LOSSY INTERACTIVE SUM MODULO TWO COMPUTATION OF BINARY SOURCES

Solmaz Torabi, and John MacLaren Walsh

Dept. of Electrical and Computer Engineering
Drexel University

ABSTRACT

In the lossy interactive function computation considered in this paper, two encoders observe discrete sources, and the receiver who has a correlated side information wishes to compute a function of all the random variables, subject to a distortion constraint. We assume that the sources are independent given the side information. To this aim, the sources take turn and broadcast their messages which are heard perfectly by other sources and also the Central Estimation Officer (CEO), but the CEO doesn’t participate in the communication. We determine the rate distortion region for this problem. Another model when the side information is not available to the decoder is also presented. An example of this model, is a lossy interactive variant model of Körner-Marton problem, for which an explicit expression for the rate distortion region as well as the optimal forward test channel for this problem are obtained. We also make a connection between the interaction in a lossless regime, and one shot Körner-Marton problem.

INDEX TERMS— Lossy Compression, function computation, interactive communication

1. INTRODUCTION

Distributed function computation where correlated sources are separately encoded and jointly decoded to reproduce a function of the sources have gained a lot of attentions. [1] established the first fundamental lossless theorem for a point to point case where a function of two sources are reconstructed with one source being uncoded and available to the decoder. This was further generalized to a distributed case [2]. However, the generalization is not yet completely solved for a general function of arbitrary correlated sources [3]. Efforts have been, since then, extended to studying special correlation among the sources [4], and computing special functions [5]. Körner and Marton in [5] provide a method for losslessly computing the binary sum of doubly symmetric binary sources (DSBS) as depicted in Fig. 2 without the need to communicate the sources directly. [6] provided inner and outer bound for the rate region of the the sources that have neither independence nor symmetric property. If the sources are allowed to cooperate, further results of the rate region is obtained [7] for independent sources. [8] considered this problem in a lossy regime with presence of the side information, which subsumes the model in [7].

A main source of difficulty after obtaining the rate distortion expression is computing the region, which usually admits cardinality bounds on its auxiliary random variables. [7] provides a convex geometric approach algorithm to compute the region for losslessly reproducing a function of distributed independent sources. However, from a practical standpoint, for small problems, it uses a large series of convex hulls, which is unlikely to be computationally feasible in many contexts. [9] also provides a Blahut Arimoto type algorithm to iteratively compute the rate distortion region in one shot scenario. In this paper we first formally define the problem (Section §2). Next, find the rate-distortion region for lossy interactive Körner-Marton type problem. We then explicitly evaluate the region for computing the binary sum of two binary symmetric sources (Section §3). We also make a connection between the interaction in a lossless regime, and one shot Körner-Marton problem. Next (Section §4) we consider the model when the side information is present, which given the side information the sources become independent, and the function of all the sources needs to be computed at the decoder. We show that the side information can optimally handle the randomness that is common between the sources, therefore, the optimal rate distortion region is can be derived. Ax example with binary sources is also provided (Section §4.1) which shows achievable forward test channels for computing binary sum of three binary sources, however, does not claim the optimality of such channels.

2. PROBLEM FORMULATION

Consider a network with two source terminals and a single sink terminal as depicted in Fig. 1. Terminal \( j = 1, 2 \) observes a random sequence \( X_j^n \) = \( (X_j(1), ..., X_j(N)) \) ∈ \( \mathcal{X}_j^n \). The sink node, the central estimation officer (CEO), observes a side information sequence \( X_3(n), n \in [N] \) which is assumed to be correlated with the source variables. The random vectors \( X_{1,3}(n) = (X_1(n), X_2(n), X_3(n)) \) are iid in time \( n \in [N] \), and, \( X_1(n) \leftrightarrow X_3(n) \leftrightarrow X_2(n) \). The sink terminal wishes to compute \( f : \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3 \rightarrow \mathcal{Z} \) in a lossy manner elementwise, estimating the sequence \( Z^n = (Z(1), ..., Z(N)) \) with \( Z(n) = f(X_1(n), X_2(n), X_3(n)) \). Let \( d : \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3 \times \mathcal{Z} \rightarrow \mathbb{R}^+ \) be a fidelity criterion for this function computation, yield-
In order to enable the CEO to estimate the function computation, the nodes take turns broadcasting messages which are perfectly received at both the other nodes and the CEO. The communication is initiated by node 1, and at round $i$, node $j = ((i - 1) \mod 2) + 1$ sends a message $M_i \in \mathcal{M}_i$, using the encoding function $\psi_i : \chi_i^N \times \bigotimes_{k=1}^{i-1} \mathcal{M}_k \to \mathcal{M}_i$ to encode its observations based on the previously overheard messages. After $t$ rounds of communication, the CEO estimates the function sequence based on the messages it has received and the side information using the decoding function $\phi : \chi_i^N \times \bigotimes_{k=1}^{t} \mathcal{M}_k \to \chi_i^N$.

