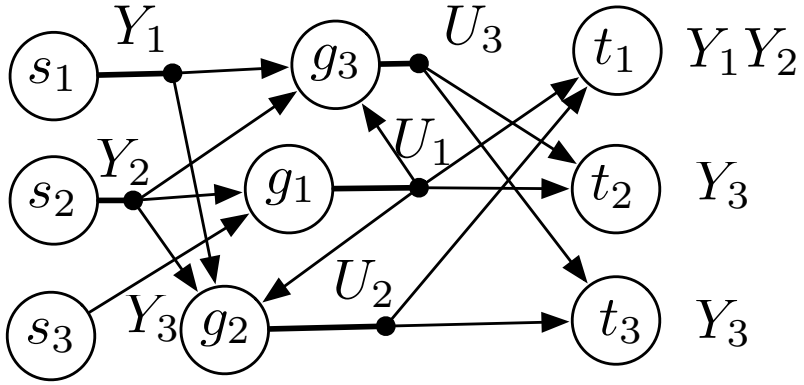
$$\mathcal{L}_5 = \{\mathbf{h} \in \Gamma_N^* : h_{Y_{\beta(t)}|U_{\mathrm{In}(t)}} = 0\}. \quad (5)$$

- Rate region (cone) in terms of edge rates and source entropies (derived from [Yan, Yeung, Zhang TranIT 2012]):

$$\mathcal{R}_*(\mathrm{A}) = \mathrm{proj}_{R_\mathcal{E}, H(Y_\mathcal{S})}(\overline{\mathrm{con}(\Gamma_\mathrm{N}^* \cap \mathcal{L}_{123})} \cap \mathcal{L}_{45}) \quad (6)$$

Rate Region Example

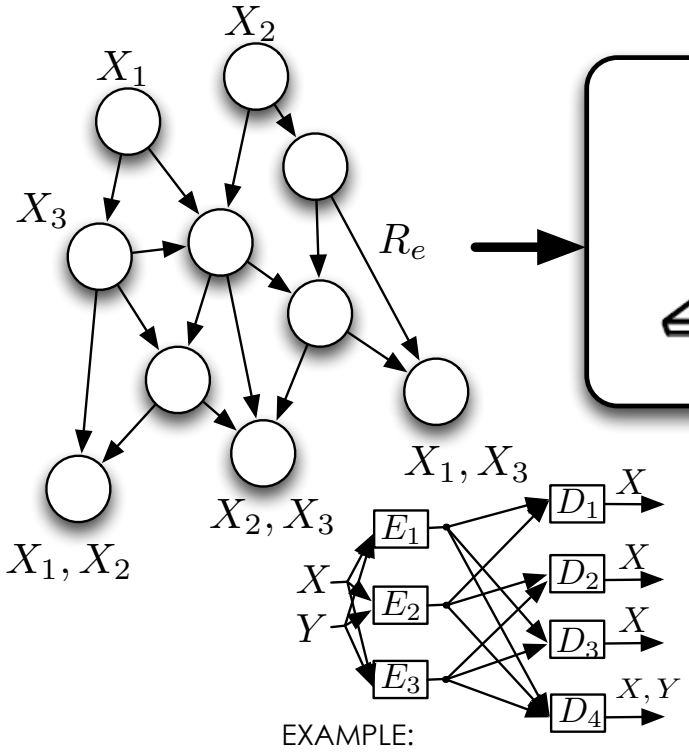
- A $(3, 3)$ network and its rate region $\mathcal{R}_*(\mathrm{A})$
- Rate region: a cone with dimensions of all variables in the network



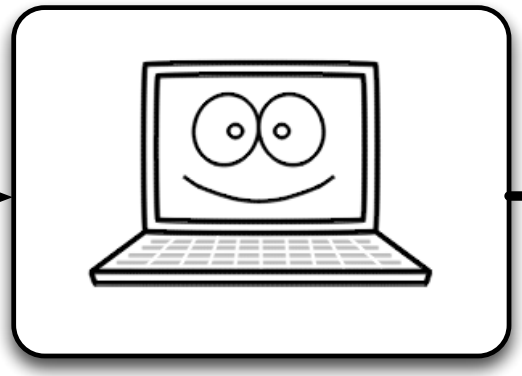
$$
\begin{aligned}
R_1 &\geq H(Y_3) \\
R_2 &\geq H(Y_1) \\
R_1 + R_2 &\geq H(Y_1) + H(Y_2) + H(Y_3) \\
R_2 + R_3 &\geq H(Y_1) + H(Y_3) \\
R_1 + R_2 + 2R_3 &\geq H(Y_1) + H(Y_2) + 2H(Y_3)
\end{aligned}
$$

Rate Regions through Algorithms & Software

NETWORK CODING PROBLEMS



MAJOR RECENT FOCUS OF ASPITRG:



CAPACITY REGION & Codes that achieve it

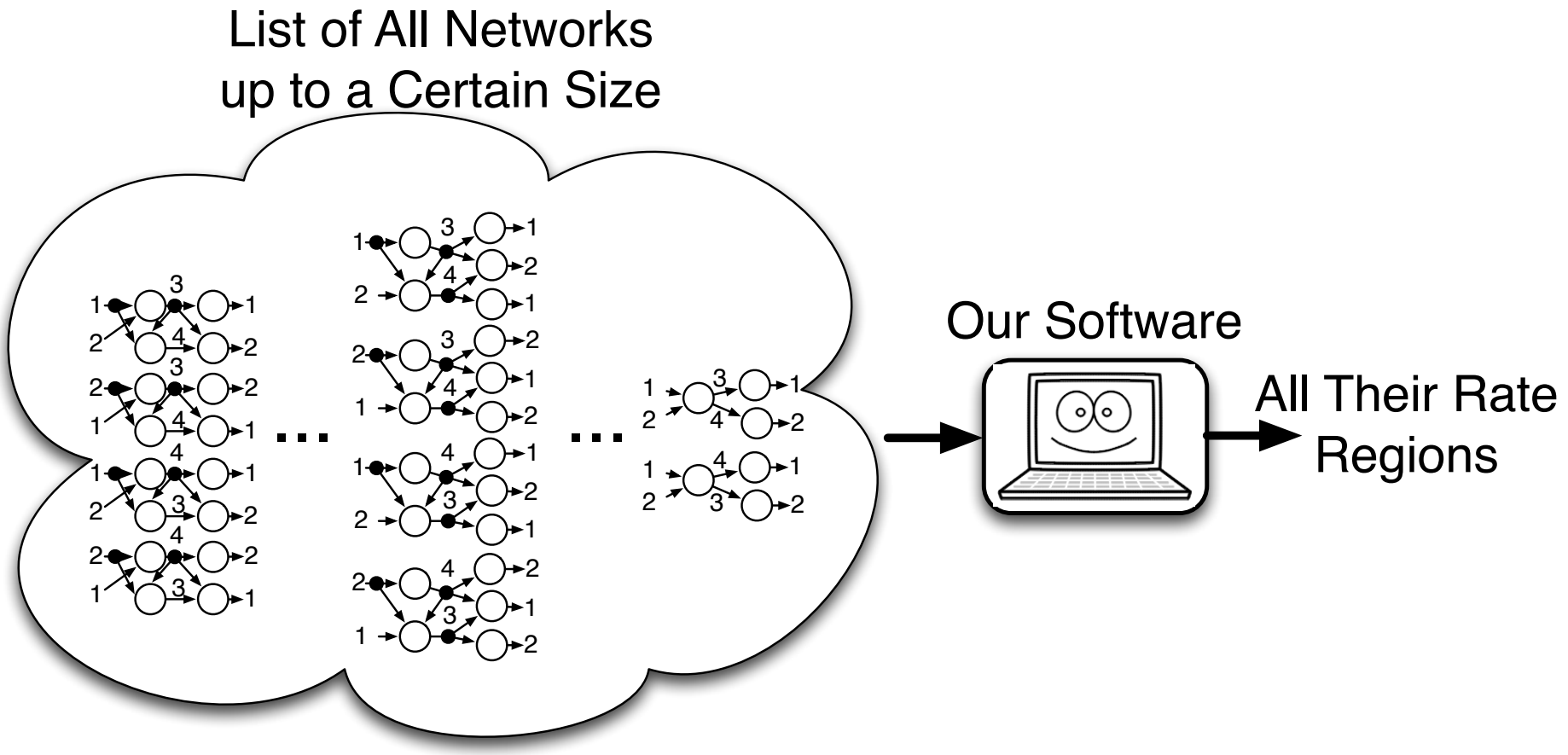
$$\mathbb{A} \begin{bmatrix} H(X_1) \\ \vdots \\ H(X_K) \\ R_1 \\ \vdots \\ R_{|\mathcal{E}|} \end{bmatrix} \geq \mathbf{0}$$

$$\begin{aligned}
R_1 + R_2 &\geq H(X) \\
R_1 + R_3 &\geq H(X) \\
R_2 + R_3 &\geq H(X) \\
R_1 + R_2 + 2R_3 &\geq 2H(X) + H(Y) \\
R_1 + 2R_2 + 2R_3 &\geq 2H(X) + H(Y) \\
2R_1 + R_2 + R_3 &\geq 2H(X) + H(Y) \\
3R_1 + 2R_2 + 2R_3 &\geq 3H(X) + 2H(Y)
\end{aligned}$$

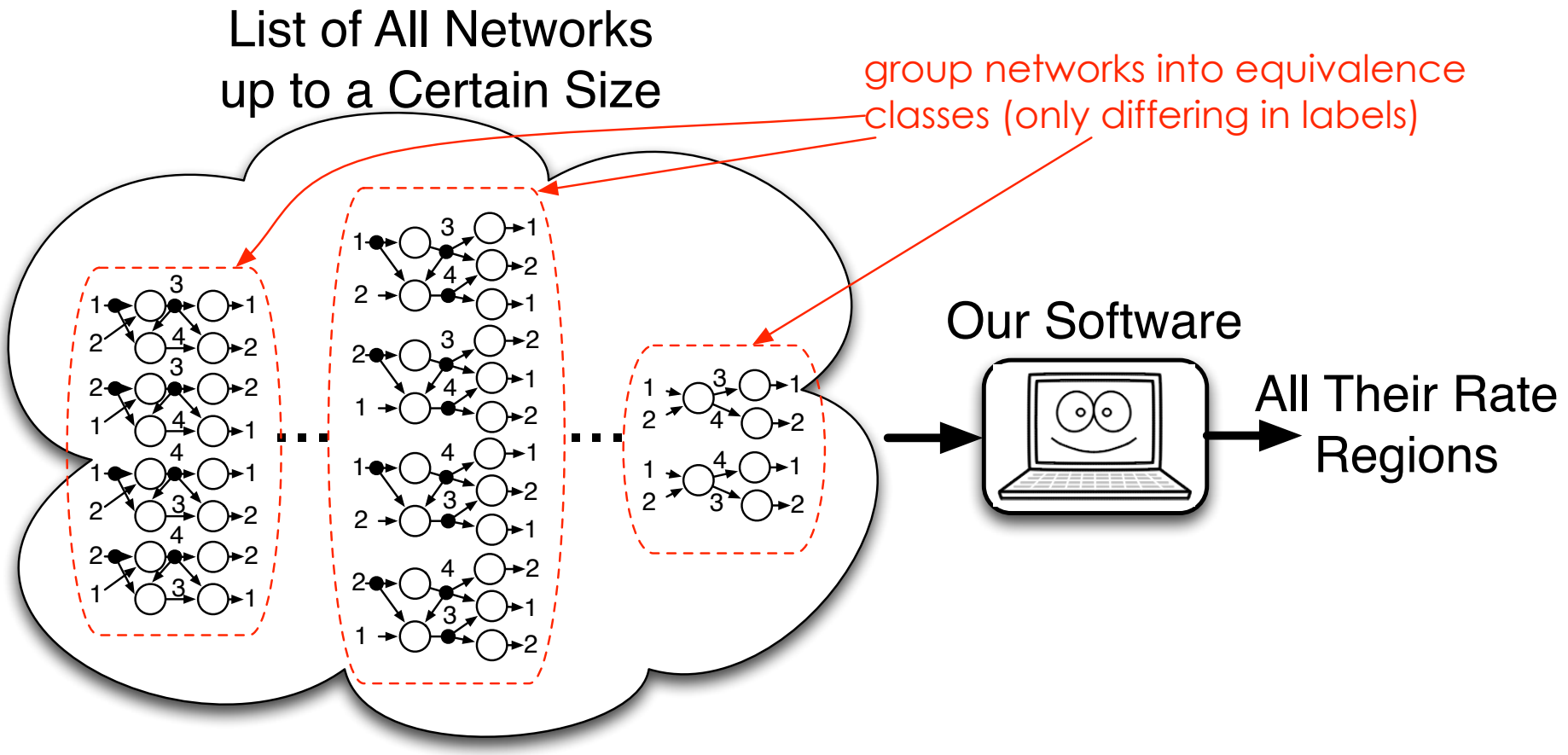
Jayant Apte, Congduan Li, and John MacLaren Walsh, "Algorithms for Computing Network Coding Rate Regions via Single Element Extensions of Matroids," in *IEEE International Symposium on Information Theory (ISIT 2014)*, Jul. 2014, pp. 2306–2310. [Online]. Available: http://www.ece.drexel.edu/walsh/Apte_ISIT_2014.pdf

