

Entropic Vectors: Polyhedral Computation & Information Geometry

John MacLaren Walsh

Department of Electrical and Computer Engineering
Drexel University
Philadelphia, PA
jwalsh@ece.drexel.edu



Thanks to NSF CCF-1016588, NSF CCF-1053702, & AFOSR FA9550-12-1-0086.

Outline

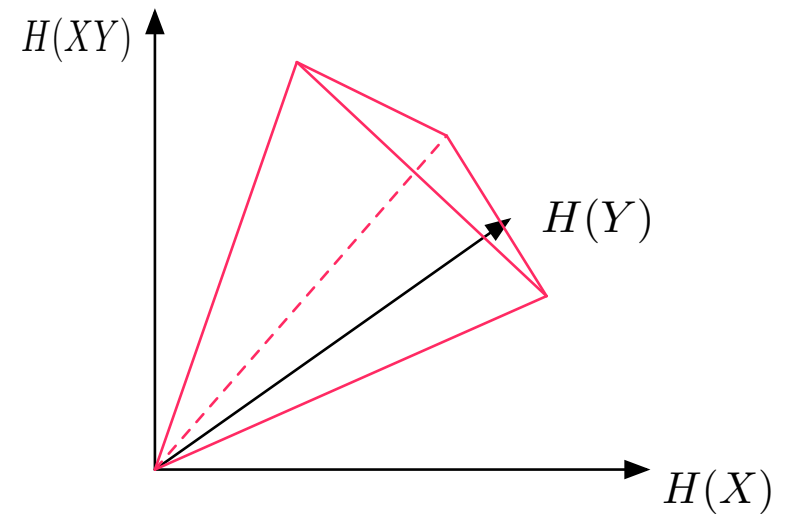
1. Entropic Vectors Review: What are they, and why should I care?
2. Entropic Vectors and Polyhedral Computation
 - (a) One Procedure to Rule Four Problems
 - i. Non-Shannon outer bounds for $\bar{\Gamma}_N^*$
 - ii. Vector Matroidal Inner bound for $\bar{\Gamma}_N^*$
 - iii. Linear Code/Subspace Rank Outer Bound
 - iv. Computing Network Coding/Distributed Storage Rate Regions
 - (b) Many Paths Lead to the Same Truth
 - (c) The Path Less Laden: Complexity Experiments & Moving Forward
3. Characterizing Extremal Entropic Vectors with Information Geometry
 - (a) Information Geometric Interpretation of Shannon type Inequalities
 - (b) Some Implications for Non-Shannon type Inequalities

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Region of Entropic Vectors $\bar{\Gamma}_N^*$ – What is it?

1. $\mathbf{X} = (X_1, \dots, X_N)$ N discrete RVs
2. every subset $\mathbf{X}_{\mathcal{A}} = (X_i, i \in \mathcal{A})$ $\mathcal{A} \subseteq \{1, \dots, N\} \equiv [N]$ has joint entropy $h(\mathbf{X}_{\mathcal{A}})$.
3. $\mathbf{h} = (h(\mathbf{X}_{\mathcal{A}}) | \mathcal{A} \subseteq [N]) \in \mathbb{R}^{2^N - 1}$ *entropic vector*
 - Example: for $N = 3$, $\mathbf{h} = (h_1, h_2, h_3, h_{12}, h_{13}, h_{23}, h_{123})$.
4. a $\mathbf{h}_o \in \mathbb{R}^{2^N - 1}$ is entropic if \exists joint PMF $p_{\mathbf{X}}$ s.t. $\mathbf{h}(p_{\mathbf{X}}) = \mathbf{h}_o$.
5. Region of entropic vectors = Γ_N^*
6. Closure $\bar{\Gamma}_N^*$ is a convex cone [1].



REV for $N = 2$:

$$H(X) \leq H(XY)$$

$$H(Y) \leq H(XY)$$

$$H(XY) \leq H(X) + H(Y)$$

$\bar{\Gamma}_N^*$ is an **unknown** non-polyhedral convex cone for $N \geq 4$.

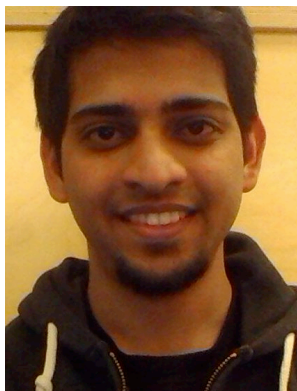
Region of Entropic Vectors $\bar{\Gamma}_N^*$ – Why should I care?

- 1:1 correspondence between every linear information inequality and supporting hyperplanes to $\bar{\Gamma}_N^*$ [1]
 - Zhang & Yeung [2, 3] showed that there were non-Shannon linear information inequalities for $N \geq 4$. [4]
 - Matùš recently definitively proved [5] there are an infinite number of information inequalities for $N \geq 4$. ($\bar{\Gamma}_4^*$ is not polyhedral)
- multi-user rate regions can be expressed as a function (linear projection) of $\bar{\Gamma}_N^*$.
 - Chan & Grant showed that for every non-Shannon inequality face of $\bar{\Gamma}_N^*$ there is a network whose capacity region depends on this non-Shannon inequality [6]
 - Network coding capacity region [1, 7] is a linear projection of $\bar{\Gamma}_N^*$ intersected with a vector subspace
- if willing to limit to distributions obeying constraints \mathcal{C} , *any* rate region in information theory is a linear projection of some $\bar{\Gamma}_N^*(\mathcal{C})$.
- Hence, most fundamental questions in information theory come back to these sets.