Congduan Li, Jayant Apte, John MacLaren Walsh, Steven Weber, "A new computational approach for determining rate regions and optimal codes for coded networks," in *The 2013 IEEE International Symposium on Network Coding (NetCod 2013)*, Jun. 2013. [Online]. Available: http://www.ece.drexel.edu/walsh/LiNetCod13.pdf

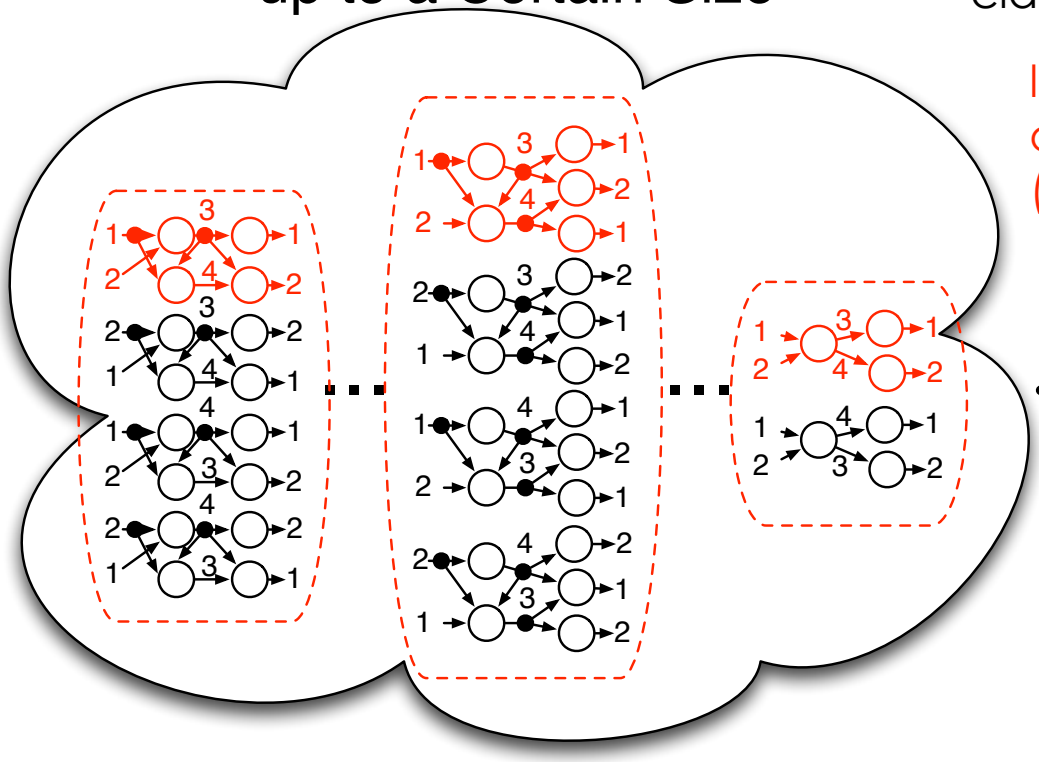
Listing Networks & Calculating Their Rate Regions



Listing Networks & Calculating Their Rate Regions



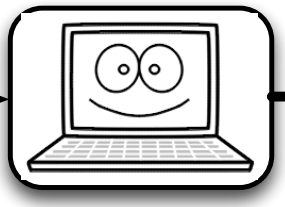
Listing Networks & Calculating Their Rate Regions



group networks into equivalence classes (only differing in labels)

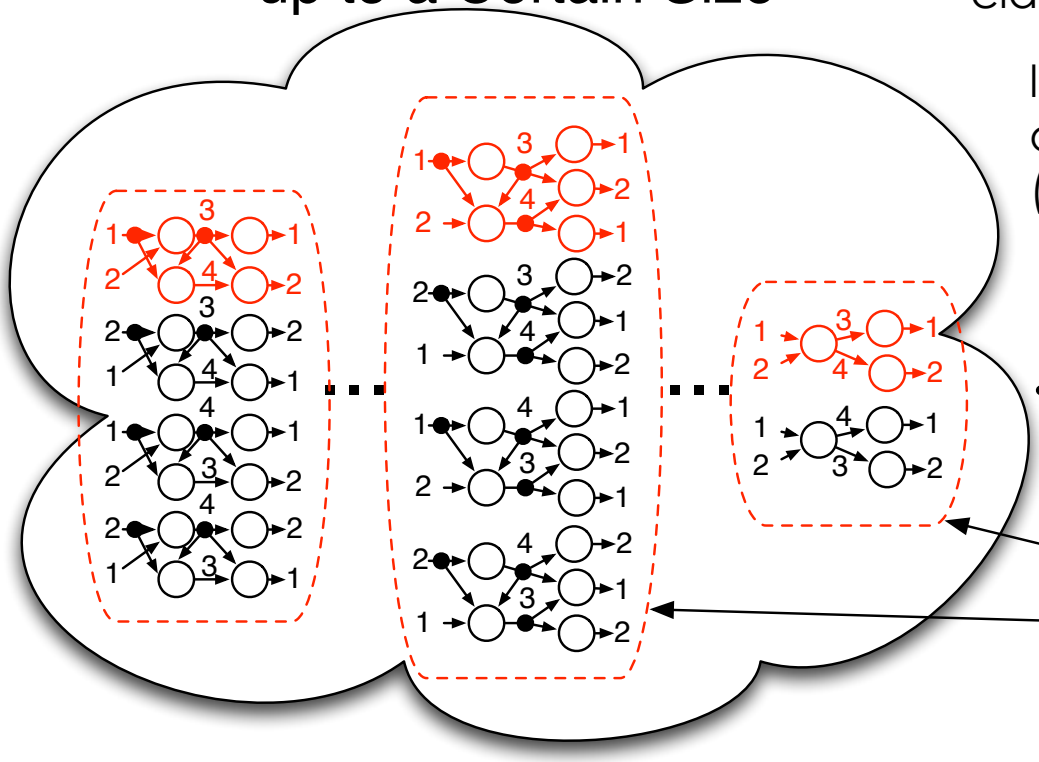
list only canonical representative of each equivalence class, (Li NetCod '15 & Trans IT sub)

Our Software



All Their Rate Regions

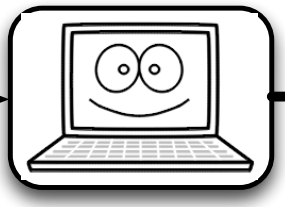
Listing Networks & Calculating Their Rate Regions



group networks into equivalence classes (only differing in labels)

list only canonical representative of each equivalence class (Li NetCod '15 & Trans IT sub)

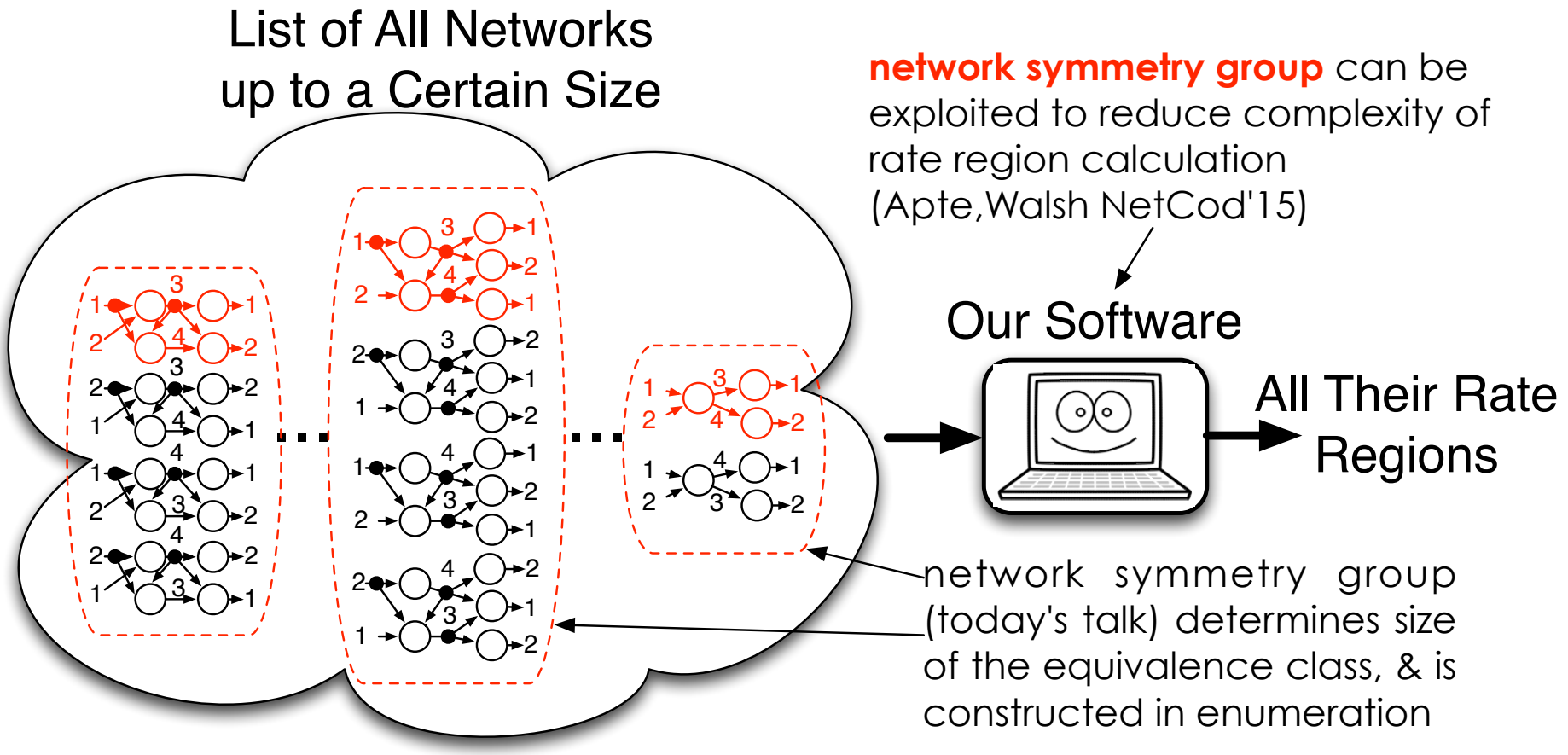
Our Software



All Their Rate Regions

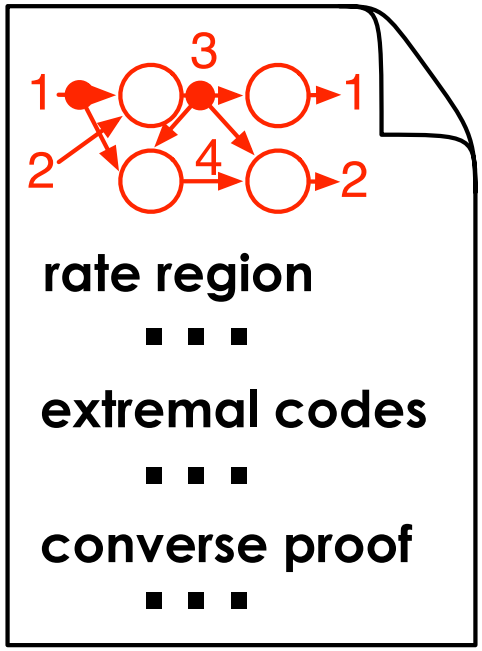
network symmetry group (today's talk) determines size of the equivalence class, & is constructed in enumeration

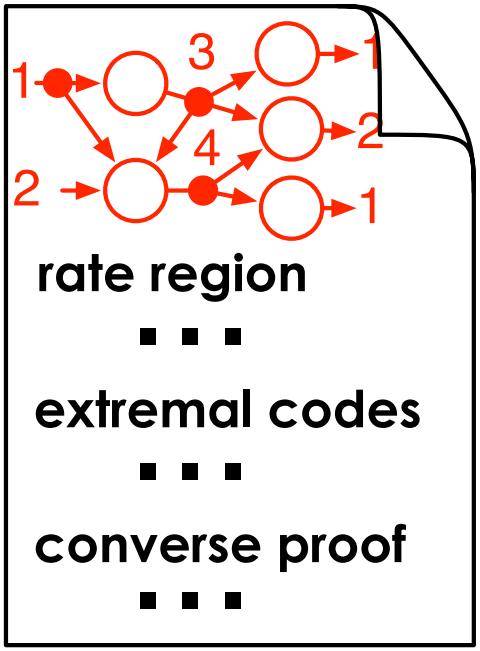
Listing Networks & Calculating Their Rate Regions