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Entropic Vectors and Polyhedral Computation (Development Team)



Jayant Apte

*efficient parallel
infinite precision
polyhedral computation*



Congduan Li

*rate regions &
rate delay tradeoffs*



Daniel Venutolo

*computing non-Shannon
inequalities*



Prof. Steven Weber

Drexel University

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Bounding the Region of Entropic Vectors $\bar{\Gamma}_N^*$ from the Outside

- **Shannon Outer Bound:** Γ_N . entropy is submodular:

$$I(\mathbf{X}_A; \mathbf{X}_B | \mathbf{X}_C) \geq 0 \quad \forall A, B, C$$

$$\Gamma_2 = \bar{\Gamma}_2^*, \Gamma_3 = \bar{\Gamma}_3^*.$$

$\Gamma_N \neq \bar{\Gamma}_N^*, N \geq 4$ $\bar{\Gamma}_N^*$ non-polyhedral convex cone

- **Non-Shannon Outer Bounds:** [8, 3, 2, 9, 10, 5, 11]

Yeung & Zhang, Dougherty & Freiling & Zeger, Matus

Start with 4 unconstr. r.v.s

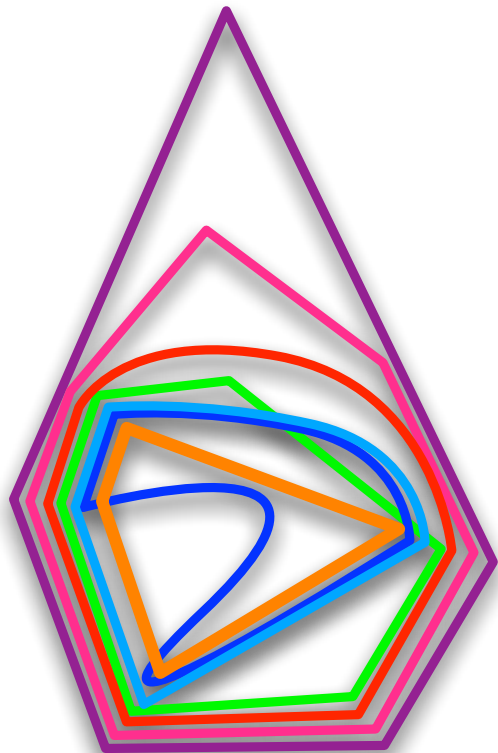
add rv. obeying distr. match & Markov. cond.

Intersect Γ_N for $N \geq 5$ w/ Markov & distr. match

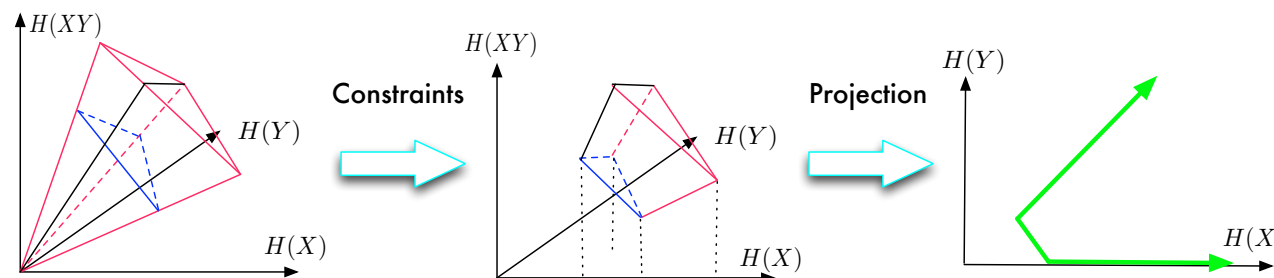
Project back to orig. 4 unconstr. vars.

obtain new information inequalities this way!

overall: Shannon \rightarrow linear eq./ineq. $\cap \rightarrow$ project



- Γ_N Shannon Outer Bound
- \mathcal{Z}_N Non-Shannon Outer Bound
- $\bar{\Gamma}_N^*$ Region of Entropic Vectors
- \mathcal{S}_N Subspace Ranks Bound
- \mathcal{M}_N^q GF(q)-Representable Matroid Bound
- Φ_4 binary entropic vectors
- $\text{conv}(\Phi_4)$ convex hull



Bounding $\bar{\Gamma}_N^*$ from the Inside, 1: Representable Matroids

Matroids: $r \in \Gamma_N \cap \mathbb{Z}^{2^N - 1}$, $r(\mathcal{A}) \leq |\mathcal{A}|$