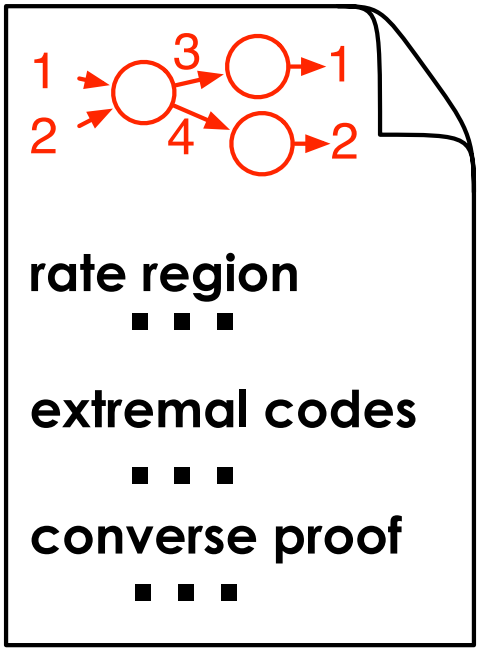


Rate Region Database

Network Coding Rate Region Database



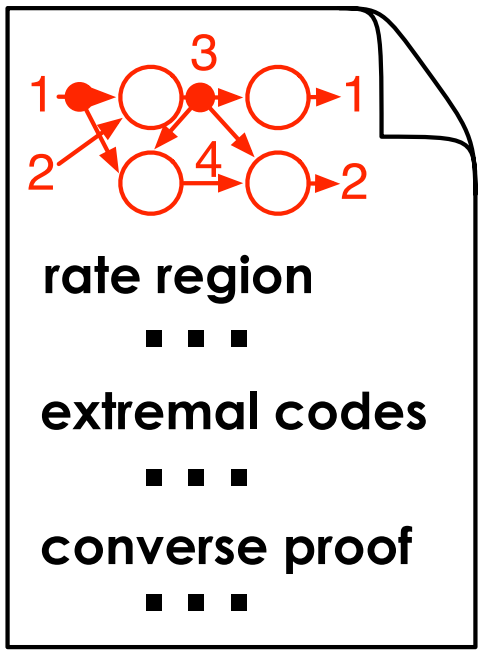


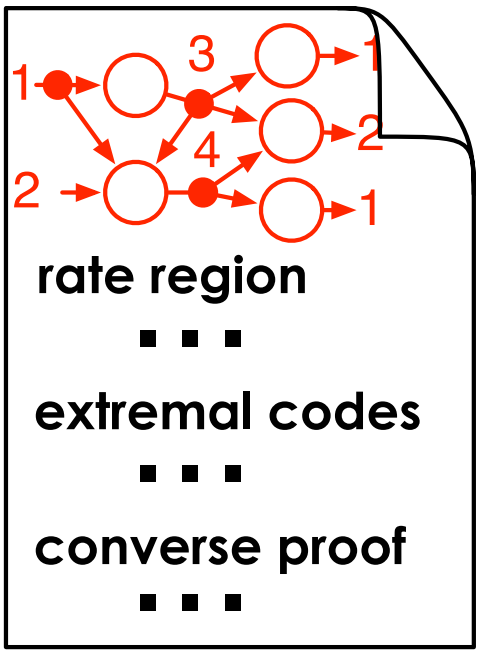


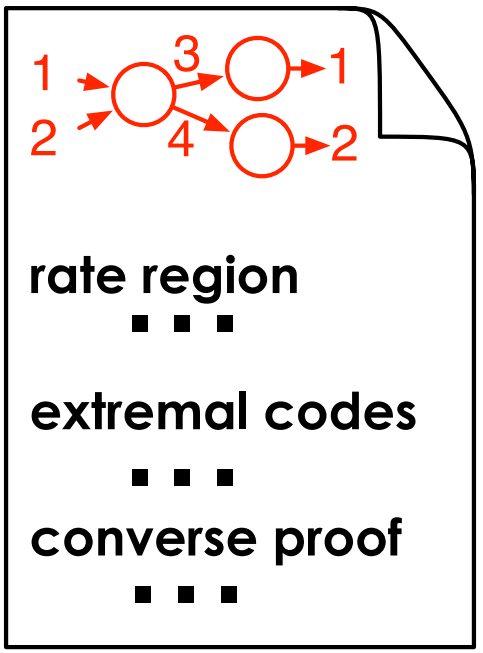
Database of ~**7000** Rate Regions of >**100k** Networks (Li Trans IT sub 2014)
Database of ~**635k** Rate Regions of > 7M or **545B** and bounds for ~**1.8T** more (Li Trans IT sub 2015)

Rate Region Database

Network Coding Rate Region Database





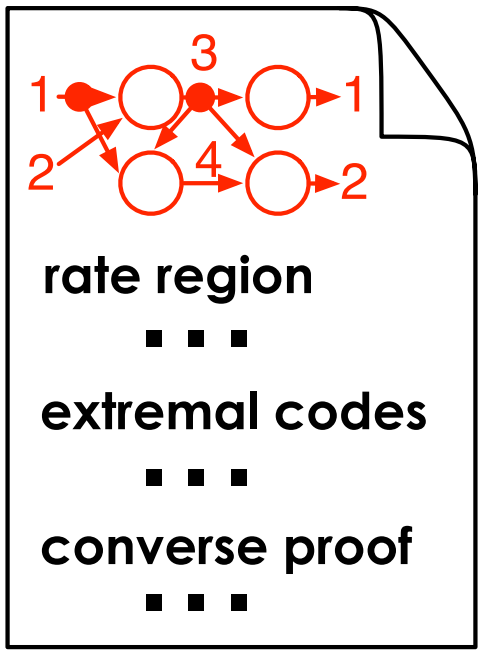


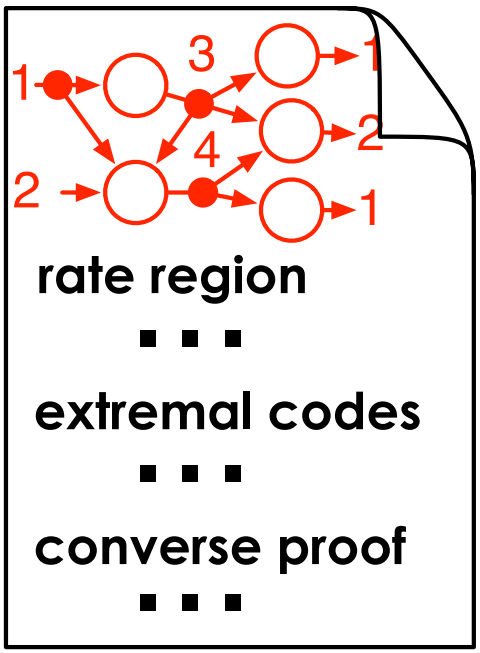
Database of ~**7000** Rate Regions of >**100k** Networks (Li Trans IT sub 2014)
Database of ~**635k** Rate Regions of > 7M or **545B** and bounds for ~**1.8T** more (Li Trans IT sub 2015)

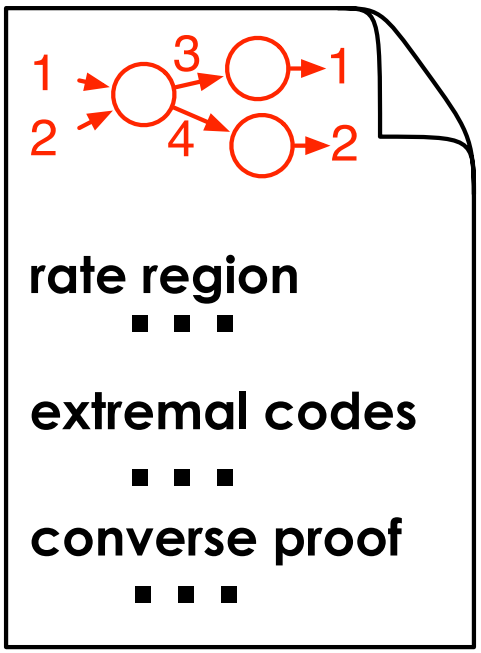
What now?

Rate Region Database

Network Coding Rate Region Database





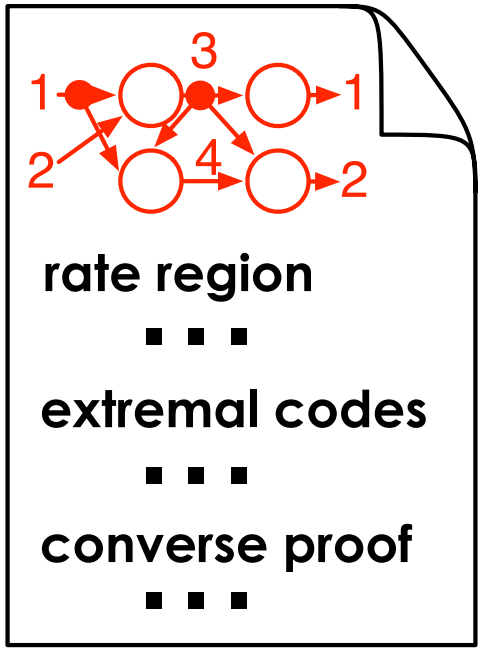


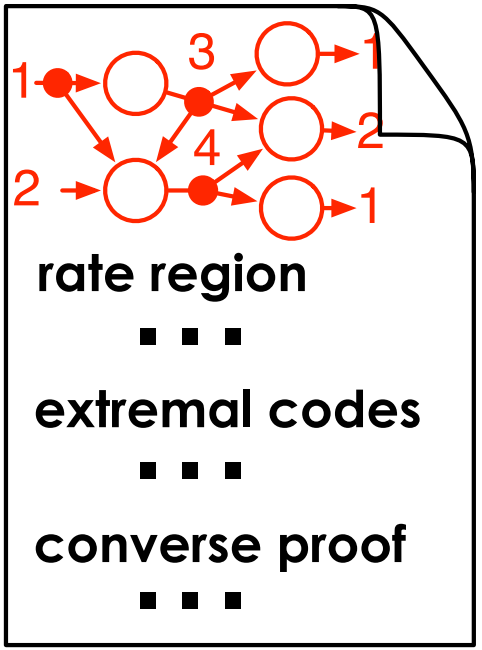
Database of ~**7000** Rate Regions of >**100k** Networks (Li Trans IT sub 2014)
Database of ~**635k** Rate Regions of > 7M or **545B** and bounds for ~**1.8T** more (Li Trans IT sub 2015)

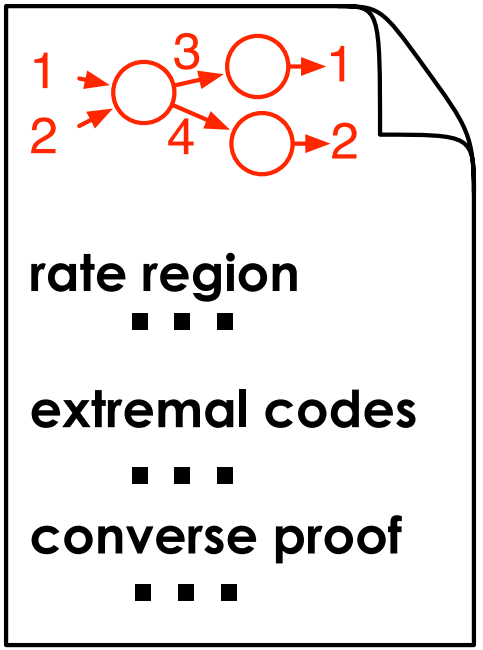
What now? Submit 635,000 transactions papers?

Rate Region Database

Network Coding Rate Region Database





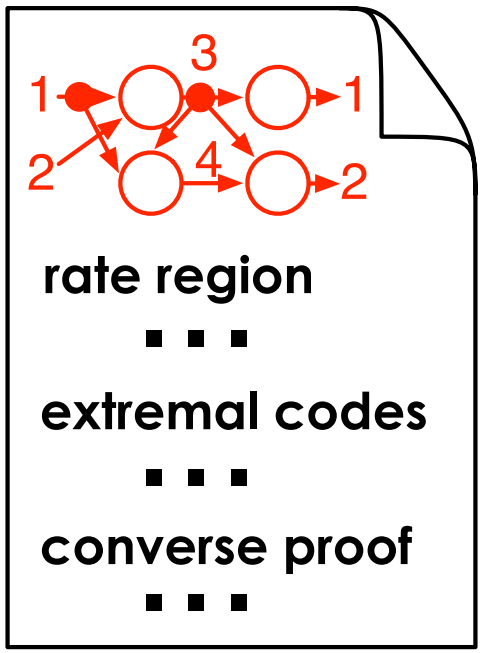


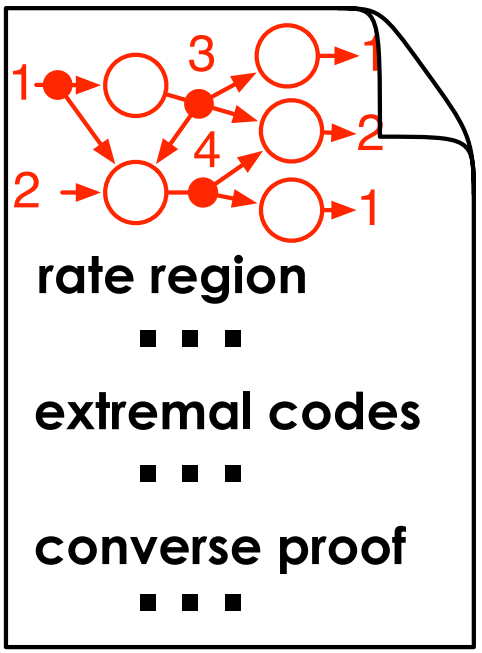
Database of ~**7000** Rate Regions of >**100k** Networks (Li Trans IT sub 2014)
Database of ~**635k** Rate Regions of > 7M or **545B** and bounds for ~**1.8T** more (Li Trans IT sub 2015)

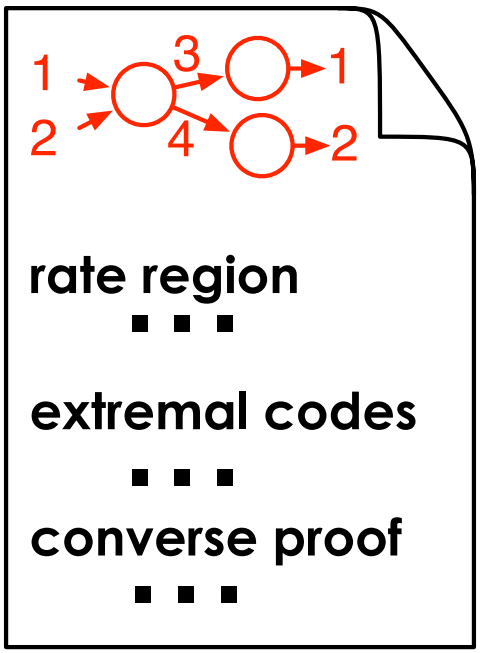
What now? Analyze, learn, and explain something!
Can't read and remember 635,000 network proofs.