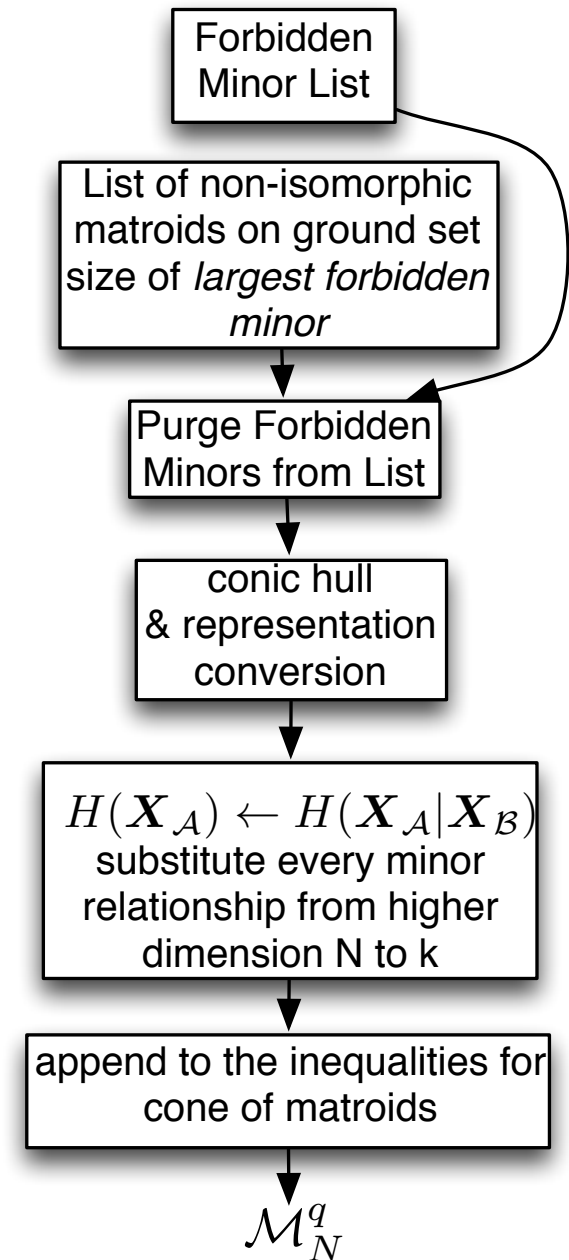
- All non-isomorphic matroids for $N \leq 9$ [12] '08
- enumerating non-iso. matroids is difficult

$GF(q)$ -Representable Matroid: $r \in \Gamma_N \cap \mathbb{Z}^{2^N - 1}$
s.t. $\exists \mathbf{A} \in GF(q)^{M \times N}$ s.t. $r(\mathcal{A}) = \text{rank}(\mathbf{A}_{:, \mathcal{A}})$

- repr. matroid = scaled EV!: $\mathbf{u} \sim \mathcal{U}(GF(q)^M)$

$$\mathbf{X} = \mathbf{u}\mathbf{A} \Rightarrow h_{\mathcal{A}} = r(\mathcal{A}) \log_2 q$$

- Key: representability \Leftrightarrow no forbidden minors:
 - *complete small list known for $q \in \{2, 3, 4\}$*
[13, 14, 15, 16, 17] eg.: $GF(2)$ repr. \Leftrightarrow no $U(2, 4)$ minor (*Tutte 1958*)
- $\bar{\Gamma}_N^*$ bound: \mathcal{M}_N^q conic hull of $GF(q)$ -repr. matroids. (*Hassibi et. al. 2010 [18]*). see right.



Bounding $\bar{\Gamma}_N^*$ from the Inside, 2: Inner Bounds from Subspace Ranks

Subspace Bounds: $r \in \Gamma_N \cap \mathbb{Z}^{2^N-1}$ projections of representable matroids, $N' \geq N$, partition $\{1, \dots, N'\} = \bigcup_{n=1}^{N'} \mathcal{G}_n$, $\mathcal{G}_n \cap \mathcal{G}_k = \emptyset$ $n \neq k$

$$r(\mathcal{A}) = \text{rank}([\mathbf{A}_{:, \mathcal{G}_n} | n \in \mathcal{A}]) \quad (1)$$

subspace ranks = scaled EV!:

$$\mathbf{X}_n = \mathbf{u} \mathbf{A}_{:, \mathcal{G}_n} \Rightarrow h_{\mathcal{A}} = r(\mathcal{A}) \log_2 q$$

\mathcal{S}_N : conic hull of all subspace ranks

\mathcal{S}_4 : $\Gamma_4 \cap$ Ingleton's [19, 20, 21]

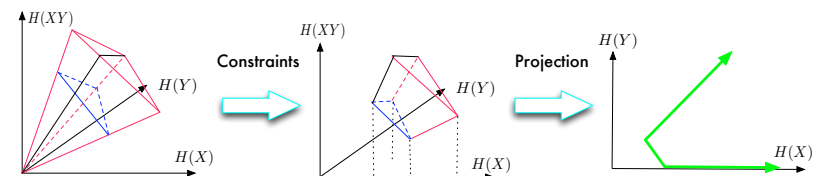
$$I(X_1; X_2) + I(X_3; X_4 | X_1) + I(X_3; X_4 | X_2) - I(X_3; X_4) \geq 0$$

\mathcal{S}_5 recently characterized by *DFZ + Kinser* [22, 23]

\mathcal{S}_N unknown for $N \geq 6$, but can inner bounded by projecting \mathcal{M}_N^q (see right)

Build inner bound for $\mathcal{S}_N \subseteq \bar{\Gamma}_N^*$:

1. Obtain $\mathcal{M}_{N'}^q$, using method from previous slide, i.e. *intersect Γ_N with inequalities from forbidden minors*
2. project (remove all but entropies where each element in \mathbf{X}_n appears together)



sound

familiar???