Rate Region Database

Network Coding Rate Region Database





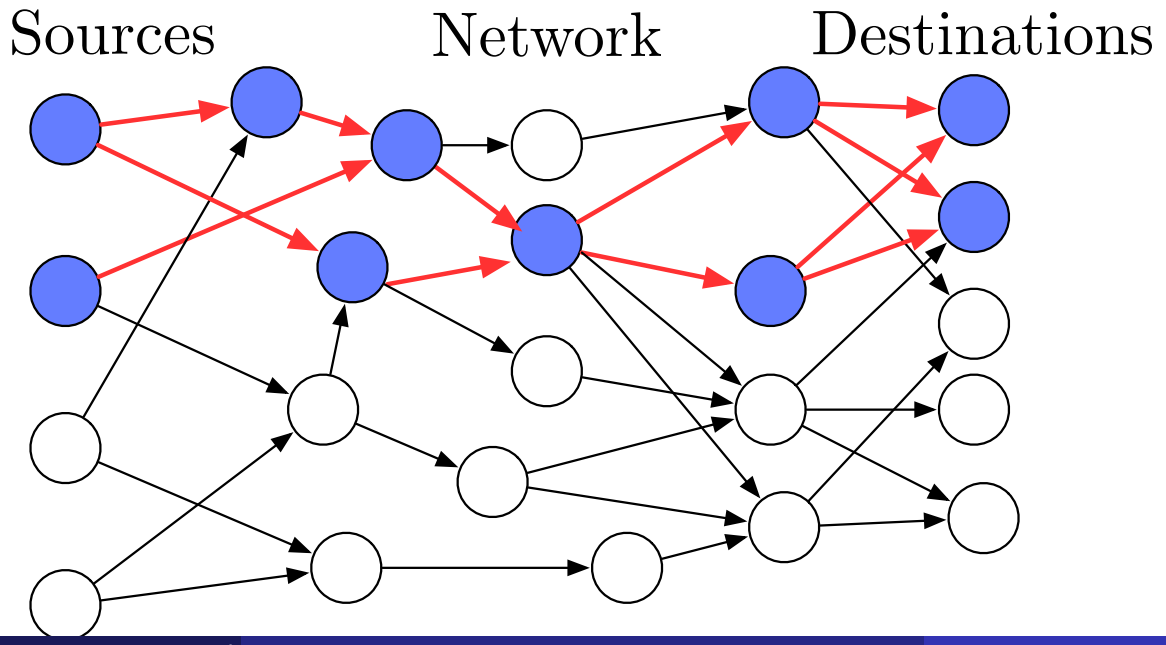


Database of ~**7000** Rate Regions of >**100k** Networks (Li Trans IT sub 2014)
Database of ~**635k** Rate Regions of > 7M or **545B** and bounds for ~**1.8T** more (Li Trans IT sub 2015)

Investigate Structure through Network Hierarchy

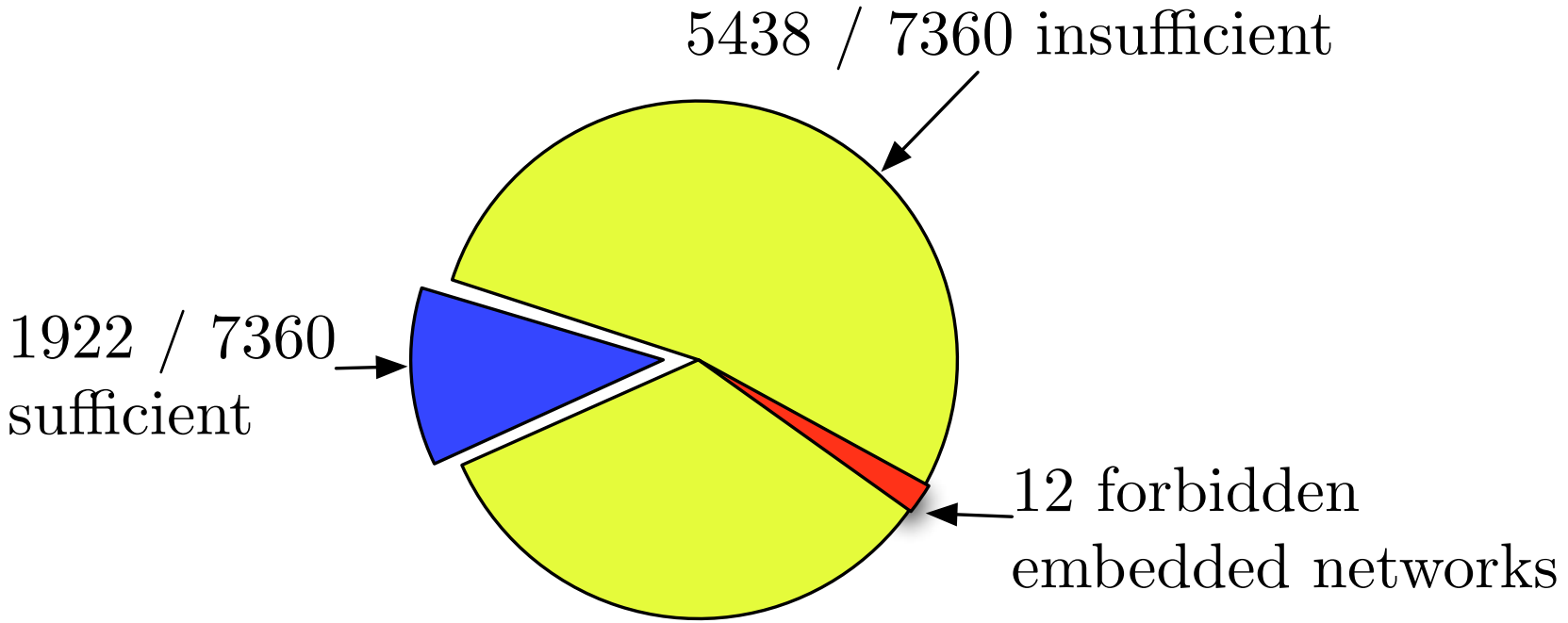
Hierarchy: Embedding Operators & Forbidden Embedded Networks

- Minor-closed graphs: finite number of forbidden minors [RobertsonSeymour1983-2004]
- Rota's conjecture in matroid theory: finite number of forbidden minors for $\mathbb{F}_q$ representability [Oxley2011]
- 3 Embedding Operators w/ Big→Small rate region expression [Li Allerton 2014, Li Trans IT sub 2014]
- Forbidden embedded networks for $\mathbb{F}_q$ codes sufficiency.



Example: Forbidden embedded networks

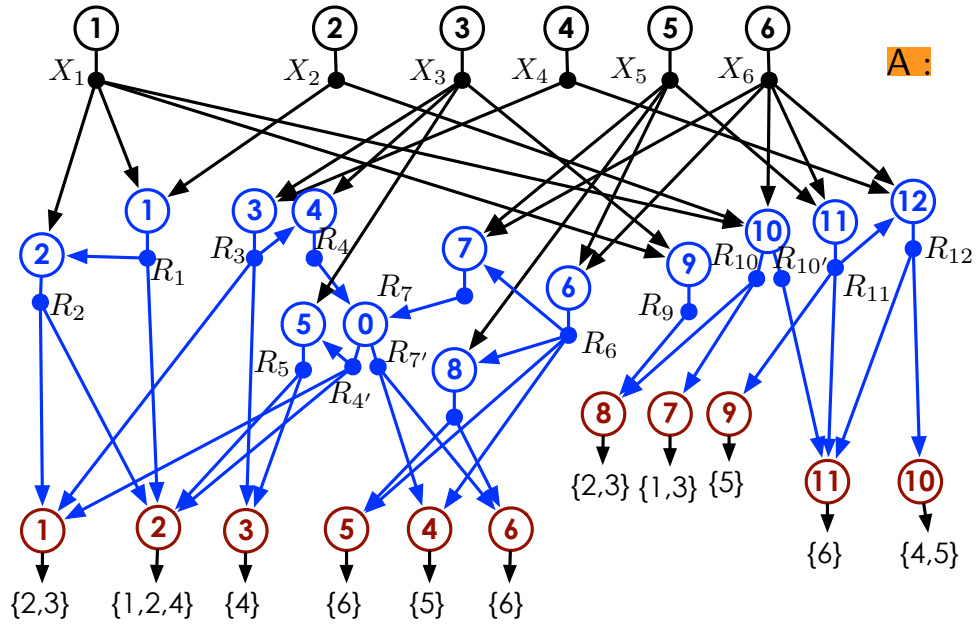
- Goal: minimal forbidden networks for sufficiency
- Scalar binary codes considered
- $k = 1, 2, 3; |\mathcal{E}| = 2, 3, 4$, 7360 non-isomorphic MDCS
- 1922 sufficient, 5438 insufficient
- 12 minimal forbidden minors [Li, *et. al* TransIT sub 2014]



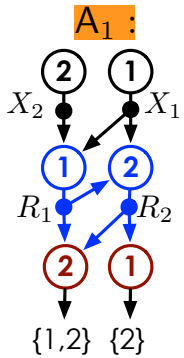
Hierarchy: Combination Operators

- Combination operators [Li ITW sub 2015, Li Trans IT sub 2015]
- Rules for combining smaller networks into large ones.
- Rate region of large expressed in terms of those of small.

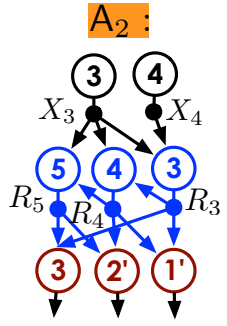
Hierarchy: Combination Operators: Ex. [Li ITW sub 2015]



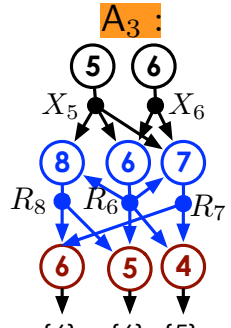
$$
\begin{array}{rcl}
R_i & \geq & H(X_2), i = 1, 2 \\
R_1 + R_2 & \geq & H(X_1) + H(X_2) \\
R_i & \geq & H(X_4), i = 3, 4, 4' \\
R_3 + R_i & \geq & H(X_3) + H(X_4), i = 4, 4' \\
R_3 + R_i + R_5 & \geq & H(X_3) + 2H(X_4), i = 4, 4' \\
R_6 & \geq & H(X_6) \\
R_i + R_8 & \geq & H(X_6), i = 7, 7' \\
R_6 + R_i & \geq & H(X_5) + H(X_6), i = 7, 7' \\
R_6 + R_i + 2R_8 & \geq & H(X_5) + 2H(X_6), i = 7, 7' \\
R_9 & \geq & H(X_3) \\
R_{10} & \geq & \sum_{i=1}^{3} H(X_i) \\
R_{11} & \geq & H(X_5) \\
R_{12} & \geq & H(X_4) + H(X_5) \\
R_{10'} + R_{11} + R_{12} & \geq & H(X_4) + 2H(X_5) + H(X_6)
\end{array}
$$



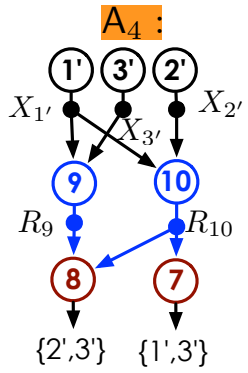
$$R_1 \geq H(X_2)$$
$$R_2 \geq H(X_2)$$
$$R_1 + R_2 \geq H(X_1) + H(X_2)$$



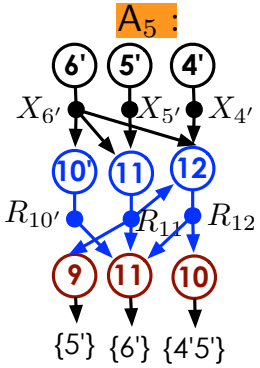
$$R_i \geq H(X_4), i = 3, 4$$
$$R_3 + R_4 \geq H(X_3) + H(X_4)$$
$$R_3 + R_4 + 2R_5 \geq H(X_3) + 2H(X_4)$$



$$R_6 \geq H(X_6)$$
$$R_7 + R_8 \geq H(X_6)$$
$$R_6 + R_7 \geq H(X_5) + H(X_6)$$
$$R_6 + R_7 + 2R_8 \geq H(X_5) + 2H(X_6)$$

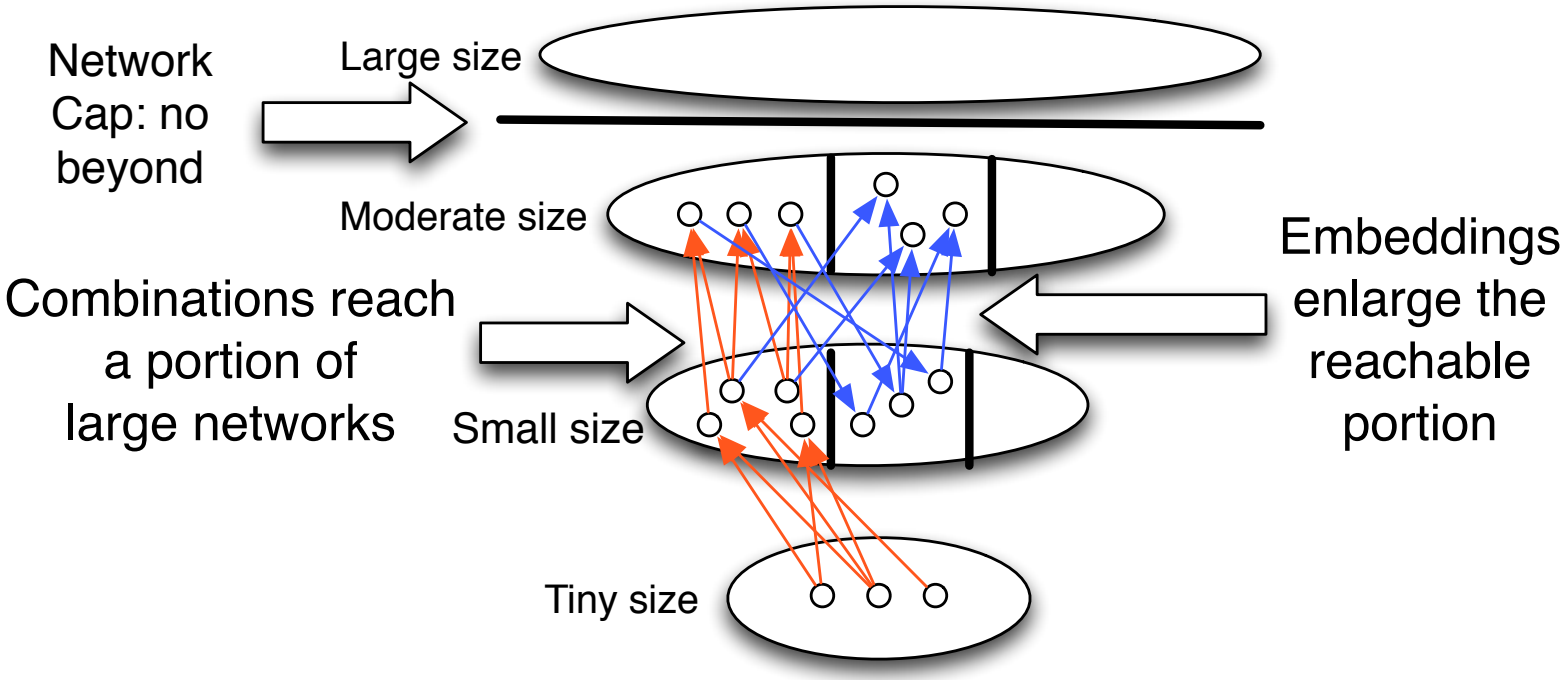


$$R_9 \geq H(X_{3'})$$
$$R_{10} \geq H(X_{1'}) + H(X_{2'}) + H(X_{3'})$$



$$R_{11} \geq H(X_{5'})$$
$$R_{12} \geq H(X_{4'}) + H(X_{5'})$$
$$R_{10'} + R_{11} + R_{12} \geq H(X_{4'}) + 2H(X_{5'}) + H(X_{6'})$$

Hierarchy: Both Embedding & Combination Ops [Li Trans IT sub 2015]



Use operators *together* to get RR for big networks. Partial Network Closure.

Hierarchy: Both Embedding & Combination Ops [Li Trans IT sub 2015]

Start with the single $(1,1)$, single $(2,1)$, and the four $(1,2)$ networks;
These 6 tiny networks can generate new 11635 networks w/ small cap!

	combination operators only			embedding and combinations		
size\cap	(3,3)	(3,4)	(4,4)	(3,3)	(3,4)	(4,4)
(1,3)	4	4	4	4	4	4
(1,4)	0	10	10	0	10	10
(2,2)	3	3	3	8	15	16
(2,3)	13	16	16	30	131	155
(2,4)	0	97	101	0	516	648
(3,2)	2	3	2	4	10	11
(3,3)	24	24	24	42	353	833
(3,4)	0	135	135	0	2361	5481
(4,2)	0	0	3	0	0	3
(4,3)	0	0	17	0	0	44
(4,4)	0	0	253	0	0	4430
all	46	292	568	88	3400	11635

Hierarchy: Both Embedding & Combination Ops [Li Trans IT sub 2015]

With the increase of cap size, number of new networks increases!

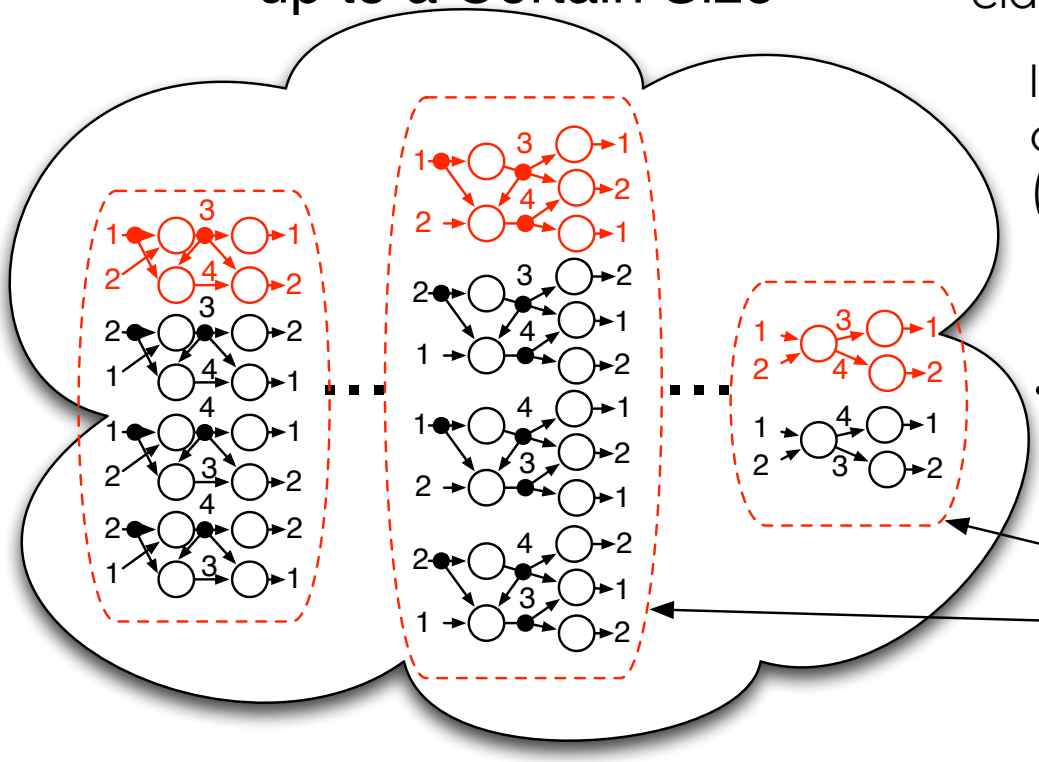
	combination operators only			embedding and combinations		
size\cap	(3,3)	(3,4)	(4,4)	(3,3)	(3,4)	(4,4)
(1,3)	4	4	4	4	4	4
(1,4)	0	10	10	0	10	10
(2,2)	3	3	3	8	15	16
(2,3)	13	16	16	30	131	155
(2,4)	0	97	101	0	516	648
(3,2)	2	3	2	4	10	11
(3,3)	24	24	24	42	353	833
(3,4)	0	135	135	0	2361	5481
(4,2)	0	0	3	0	0	3
(4,3)	0	0	17	0	0	44
(4,4)	0	0	253	0	0	4430
all	46	292	568	88	3400	11635

Hierarchy: Both Embedding & Combination Ops [Li Trans IT sub 2015]

Embedding operations are important in the process!

	combination operators only			embedding and combinations		
size\cap	(3,3)	(3,4)	(4,4)	(3,3)	(3,4)	(4,4)
(1,3)	4	4	4	4	4	4
(1,4)	0	10	10	0	10	10
(2,2)	3	3	3	8	15	16
(2,3)	13	16	16	30	131	155
(2,4)	0	97	101	0	516	648
(3,2)	2	3	2	4	10	11
(3,3)	24	24	24	42	353	833
(3,4)	0	135	135	0	2361	5481
(4,2)	0	0	3	0	0	3
(4,3)	0	0	17	0	0	44
(4,4)	0	0	253	0	0	4430
all	46	292	568	88	3400	11635

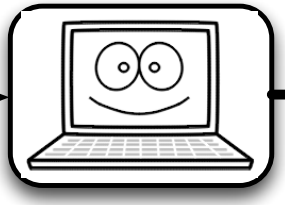
Listing Networks & Calculating Their Rate Regions



group networks into equivalence classes (only differing in labels)

list only canonical representative of each equivalence class (Li NetCod '15 & Trans IT sub)

Our Software



All Their Rate Regions

network symmetry group (today's talk) determines size of the equivalence class, & is constructed in enumeration

Outline

1. Motivation, Context, & Bigger Picture
2. Formalization via Groups
   - Group Action
   - Network Symmetry Group
   - Symmetries of Network Codes and Rate Regions
3. Graph Theoretic Characterization
   - Two Special Network Coding Problem Classes
   - Automorphism Groups
   - Characterization for k-Unicast Networks
   - Characterization for MSNC Networks
   - Computation
4. Applications
   - Computation of Polyhedral Bounds on Rate Regions
   - A generalization of partition-symmetrical entropy functions

Drexel UNIVERSITY

Group Action

Action of a finite group on a set

Definition

[a] The *(left) group action* of a group $G \leq S_n$ on set S is the map $\phi : G \times S \to S : (g, x) \mapsto \phi(g, x)$ that satisfies:

G1 $\phi((g \circ h), x) = \phi(g, \phi(h, x)),\ \forall g, h \in G, x \in S$

G2 $\phi(e, x) = x,\ \forall x \in S$

[a] $\leq$ denotes subgroups, S_n denotes symmetric group

- Denote $\phi(g, x)$ as x^g and, $\phi(g, X)$ as X^g for $x \in S$, and $X \subseteq S$
- For $X \subseteq S$, an element $g \in G$ is said to stabilize X setwise if $X^g = X$
- The collection of all group elements $g \in G$ that setwise stabilize a subset $X \subseteq S$ forms a group called stabilizer subgroup, denoted as G_X.