Shannon \rightarrow lin. \cap \rightarrow project

Bounding $\bar{\Gamma}_N^*$ from the Inside, 3: Outer Bounds for Subspace Ranks \mathcal{S}_N

- Hammer et. al.: key [20] r.v.s X, Y made with subspaces: \exists *common information* Z s.t.

$$H(Z|X) = H(Z|Y) = 0 \quad (2)$$

$$H(Z) = I(X; Y) \quad (3)$$

- Common information Z obtained by looking at component along intersection of subspaces $X Y$
- Not all RVs have common information, but rvs from subspaces do

Build outer bound for $\mathcal{S}_N \subsetneq \bar{\Gamma}_N^*$:

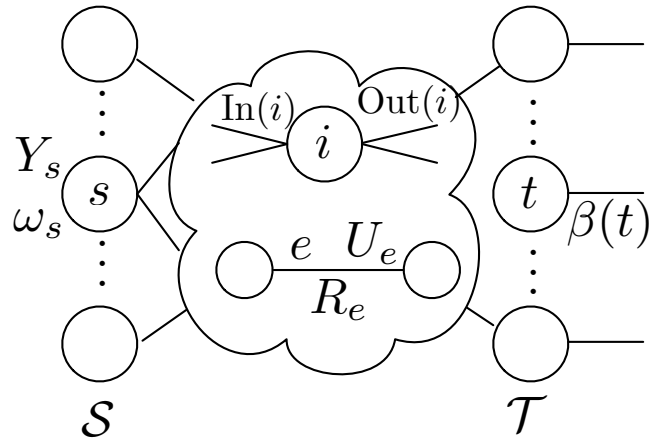
1. Shannon
2. Intersect with common information equalities (2)
3. project out the common informations Z

sound familiar?

Shannon \rightarrow lin. $\cap \rightarrow$ project

Networking Coding and Distributed Storage Rate Regions

Network Coding



(Roughly) intersect $\bar{\Gamma}_N^*$ with

$$h_{Y_s} \geq \omega_s, s \in \mathcal{S} \quad h_{Y_S} = \sum_{s \in \mathcal{S}} h_{Y_s},$$

$$h_{U_{\text{Out}(s)}|Y_s} = 0, s \in \mathcal{S}$$

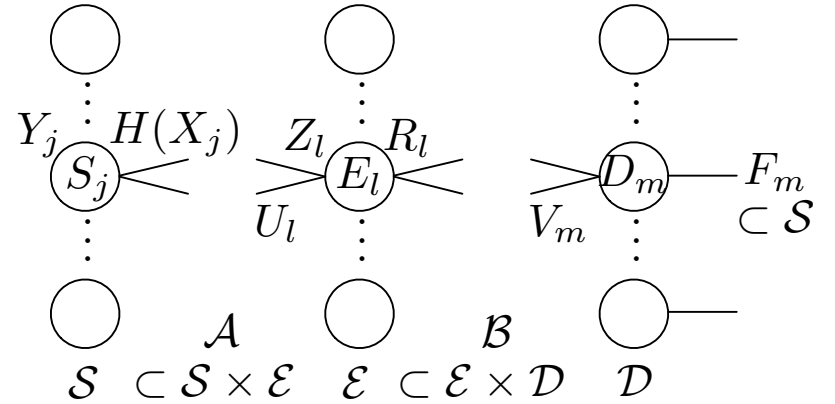
$$h_{U_{\text{Out}(i)}|U_{\text{In}(i)}} = 0, i \in \mathcal{V} \setminus (\mathcal{S} \cup \mathcal{T})$$

$$h_{U_e} \leq R_e, e \in \mathcal{E}$$

$$h_{Y_{\beta(t)}|U_{\text{In}(t)}} = 0, t \in \mathcal{T}$$

and project onto ω_s, R_e

Distributed Storage (MDCS DSCSC)



(Roughly) intersect $\bar{\Gamma}_N^*$ with

$$h_{Y_j, j \in \mathcal{S}} = \sum_{j \in \mathcal{S}} h_{Y_j}$$

$$h_{Z_l|(Y_j, j \in U_l)} = 0, l \in \mathcal{E}$$

$$h_{(Y_j, j \in F_m)|(Z_l, l \in V_m)} = 0, m \in \mathcal{D}$$

$$h_{Y_j} > H(X_j), j \in \mathcal{S}$$

$$h_{Z_l} \leq R_l, l \in \mathcal{E}$$

and project onto $\{H(X_i), R_l\}$

Substituting inner/outer bounds for $\bar{\Gamma}_N^*$, we arrive again at

Shannon \rightarrow linear equality/inequality $\cap \rightarrow$ project

One Procedure to Rule Four Problems

We've covered:

1. Non-Shannon Outer bounds for $\bar{\Gamma}_N^*$
2. Vector Matroidal Inner Bounds for $\bar{\Gamma}_N^*$
3. Outer Bound for \mathcal{S}_N (Conic hull of subspace ranks)
4. Network Coding/Distributed Storage Rate Regions