Group Action on Network Constraints

- Recall network constraints $\mathcal{L}_{12345} = \mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_3 \cap \mathcal{L}_4 \cap \mathcal{L}_5$
- Let $[n]$ be the set of subscripts of RVs in $\mathcal{X}_n$ and $\hat{\mathcal{L}}$ be the set of sets of constraints arising from different network coding problems
- The natural action of $G \leq S_n$ on $[n]$ induces following sequence of actions: $\text{Act}([n]) \xrightarrow{\text{induced}} \text{Act}(\mathcal{X}_n) \xrightarrow{\text{induced}} \text{Act}(2^{X_n}) \xrightarrow{\text{induced}} \text{Act}(\hat{\mathcal{L}})$

Example

Let $g = (1,2)(3,4)(7,8) \in G \leq S_8$ (natural action on $[8]$ and $\mathcal{X}_8$)
$\{1,3\} \mapsto \{2,4\}$, $\{1,3,4,8\} \mapsto \{2,4,3,7\}$ (induced action on 2^{X_8})

$$\left.\begin{array}{c} \{h_{\{1,3\}} = h_{\{1,3,4,8\}}\} \mapsto \{h_{\{2,4\}} = h_{\{2,4,3,7\}}\} \\ \{h_8 \leq R_8\} \mapsto \{h_7 \leq R_7\} \end{array}\right\} \text{(induced action on } \hat{\mathcal{L}})$$

Rate Region

- Rate region: all possible rate and source entropy vectors satisfying all network constraints.
- Collect the N network random variables and their joint entropies.
- Define Γ_N^*: $2^N - 1$-dim. cone, region of valid entropy vectors. (revisit later)
- Constraints from network A:

$$\mathcal{L}_1 = \{\mathbf{h} \in \Gamma_N^* : h_{Y_S} = \Sigma_{s \in \mathcal{S}} h_{Y_s}\} \quad (7)$$

$$\mathcal{L}_2 = \{\mathbf{h} \in \Gamma_N^* : h_{X_{\mathrm{Out}(k)}|Y_s} = 0\} \quad (8)$$

$$\mathcal{L}_3 = \{\mathbf{h} \in \Gamma_N^* : h_{X_{\mathrm{Out}(i)}|X_{\mathrm{In}(i)}} = 0\} \quad (9)$$

$$\mathcal{L}_4 = \{(\mathbf{h}^T, \mathbf{R}^T)^T \in \mathbb{R}_+^{2^N-1+|\mathcal{E}|} : R_e \geq h_{U_e}, e \in \mathcal{E}\} \quad (10)$$

$$\mathcal{L}_5 = \{\mathbf{h} \in \Gamma_N^* : h_{Y_{\beta(t)}|U_{\mathrm{In}(t)}} = 0\}. \quad (11)$$

- Rate region (cone) in terms of edge rates and source entropies (derived from [Yan, Yeung, Zhang TranIT 2012]):

$$\mathcal{R}_*(\mathrm{A}) = \mathrm{proj}_{R_\mathcal{E}, H(Y_\mathcal{S})}(\overline{\mathrm{con}(\Gamma_\mathrm{N}^* \cap \mathcal{L}_{123})} \cap \mathcal{L}_{45}) \quad (12)$$

Group Action on Network Constraints

- Recall network constraints $\mathcal{L}_{12345} = \mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_3 \cap \mathcal{L}_4 \cap \mathcal{L}_5$
- Let $[n]$ be the set of subscripts of RVs in $\mathcal{X}_n$ and $\hat{\mathcal{L}}$ be the set of sets of constraints arising from different network coding problems
- The natural action of $G \leq S_n$ on $[n]$ induces following sequence of actions: $\text{Act}([n]) \xrightarrow{\text{induced}} \text{Act}(\mathcal{X}_n) \xrightarrow{\text{induced}} \text{Act}(2^{X_n}) \xrightarrow{\text{induced}} \text{Act}(\hat{\mathcal{L}})$

Example

Let $g = (1,2)(3,4)(7,8) \in G \leq S_8$ (natural action on $[8]$ and $\mathcal{X}_8$)
$\{1,3\} \mapsto \{2,4\}$, $\{1,3,4,8\} \mapsto \{2,4,3,7\}$ (induced action on 2^{X_8})

$$\left.\begin{array}{c} \{h_{\{1,3\}} = h_{\{1,3,4,8\}}\} \mapsto \{h_{\{2,4\}} = h_{\{2,4,3,7\}}\} \\ \{h_8 \leq R_8\} \mapsto \{h_7 \leq R_7\} \end{array}\right\} \text{(induced action on } \hat{\mathcal{L}})$$

Outline

1. Motivation, Context, & Bigger Picture
2. Formalization via Groups
   - Group Action
   - Network Symmetry Group
   - Symmetries of Network Codes and Rate Regions
3. Graph Theoretic Characterization
   - Two Special Network Coding Problem Classes
   - Automorphism Groups
   - Characterization for k-Unicast Networks
   - Characterization for MSNC Networks
   - Computation
4. Applications
   - Computation of Polyhedral Bounds on Rate Regions
   - A generalization of partition-symmetrical entropy functions

Drexel UNIVERSITY

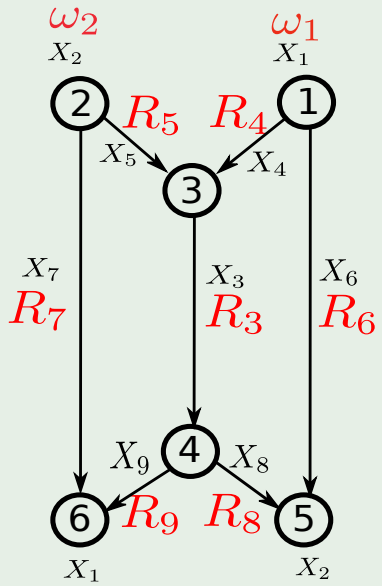
Network Symmetry Group

Definition (Network Symmetry Group)

The *network symmetry group* $G^{\mathcal{I}}$ of a MSNC instance $\mathcal{I} = (\mathcal{G} = (V, E), \mathcal{S}, \mathcal{T}, \beta)$ is the subgroup of S_n, $n = |\mathcal{S}| + |E|$, that stabilizes $\mathcal{L}_{12345}$ setwise under its induced action on $\hat{\mathcal{L}}$.

Network Symmetry Group

Example



(1,2)(3)(4,5)(6,7)(8,9)

$\mathcal{L}_1$ source independence

$h_1+h_2=h_{\{1,2\}} \longmapsto h_1+h_2=h_{\{1,2\}}$

$\mathcal{L}_2$ node constraints

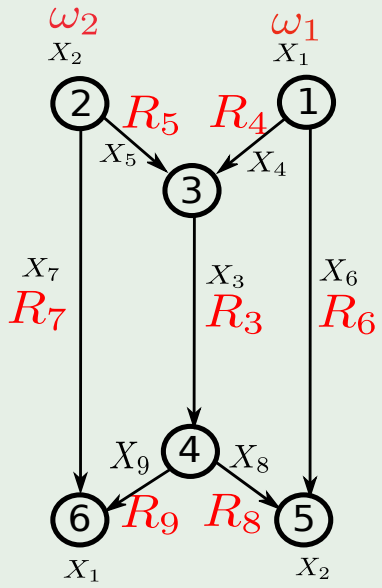
$h_1=h_{\{1,4,6\}} \longmapsto h_2=h_{\{2,5,7\}}$
$h_2=h_{\{2,5,7\}} \longmapsto h_1=h_{\{1,4,6\}}$
$h_{\{4,5\}}=h_{\{3,4,5\}} \longmapsto h_{\{4,5\}}=h_{\{3,4,5\}}$
$h_3=h_{\{3,8,9\}} \longmapsto h_3=h_{\{3,8,9\}}$
$h_{\{6,8\}}=h_{\{2,6,8\}} \longmapsto h_{\{7,9\}}=h_{\{1,7,9\}}$
$h_{\{7,9\}}=h_{\{1,7,9\}} \longmapsto h_{\{6,8\}}=h_{\{2,6,8\}}$

$\mathcal{L}_3$ rate constraints

$h_1 \geq \omega_1 \longmapsto h_2 \geq \omega_2$
$h_2 \geq \omega_2 \longmapsto h_1 \geq \omega_1$
$h_3 \leq R_3 \longmapsto h_3 \leq R_3$
$h_4 \leq R_4 \longmapsto h_5 \leq R_5$
$h_5 \leq R_5 \longmapsto h_4 \leq R_4$
$h_6 \leq R_6 \longmapsto h_7 \leq R_7$
$h_7 \leq R_7 \longmapsto h_6 \leq R_6$
$h_8 \leq R_8 \longmapsto h_9 \leq R_9$
$h_9 \leq R_9 \longmapsto h_8 \leq R_8$

Network Symmetry Group

Example



(1,2)(3)(4,5)(6,7)(8,9)

$\mathcal{L}_1$ source independence

$h_1+h_2=h_{\{1,2\}} \longmapsto h_1+h_2=h_{\{1,2\}}$

$\mathcal{L}_2$ node constraints

$h_1=h_{\{1,4,6\}} \longmapsto h_2=h_{\{2,5,7\}}$
$h_2=h_{\{2,5,7\}} \longmapsto h_1=h_{\{1,4,6\}}$
$h_{\{4,5\}}=h_{\{3,4,5\}} \longmapsto h_{\{4,5\}}=h_{\{3,4,5\}}$
$h_3=h_{\{3,8,9\}} \longmapsto h_3=h_{\{3,8,9\}}$
$h_{\{6,8\}}=h_{\{2,6,8\}} \longmapsto h_{\{7,9\}}=h_{\{1,7,9\}}$
$h_{\{7,9\}}=h_{\{1,7,9\}} \longmapsto h_{\{6,8\}}=h_{\{2,6,8\}}$

$\mathcal{L}_3$ rate constraints

$h_1 \geq \omega_1 \longmapsto h_2 \geq \omega_2$
$h_2 \geq \omega_2 \longmapsto h_1 \geq \omega_1$
$h_3 \leq R_3 \longmapsto h_3 \leq R_3$
$h_4 \leq R_4 \longmapsto h_5 \leq R_5$
$h_5 \leq R_5 \longmapsto h_4 \leq R_4$
$h_6 \leq R_6 \longmapsto h_7 \leq R_7$
$h_7 \leq R_7 \longmapsto h_6 \leq R_6$
$h_8 \leq R_8 \longmapsto h_9 \leq R_9$
$h_9 \leq R_9 \longmapsto h_8 \leq R_8$

Network Symmetry Group of Butterfly Network is of order 2 with (1,2)(3)(4,5)(6,7)(8,9) being the only non-trivial symmetry

Outline

1. Motivation, Context, & Bigger Picture
2. Formalization via Groups
   - Group Action
   - Network Symmetry Group
   - Symmetries of Network Codes and Rate Regions
3. Graph Theoretic Characterization
   - Two Special Network Coding Problem Classes
   - Automorphism Groups
   - Characterization for k-Unicast Networks
   - Characterization for MSNC Networks
   - Computation
4. Applications
   - Computation of Polyhedral Bounds on Rate Regions
   - A generalization of partition-symmetrical entropy functions

Drexel UNIVERSITY

Symmetries of Network Codes

Definition

A *network code* $(\{f_e\}, \{g_t\})$ for a MSNC instance $\mathcal{I}$ is an assignment of a function f_e to each edge $e \in E$ and a function g_t to each sink $t \in \mathcal{T}$.

Symmetries of Network Codes

Definition

A *network code* $(\{f_e\}, \{g_t\})$ for a MSNC instance $\mathcal{I}$ is an assignment of a function f_e to each edge $e \in E$ and a function g_t to each sink $t \in \mathcal{T}$.

Theorem

Let $G^{\mathcal{I}}$ be the NSG associated with MSNC instance $\mathcal{I}$. Then, for any $g \in G^{\mathcal{I}}$, if a network code $(\{f_i\}, \{g_t\})$ satisfies $\mathcal{L}_{12345}$, so does $(\{f_{ig}\}, \{g_{tg}\})$ for every $g \in G^{\mathcal{I}}$.

Drexel UNIVERSITY

Symmetries of Rate Regions

Theorem

If $\boldsymbol{\omega}, \boldsymbol{r}$ is an achievable source information rates and edge rates vector pair for a MSNC instance $\mathcal{I}$, so are $[\omega_{sg} | s \in \mathcal{S}], [R_{eg} | e \in E]$ for every $g \in G^{\mathcal{I}}$.

Symmetries of Rate Regions

Theorem

If $\boldsymbol{\omega}, \boldsymbol{r}$ is an achievable source information rates and edge rates vector pair for a MSNC instance $\mathcal{I}$, so are $[\omega_{sg} | s \in \mathcal{S}], [R_{eg} | e \in E]$ for every $g \in G^{\mathcal{I}}$.