(Our Point) There were $>$ four jokes... but only one punchline

Shannon \rightarrow linear equality/inequality \cap \rightarrow project

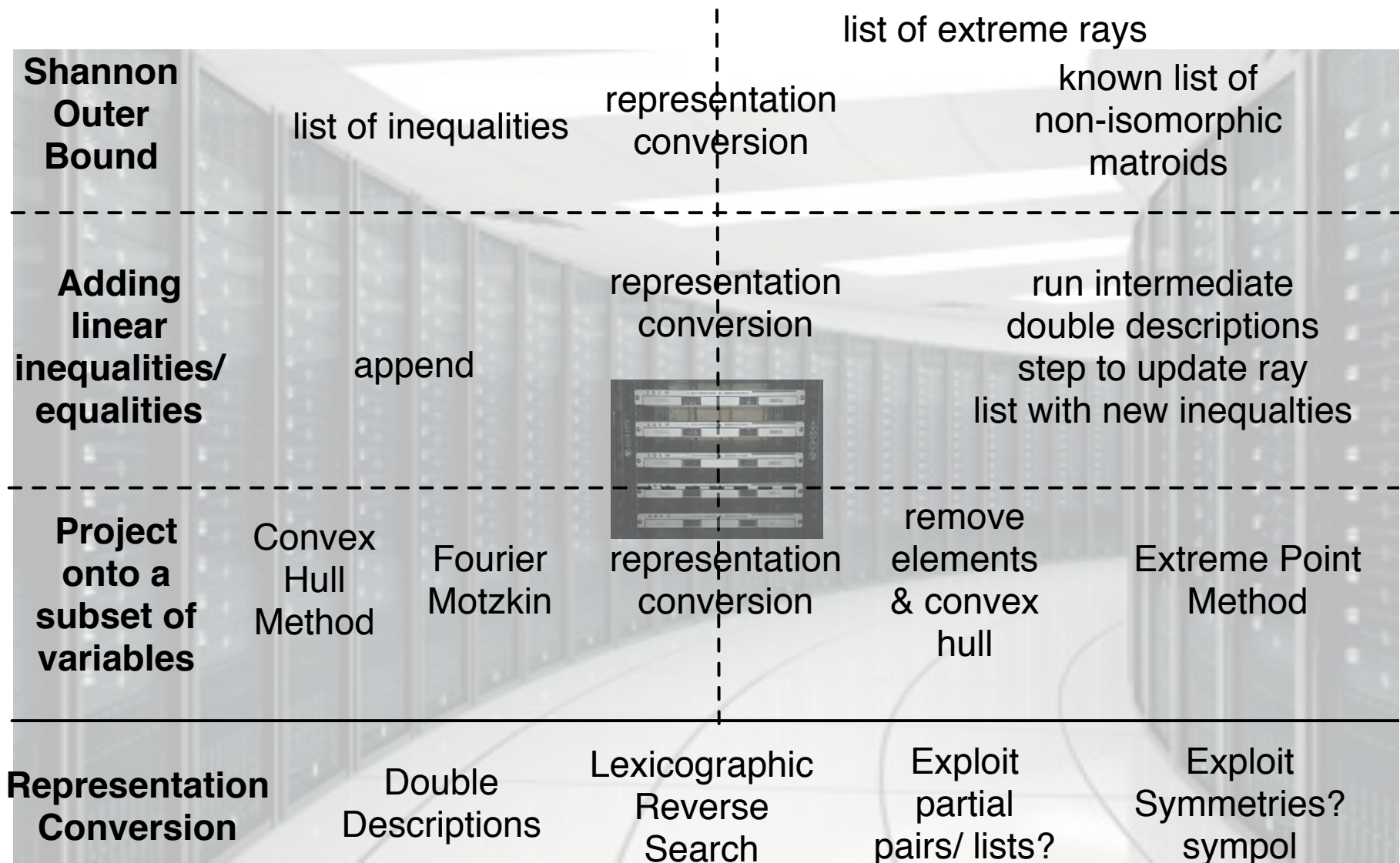
Thus, let's have a look at the general computational structure in this agenda.

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Many Paths Lead to the Same Truth

∃ wide variety of techniques (yielding mathematically equivalent results) for each step:



Parallelization: message passing VS massively parallel GPU

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The Path Less Laden: Complexity Experiments

- mathematically equal \neq computationally equal

Γ_N^{bin}	$N = 5$	6	7	8
Algorithm 2	1.3 s	11 s	150 s	3800 s
Algorithm 3	1.6 s	45 s	3000 s	36000 s
Algorithm 4	1.4 s	23 s	2000 s	25000 s

- Algorithm 2 uses substitution of conditional entropies into lower dimensional forbidden minor set to get inequality representation.
- Algorithm 3 & 4 work with non-isomorphic matroid list and remove minors (4: minor checking, 3: using 2's inequalities) to get extreme representation.

The Path Less Laden: Complexity Experiments

Rate Region	2-level-3-encoder	3-level-3-encoder
Algorithm 6	46 s	3600 s
Algorithm 7	2.9 s	47 s
Algorithm 8	2.7 s	35 s

- Algorithm 6: uses algorithm 2 to get inequalities of inner bound, appends rate region equalities/inequalities, and projects using Fourier Motzkin. (inequality based)
- Algorithm 7 & 8 : adds rate regions ineq.s & eq.s to alg. 3 and 4. inner bound extreme rays via steps of double descriptions, then projects

total time winners are not always concatenation of the winners at each stage

The Path Less Laden: Moving Forward

- There is an absurd amount of symmetry in these problems!
 - Labeling of variables, but also
 - Shannon is only one inequality! conditional mutual info ≥ 0
- Key factor in the computation is exploiting the known symmetry to ease the computation
- 2010 Diploma Thesis from Thomas Rehn Univ. Magdeburg (now at Uni. Rostock)
 - sympol
 - “representation conversion up to symmetries”. D. Bremner & A. Schurmann
- Also, given that there is no unique “best path” and “best representation”, yet a rich theory regarding which algorithms are good for which structures, need parallel tools that try the right candidates out then select which ones finish first
 - area of active software development (what our team is developing)
- given proof nature of results, need infinite precision arithmetic

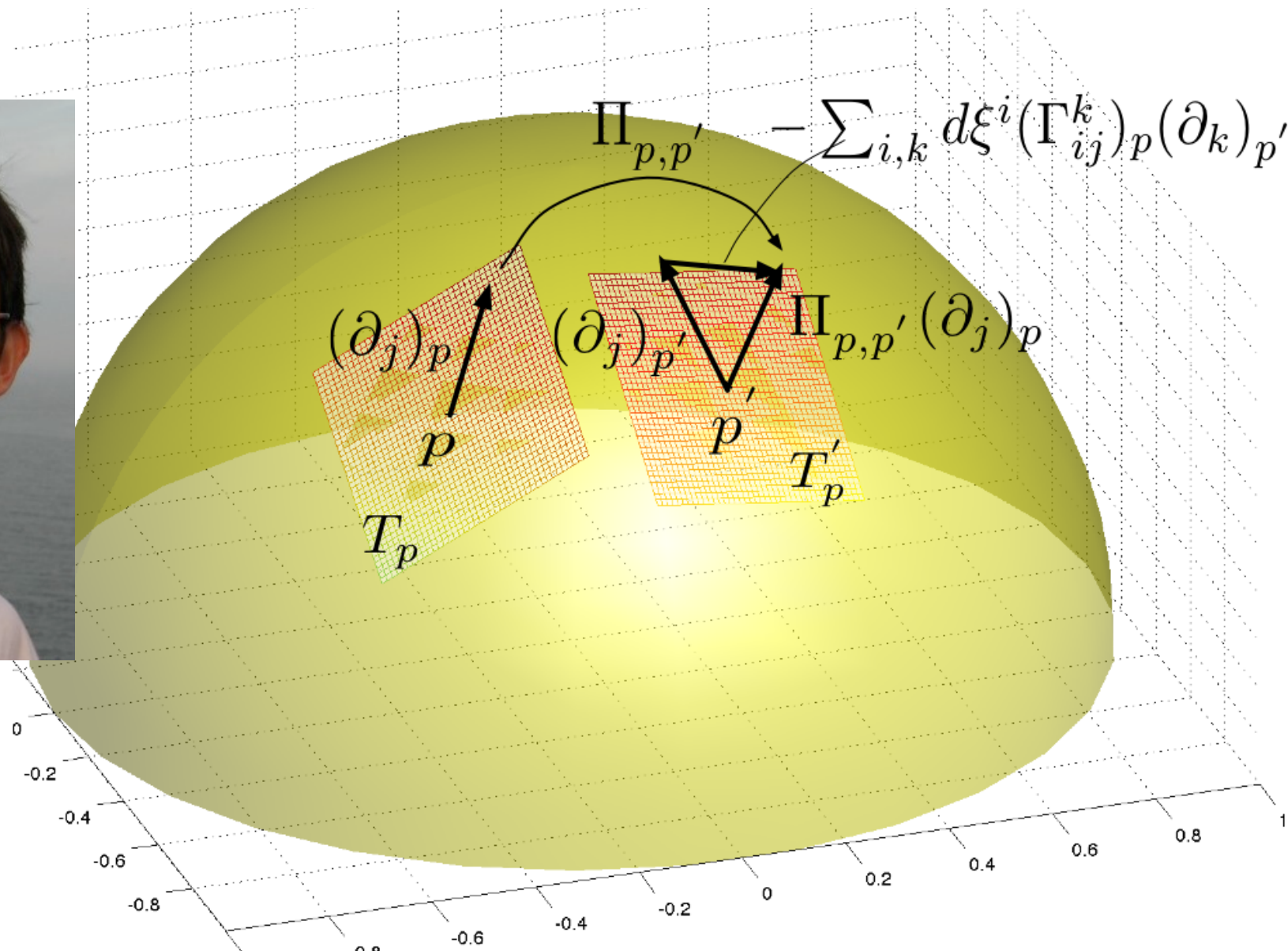
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Characterizing Extremal Entropic Vectors with Information Geometry (Lead Student)



Yunshu Liu



Characterizing Extremal Entropic Vectors with Information Geometry: Motivation & Goals

Since $\bar{\Gamma}_N^*$ is a non-polyhedral convex cone:

- Need a tool that is not limited to (tightening of) polyhedral bounds
- Need a tool to handle non-linear codes!: $\mathcal{S}_N \subsetneq \bar{\Gamma}_N^* \forall N \geq 4$
 - Bye bye representable matroids. more general matroids promising path, but:
 - * a pain to enumerate (list gigantic and unknown $N \geq 10$)
 - * discrete \implies conic hulls & rep. conv. nec. for REV. also expensive
 - * algebraic matroids are far less understood than representable. other tools?

Extreme rays of $\bar{\Gamma}_N^*$ and \cap correspond to efficient codes, hence:

- Want to parameterize the EVs and PMFs on boundary of $\bar{\Gamma}_N^*$
 - (esp. new extreme rays not shared with Shannon)

Information geometry:

- endows differential geometric structure to set of joint PMFs (parameterizations!)
- coord.'s flatness & certain affine sets = familiar properties
 - marginals, independence, conditional independence
- studies divergences (incl. KL) & projections that are easily related to entropy

Hence, information geometry seems a natural candidate to deal with these questions

Key Relevant Quantities from Information Geometry

m-coordinates:

$$\boldsymbol{\eta} = \left[p_{\mathbf{X}}(v_{i_1,1}, \dots, v_{i_N,N}) \mid i_k \in \{2, \dots, |\mathcal{X}_k|\}, k \in \{1, \dots, N\} \right]$$

e-coordinates: $\prod_{n=1}^N |\mathcal{X}_n| - 1$ elements take the form

$$\boldsymbol{\theta} = \left[\log \left(\frac{p_{\mathbf{X}}(v_{i_1,1}, \dots, v_{i_N,N})}{p_{\mathbf{X}}(v_{1,1}, \dots, v_{1,N})} \right) \mid \begin{array}{l} i_k \in \{2, \dots, |\mathcal{X}_k|\}, \\ k \in \{1, \dots, N\} \end{array} \right]$$

m-autoparallel submanifold (affine subset of m-coords) fix \mathbf{A}, \mathbf{b} all $\boldsymbol{\eta}$ of the form

$$\boldsymbol{\eta} = \mathbf{A}\boldsymbol{p} + \mathbf{b} \tag{4}$$

example: all joint distributions $p_{X,Y}$ having a given marginal p_X

e-autoparallel submanifold (affine subset of e-coords) fix \mathbf{A}, \mathbf{b} all $\boldsymbol{\theta}$ of the form