Example

(ω_1, ω_2) is achievable rate tuple iff
(ω_2, ω_1) is achievable rate tuple

(bpnu: bits per network use)

Outline

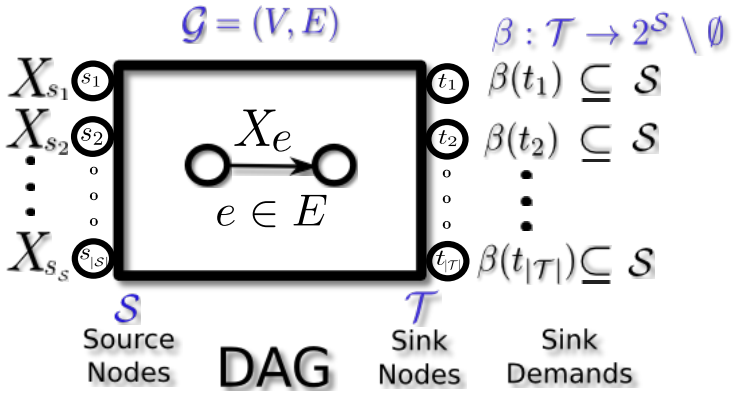
1. Motivation, Context, & Bigger Picture
2. Formalization via Groups
   - Group Action
   - Network Symmetry Group
   - Symmetries of Network Codes and Rate Regions
3. Graph Theoretic Characterization
   - Two Special Network Coding Problem Classes
   - Automorphism Groups
   - Characterization for k-Unicast Networks
   - Characterization for MSNC Networks
   - Computation
4. Applications
   - Computation of Polyhedral Bounds on Rate Regions
   - A generalization of partition-symmetrical entropy functions

Drexel UNIVERSITY

Two Special Problems

MSNC

- Multisource Network Coding
- Described by tuple $(\mathcal{G}, \mathcal{S}, \mathcal{T}, \beta)$

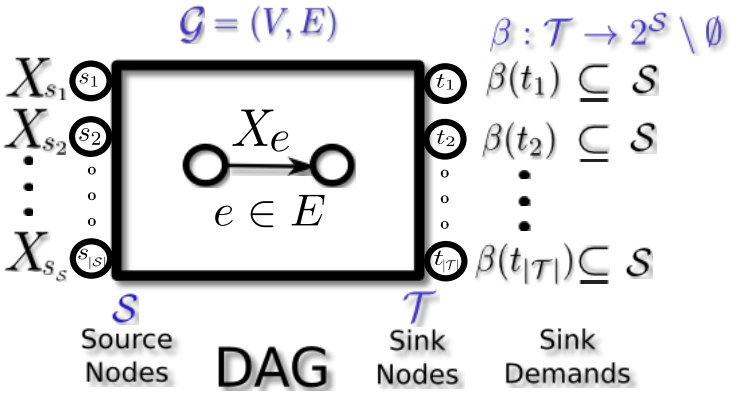


- $\beta : \mathcal{T} \to 2^{\mathcal{S}} \setminus \emptyset$ allows arbitrary subset demands

Two Special Problems

MSNC

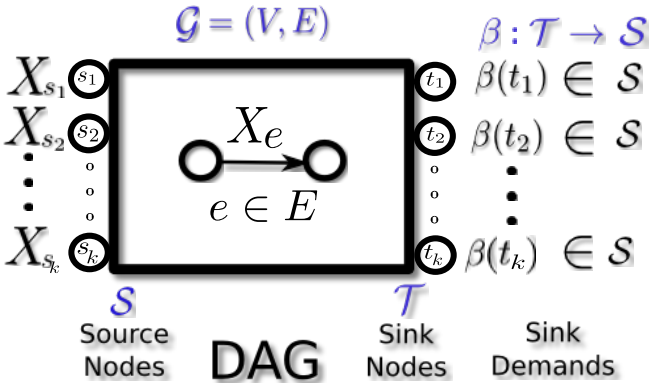
- Multisource Network Coding
- Described by tuple $(\mathcal{G}, \mathcal{S}, \mathcal{T}, \beta)$



- $\beta : \mathcal{T} \to 2^{\mathcal{S}} \setminus \emptyset$ allows arbitrary subset demands

k-UNC

- k-Unicast Network Coding
- Described by tuple $(\mathcal{G}, \mathcal{S}, \mathcal{T}, \beta)$



- $\beta : \mathcal{T} \to S$ is a bijection

Outline

1. Motivation, Context, & Bigger Picture
2. Formalization via Groups
   - Group Action
   - Network Symmetry Group
   - Symmetries of Network Codes and Rate Regions
3. Graph Theoretic Characterization
   - Two Special Network Coding Problem Classes
   - Automorphism Groups
   - Characterization for k-Unicast Networks
   - Characterization for MSNC Networks
   - Computation
4. Applications
   - Computation of Polyhedral Bounds on Rate Regions
   - A generalization of partition-symmetrical entropy functions

Drexel UNIVERSITY

Automorphism Groups

- Several combinatorial objects can be described as collections of subsets of some set eg. graphs, digraphs, linear codes and so on...
- Automorphism group of a combinatorial object is the group containing all permutations that keep the collection of subsets invariant

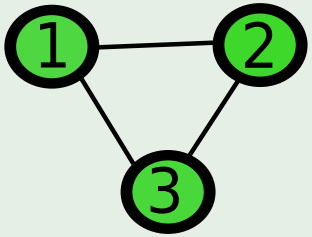
Drexel UNIVERSITY

Automorphism Groups

- Several combinatorial objects can be described as collections of subsets of some set eg. graphs, digraphs, linear codes and so on...
- Automorphism group of a combinatorial object is the group containing all permutations that keep the collection of subsets invariant

Example

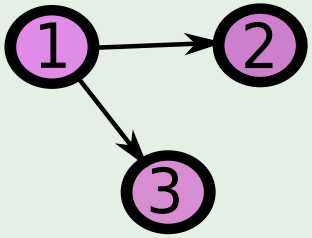
V=S, E={{1,2},{2,3},{1,3}}



S={1,2,3}
V is S,
E is a collection of 2 subsets of V

Aut(G) contains all 3! permutations of {1,2,3}

V=S, E={(1,2),(1,3)}



S={1,2,3}
V is S,
E is a subset of $\mathrm{V}\times\mathrm{V}$

Aut(G) contains all identity and (2,3)

Outline

1. Motivation, Context, & Bigger Picture
2. Formalization via Groups
   - Group Action
   - Network Symmetry Group
   - Symmetries of Network Codes and Rate Regions
3. Graph Theoretic Characterization
   - Two Special Network Coding Problem Classes
   - Automorphism Groups
   - Characterization for k-Unicast Networks
   - Characterization for MSNC Networks
   - Computation
4. Applications
   - Computation of Polyhedral Bounds on Rate Regions
   - A generalization of partition-symmetrical entropy functions

Drexel UNIVERSITY

NSG via Graphs for k-Unicast

Construction

1. Start with the underlying DAG $\mathcal{G} = (V, E)$
2. Add a feedback edges $F = \{(t, \beta(t)) \mid t \in \mathcal{T}\}$ to create circulation graph $\mathcal{G}_c = (V, E \cup F_c)$
3. Take line graph: vertices are edges of $\mathcal{G}_c$, edges are length 2 directed paths in $\mathcal{G}_c$ to create dual circulation graph $\mathcal{G}_c^*$
4. Associate random variables with vertices of $\mathcal{G}_c^*$

NSG via Graphs for k-Unicast

Construction

1. Start with the underlying DAG $\mathcal{G} = (V, E)$
2. Add a feedback edges $F = \{(t, \beta(t)) \mid t \in \mathcal{T}\}$ to create circulation graph $\mathcal{G}_c = (V, E \cup F_c)$
3. Take line graph: vertices are edges of $\mathcal{G}_c$, edges are length 2 directed paths in $\mathcal{G}_c$ to create dual circulation graph $\mathcal{G}_c^*$
4. Associate random variables with vertices of $\mathcal{G}_c^*$

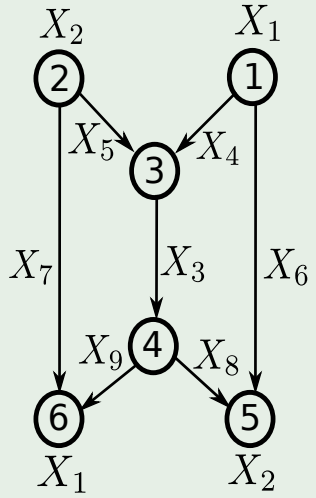
Theorem

The network symmetry group of a k-UNC instance is $G^{\mathcal{I}} = G_{F_c} \cap Aut(\mathcal{G}_c^*)$, the group of permutations of $\mathcal{X}_n$ induced by the subgroup of automorphism group of its dual circulation graph that setwise stabilizes the feedback edge set.

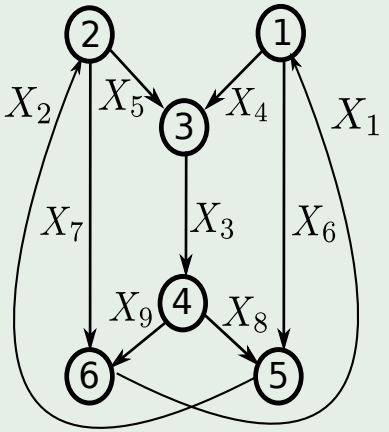
NSG via Graphs for k-Unicast

Example

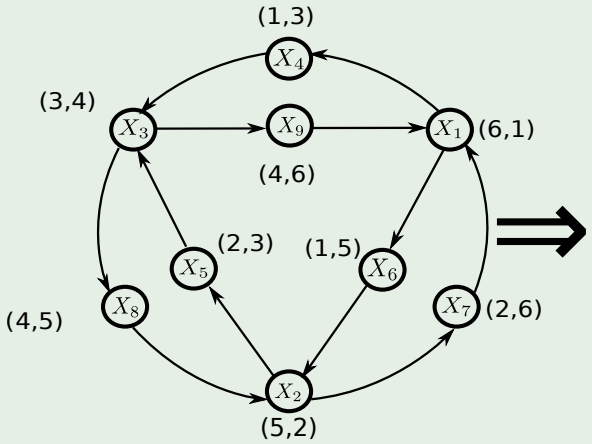
A k-UNC instance



circulation graph $\mathcal{G}_c^*$



line graph of circulation graph $\mathcal{G}_c^*$



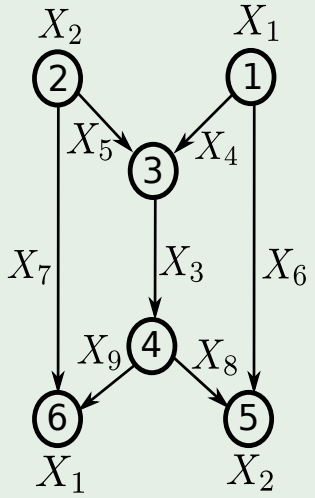
Problem Symmetries

symmetries are the subgroup of $\mathsf{Aut}(\mathcal{G}_c^*)$ that setwise stabilizes the feedback edges {(6,1),(5,2)}

NSG via Graphs for k-Unicast

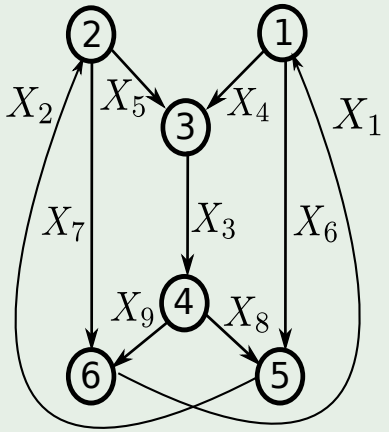
Example

A k-UNC instance



$\Rightarrow$

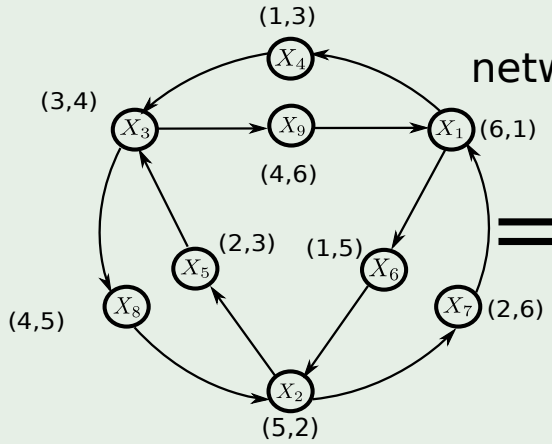
circulation graph $\mathcal{G}_c^*$



$\Rightarrow$

$G^{\mathcal{I}} = \{(1), (1,2)(3)(4,5)(6,7)(8,9)\}$

line graph of circulation graph $\mathcal{G}_c^*$



$\Rightarrow$

network symmetry group $G^{\mathcal{I}}$ contains identity and $(1,2)(3)(4,5)(6,7)(8,9)$

Outline

1. Motivation, Context, & Bigger Picture
2. Formalization via Groups
   - Group Action
   - Network Symmetry Group
   - Symmetries of Network Codes and Rate Regions
3. Graph Theoretic Characterization
   - Two Special Network Coding Problem Classes
   - Automorphism Groups
   - Characterization for k-Unicast Networks
   - Characterization for MSNC Networks
   - Computation
4. Applications
   - Computation of Polyhedral Bounds on Rate Regions
   - A generalization of partition-symmetrical entropy functions

Drexel UNIVERSITY

NSG via Graphs for MSNC Networks

- A slightly more complicated characterization

1. Start with underlying DAG $\mathcal{G} = (V, E)$
2. Add new vertices $\mathcal{S}'$, edges $E_{\mathcal{S}}$ and F_c to create circulation graph $\mathcal{G}_c$
3. Take line graph to create dual circulation graph $\mathcal{G}_c^*$
4. Compute AUT$(\mathcal{G}_c^*)$
5. Obtain network symmetry group as an induced group

NSG via Graphs for MSNC Networks

- A slightly more complicated characterization

1. Start with underlying DAG $\mathcal{G} = (V, E)$
2. Add new vertices $\mathcal{S}'$, edges $E_{\mathcal{S}}$ and F_c to create circulation graph $\mathcal{G}_c$
3. Take line graph to create dual circulation graph $\mathcal{G}_c^*$
4. Compute AUT$(\mathcal{G}_c^*)$
5. Obtain network symmetry group as an induced group

Theorem

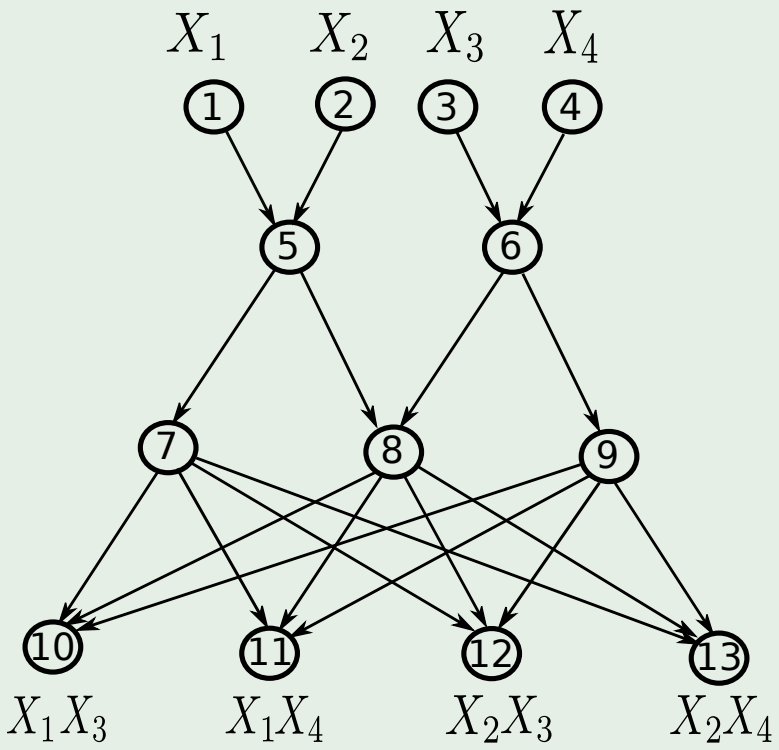
The network symmetry group of a MSNC instance is the group of permutations induced onto $E \cup E_{\mathcal{S}}$ by the subgroup of $Aut(\mathcal{G}_c^*)$ that stabilizes $E_{\mathcal{S}}$ setwise.