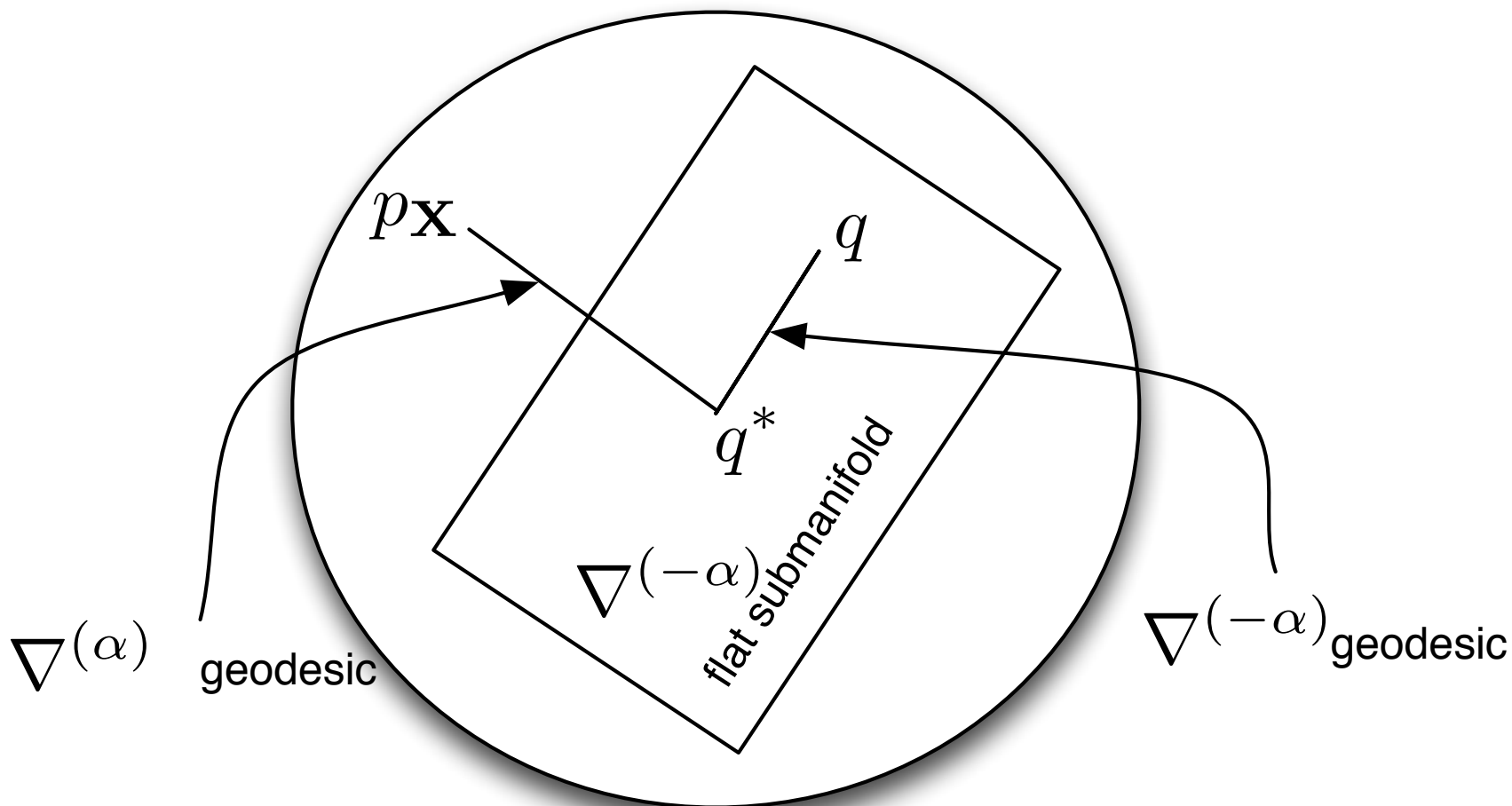
$$\boldsymbol{\theta} = \mathbf{A}\boldsymbol{\lambda} + \mathbf{b} \tag{5}$$

example: all joint distributions $p_{X,Y}$ with X, Y independent.

e-geodesic/m-geodesic: one dimensional affine manifolds

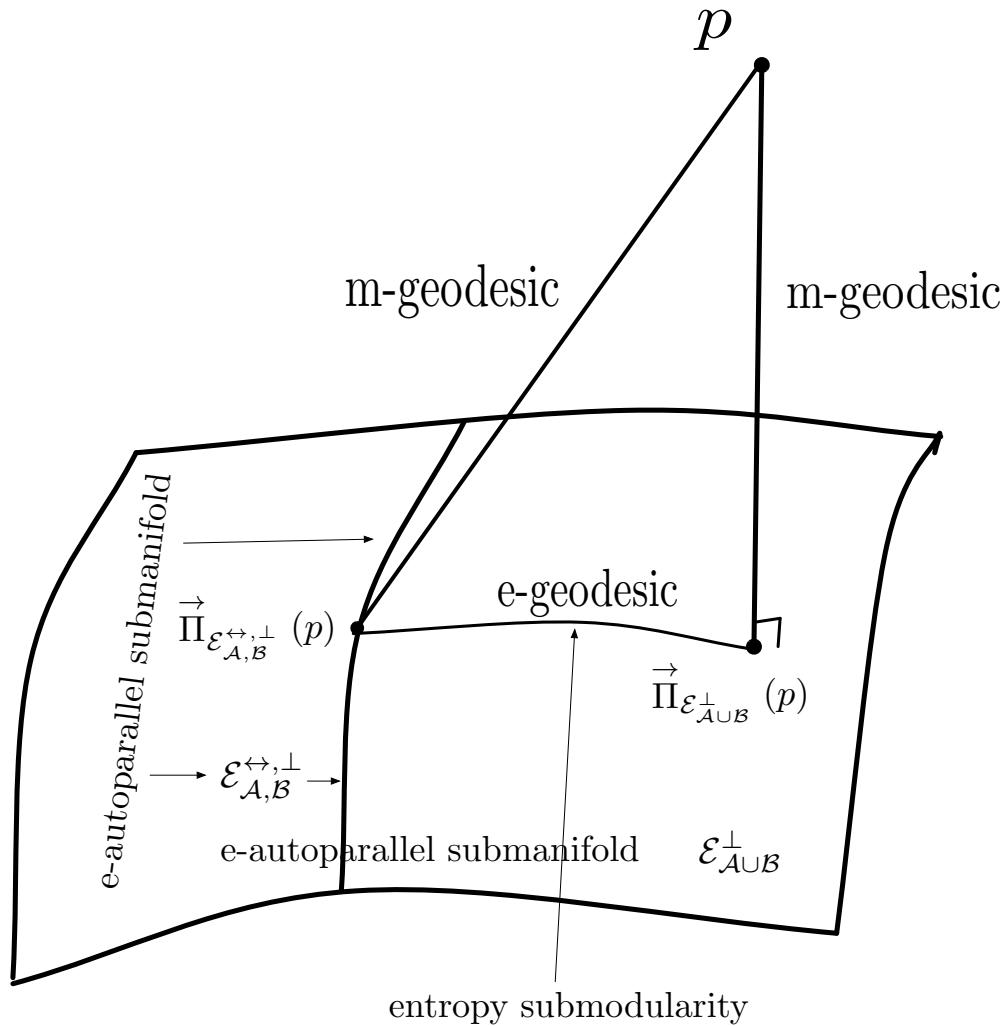
Key Relevant Quantities from Information Geometry, cont'd

$$\vec{\Pi}_{\mathcal{E}}(p_{\mathbf{X}}) = q^* = \arg \min_{q \in \mathcal{E}} D(p_{\mathbf{X}} | q) \quad (6)$$



$$D^{(\alpha)}(p_{\mathbf{X}} || q) = D^{(\alpha)}(p_{\mathbf{X}} || q^*) + D^{(\alpha)}(q^* || q)$$

Information Geometric Properties of Distributions on Shannon Facets



$$h_A + h_B \geq h_{A \cup B} + h_{A \cap B}$$

$$\mathcal{E}_{A,B}^{\leftrightarrow,\perp} = \left\{ \theta \mid p_{\mathbf{X}} = p_{\mathbf{X}_{A \setminus B} \mid \mathbf{X}_{A \cap B}} p_{\mathbf{X}_B} p_{\mathbf{X}_{(A \cup B)^c}} \right\}$$

$$\mathcal{E}_{A,B}^\perp = \left\{ \theta \mid p_{\mathbf{X}} = p_{\mathbf{X}_{A \cup B}} p_{\mathbf{X}_{(A \cup B)^c}} \right\}$$

- Shannon outer bound:
 $I(\mathbf{X}_A; \mathbf{X}_B \mid \mathbf{X}_C) \geq 0$
- Hence, on the Shannon facet:
 $I(\mathbf{X}_A; \mathbf{X}_B \mid \mathbf{X}_C) = 0$
- means $\mathbf{X}_A \leftrightarrow \mathbf{X}_C \leftrightarrow \mathbf{X}_B$
- This is an e-autoparallel submanifold of $p_{\mathbf{X}_{A \cup B \cup C}}$!
- \implies those $p_{\mathbf{X}_{A \cup B \cup C}}$ on this boundary (affine set) of entropy have a parameterization in which they are also affine (known \mathbf{A}, \mathbf{b})
- Sometimes $\mathbf{X} \neq \mathbf{X}_{A \cup B \cup C}$, so also need the structure having a particular marginal $p_{\mathbf{X}_{A \cup B \cup C}}$ (m-autoparallel)
- mutually dual foliations

Some Implications for Non-Shannon type Inequalities

- At $N = 4$, there is one extreme ray form (6 permutations) of Γ_N that is non-entropic [20, 24]
- Every Shannon facet contains at least one of these 6 non-entropic extreme rays [20, 24]
- Can easily parameterize distributions for entropic parts of that facet with information geometry.
- Use these parameterizations to (numerically) get better inner bounds (current)

another stab, towards characterizing distributions assoc. w/ non-Shannon extreme rays:

- Because $\mathcal{S}_4 \subset \bar{\Gamma}_N^*$, $\bar{\Gamma}_4^\circ \subset \mathcal{S}_4^\circ$. Hence, any non-Shannon information inequality is weighted sum of Ingleton with basic (Shannon) inequalities
- Ingleton, in turn, only has one negative mut. inf.
- \implies existence of a distribution giving entropic point w/ = in non-Shannon ineq
 \Leftrightarrow existence of point achieving equality between projection divergence,, and a weighted sum of other projection divergences

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