Drexel UNIVERSITY

NSG via Graphs for MSNC Networks

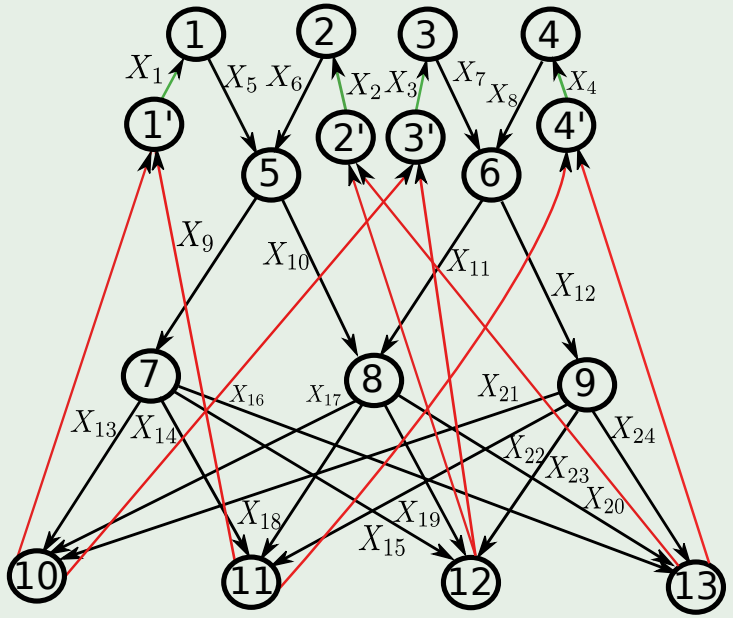
Example

An MSNC instance



$\Rightarrow$

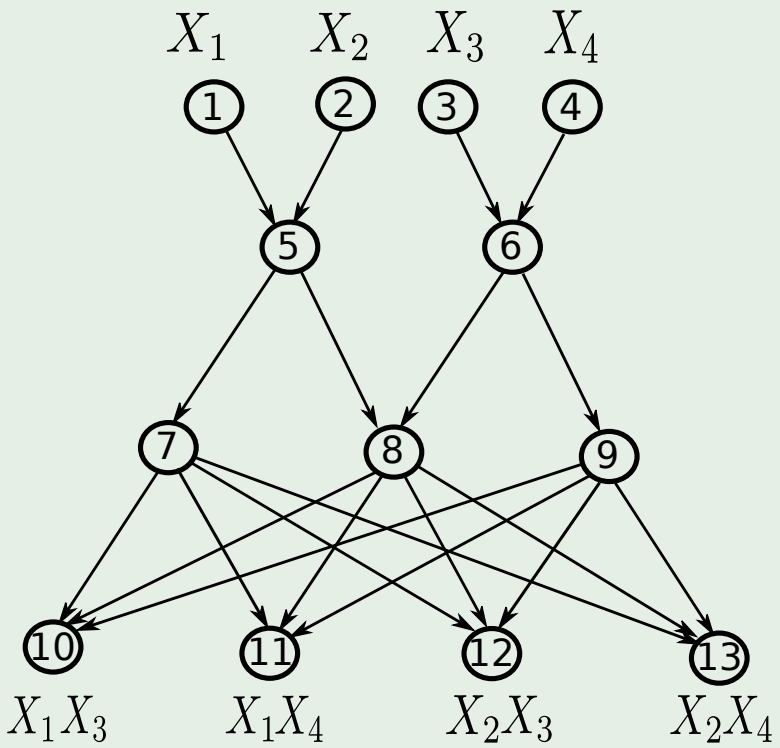
Circulation Graph $\mathcal{G}_c$



NSG via Graphs for MSNC Networks

Example

An MSNC instance



$\Rightarrow$

The Network Symmetry group is generated by:

$$
\begin{aligned}
g_1 =& (1,3)(2,4)(5,7)(6,8) \\
& (9,12)(10,11)(13,21) \\
& (14,23)(15,22)(16,24)(18,19) \\
g_2 =& (3,4)(7,8)(13,14)(15,16) \\
& (17,18)(19,20)(21,22)(23,24) \\
g_3 =& (1,2)(5,6)(13,15)(14,16) \\
& (17,19)(18,20)(21,23)(22,24)
\end{aligned}
$$

A group of order 8

Outline

1. Motivation, Context, & Bigger Picture
2. Formalization via Groups
   - Group Action
   - Network Symmetry Group
   - Symmetries of Network Codes and Rate Regions
3. Graph Theoretic Characterization
   - Two Special Network Coding Problem Classes
   - Automorphism Groups
   - Characterization for k-Unicast Networks
   - Characterization for MSNC Networks
   - Computation
4. Applications
   - Computation of Polyhedral Bounds on Rate Regions
   - A generalization of partition-symmetrical entropy functions

Drexel UNIVERSITY

Computing NSGs

- One can use off-the-shelf (di)graph automorphism group computation software for computing NSGs
- `nauty`: No Automorphisms, Yes? implements Brendan McKay's canonical labeling algorithm
- We used `SageMath`'s implementation of the same algorithm
- Can also be enumerated along with canonical networks [Li, Walsh, Weber, NetCod 2015, Trans IT sub 2015]

Drexel UNIVERSITY

Outline

1. Motivation, Context, & Bigger Picture
2. Formalization via Groups
   - Group Action
   - Network Symmetry Group
   - Symmetries of Network Codes and Rate Regions
3. Graph Theoretic Characterization
   - Two Special Network Coding Problem Classes
   - Automorphism Groups
   - Characterization for k-Unicast Networks
   - Characterization for MSNC Networks
   - Computation
4. Applications
   - Computation of Polyhedral Bounds on Rate Regions
   - A generalization of partition-symmetrical entropy functions

Drexel UNIVERSITY

Computation of Polyhedral Bounds on Rate Regions

- Let $\mathcal{H}'_n \triangleq \mathbb{R}^{2^n-1+n}$
- $\text{Act}\,([n]) \xrightarrow{\text{induced}} \text{Act}(\mathcal{X}_n) \xrightarrow{\text{induced}} \text{Act}(2^{X_n}) \xrightarrow{\text{induced}} \text{Act}(\mathcal{H}'_n)$

Computation of Polyhedral Bounds on Rate Regions

- Let $\mathcal{H}'_n \triangleq \mathbb{R}^{2^n-1+n}$
- $\text{Act}\,([n]) \xrightarrow{\text{induced}} \text{Act}(\mathcal{X}_n) \xrightarrow{\text{induced}} \text{Act}(2^{X_n}) \xrightarrow{\text{induced}} \text{Act}(\mathcal{H}'_n)$
- Let $\mathcal{L}^{\cap}_{12345}$ be the intersection of hyperplanes and halfspaces in $\mathcal{H}'_n$ associated with network constraints
- $\Gamma_n \cap \mathcal{L}^{\cap}_{12345}$ is stabilized setwise under action of NSG on $\mathcal{H}'_n$
- NSG gives polyhedral symmetry group of $\Gamma_n \cap \mathcal{L}^{\cap}_{12345}$
- NSG gives polyhedral symmetry group of $\text{proj}_{\boldsymbol{\omega},\boldsymbol{r}}(\Gamma_n \cap \mathcal{L}^{\cap}_{12345})$

Computation of Polyhedral Bounds on Rate Regions

- Let $\mathcal{H}'_n \triangleq \mathbb{R}^{2^n-1+n}$
- Act $([n]) \xrightarrow{\text{induced}} \text{Act}(\mathcal{X}_n) \xrightarrow{\text{induced}} \text{Act}(2^{X_n}) \xrightarrow{\text{induced}} \text{Act}(\mathcal{H}'_n)$
- Let $\mathcal{L}^{\cap}_{12345}$ be the intersection of hyperplanes and halfspaces in $\mathcal{H}'_n$ associated with network constraints
- $\Gamma_n \cap \mathcal{L}^{\cap}_{12345}$ is stabilized setwise under action of NSG on $\mathcal{H}'_n$
- NSG gives polyhedral symmetry group of $\Gamma_n \cap \mathcal{L}^{\cap}_{12345}$
- NSG gives polyhedral symmetry group of $\text{proj}_{\boldsymbol{\omega},\boldsymbol{r}}(\Gamma_n \cap \mathcal{L}^{\cap}_{12345})$
- Extend to any outer/inner polyhedral bound Γ^k_n, $k \in \{\text{in},\text{out}\}$ provided that Γ^k_n is stabilised setwise under induced action of S_n on $\mathcal{H}'_n$
- e.g. ZY, DFZ non-Shannon outer bounds, matroid inner bounds, subspace inner bounds

Outline

1. Motivation, Context, & Bigger Picture
2. Formalization via Groups
   - Group Action
   - Network Symmetry Group
   - Symmetries of Network Codes and Rate Regions
3. Graph Theoretic Characterization
   - Two Special Network Coding Problem Classes
   - Automorphism Groups
   - Characterization for k-Unicast Networks
   - Characterization for MSNC Networks
   - Computation
4. Applications
   - Computation of Polyhedral Bounds on Rate Regions
   - A generalization of partition-symmetrical entropy functions

Drexel UNIVERSITY

A generalization of partition-symmetrical entropy functions

- Let $\mathrm{Fix}_G(S)$ for $S \subseteq \mathcal{H}'_n$ be the subset of S fixed under the induced action of G on $\mathcal{H}'_n$
- Let p be the partition of $\mathcal{X}_n$ arising from orbits of $G^{\mathcal{I}}$, Σ_p be the partition respecting subgroup of S_n

A generalization of partition-symmetrical entropy functions

- Let $\mathrm{Fix}_G(S)$ for $S \subseteq \mathcal{H}'_n$ be the subset of S fixed under the induced action of G on $\mathcal{H}'_n$
- Let p be the partition of $\mathcal{X}_n$ arising from orbits of $G^{\mathcal{I}}$, Σ_p be the partition respecting subgroup of S_n
- Qi Chen and R. W. Yeung [arXiv] defined following regions:

$$\Psi_p = \mathrm{Fix}_{\Sigma_p}(\Gamma_n), \Psi^*_p = \mathrm{Fix}_{\Sigma_p}(\Gamma^*_n) \tag{13}$$

- For a MSNC instance $\mathcal{I}$, one can also define:

$$\Psi_{\mathcal{I}} = \mathrm{Fix}_{G^{\mathcal{I}}}(\Gamma_n), \Psi^*_{\mathcal{I}} = \mathrm{Fix}_{G^{\mathcal{I}}}(\Gamma^*_n) \tag{14}$$

Theorem

For a MSNC instance $\mathcal{I}$ with NSG $G^{\mathcal{I}}$ and associated orbits in $\mathcal{X}_n$ denoted as p, $\Psi_p \subseteq \Psi_{\mathcal{I}}$ and $\Psi^*_p \subseteq \Psi^*_{\mathcal{I}}$

Conclusions

- Defined a network symmetry group
- Showed that it gives symmetries of rate regions & codes
- Showed how to calculate it with graph automorphism based tools
- Mentioned applications in
  - enumeration [Li Walsh Weber Trans IT Sub 2015],
  - rate region calculation [Apte Walsh NetCod'15], and
  - partition-symmetrical entropy functions
- Provided context in computational network coding research

Drexel UNIVERSITY