**Definition 1.** A rate-distortion tuple $(R, D) = (R_1, R_2, ..., R_t, D)$ is admissible for $t$-message interactive function computation, if $\forall \epsilon > 0$, and $\forall N > n(\epsilon, t)$, there exist encoders $\psi_i, i \in \{1, ..., t\}$ and a decoder $\phi_{\psi}$ with parameters $(t, N, |\mathcal{M}_1|, ..., |\mathcal{M}_t|)$ satisfying
\[
\frac{1}{N}\log_2 |\mathcal{M}_i| \leq R_i \forall i = 1, ..., t \quad \text{and} \quad \mathbb{E}[d(N)(\chi_1^N, \hat{\chi}_1^N)] \leq D + \epsilon \\
\text{with } \hat{\chi}_1^N = \phi(M_1, ..., M_t, \chi_1^N).
\]

Finally, define the collection of admissible rate and distortion vectors $(R, D)$ to be $\mathcal{RD}^t$. We use this notations again in Section §4, where the complete characterization of the rate region of Fig. 1 is provided.

\[
\begin{aligned}
\begin{array}{c}
\chi_1^N \\
\chi_2^N \\
\chi_3^N
\end{array}
\end{aligned}
\quad
\begin{aligned}
\begin{array}{c}
\text{Enc} & R_1 & D = 0 \\
\text{Enc} & R_2
\end{array}
\end{aligned}
\quad
\begin{aligned}
\begin{array}{c}
\hat{\chi}_1^N \\
\hat{\chi}_2^N
\end{array}
\end{aligned}
\quad
\begin{aligned}
\begin{array}{c}
\chi_1^N \\
\chi_2^N \\
\chi_3^N
\end{array}
\end{aligned}
\quad
\begin{aligned}
\begin{array}{c}
\text{Enc} & R_1 & D \\
\text{Enc} & R_2
\end{array}
\end{aligned}
\quad
\begin{aligned}
\begin{array}{c}
\hat{\chi}_1^N \\
\hat{\chi}_2^N
\end{array}
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\begin{aligned}
\begin{array}{c}
\chi_1^N \\
\chi_2^N \\
\chi_3^N
\end{array}
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\begin{aligned}
\begin{array}{c}
\text{Enc} & R_1 & D \\
\text{Enc} & R_2
\end{array}
\end{aligned}
\quad
\begin{aligned}
\begin{array}{c}
\hat{\chi}_1^N \\
\hat{\chi}_2^N
\end{array}
\end{aligned}
\quad
\begin{aligned}
\begin{array}{c}
\chi_1^N \\
\chi_2^N \\
\chi_3^N
\end{array}
\end{aligned}
\end{aligned}

\text{Fig. 2: Körner-Marton Problem}

\text{Fig. 3: Lossy Interactive Körner-Marton Problem}

\text{Fig. 4: Lossy Interactive Binary sum of two binary sources with presence of a side information}

3. INTERACTIVE LOSSY KÖRNER-MARTON PROBLEM

In Fig. 2, we have two dependent sources, $X_1 = X \sim \text{Bern}(1/2)$, and $X_2 = X + Z_1$ with $Z_1 \sim \text{Bern}(p_1)$ being independent of $X$, and the CEO wished to compute the binary sum of the sources, $X_1 \oplus X_2 = Z_1$ losslessly. Therefore, $(X_1, X_2) \sim \text{DSBS}(p_1)$. Hence the joint distributions of random variables are as follows.

\[
p(x_1, x_2) = \begin{cases} \frac{p_1}{2} & \frac{1-p_1}{2} \\
\frac{p_2}{2} & \frac{1-p_2}{2}
\end{cases}.
\]

This is known as Körner-Marton problem. It’s that to compute this function the set of rate required are $R_1 \geq h(p_1)$, and $R_2 \geq h(p_2)$, which improves the Slepian-Wolf rate region. This is among one of the first examples that to show compression for computing a function without the need to communicating the sources. Fig. 3, shows an interactive lossy variant model of Körner and Marton problem, where the binary sum should be computed subject to the Hamming distortion. That is, for $z_1, \hat{z}_1 \in \{0, 1\}$, we have $d(z_1, \hat{z}_1) = 0$ if $z_1 = \hat{z}_1$, and $d(z_1, \hat{z}_1) = 1$ if $z \neq \hat{z}_1$. Therefore, the expected distortion is the mod 2 sum, that is, $\mathbb{E}(d(Z, \hat{Z})) = p(Z_1 \neq \hat{Z}_1) = h^{-1}(H(Z_1 \oplus \hat{Z}_1))$. Inspired from [4], since one source is a deterministic function of the other source, and the decoder seeks to reproduce a random variable independent of one of the sources, the complete rate distortion region can be characterized in Theorem 1. Note that, this model is not entirely similar to the model in [10], since here there is a third party (CEO) who reconstructs $Z_1$ with distortion $D$, hence, the reproduction must be formed from the quantizations $U_1$, and $U_2$ only. Whereas, in the Kaspi’s two way source coding theorem, each decoding function depends on one of the sources as well. This makes the derivation of the region different, since the decoder in Fig. 3 has less information than as in Kaspi’s. In the next theorem the complete characterization of the rate distortion region is derived.

**Theorem 1.** For the model in Fig. 3 where the first node observes $X_1 = X$, and the second node observes $X_2 = X \oplus Z_1$, and the CEO reconstructs $X_1 \oplus X_2 = Z_1$ with distortion $D$, with a constraint that the second message deterministically depends on the first message, the complete characterization of the rate distortion region can be derived as follows.

\[
\mathcal{RD}^2 = \left\{ (R, D) \left| \begin{array}{c}
R_1 \geq I(X; U_1) \\
R_2 \geq I(X \oplus Z_1; U_2| U_1) \\
\mathbb{E}[d(Z_1, \hat{Z}_1)] \leq D \\
U_1 \leftrightarrow X \leftrightarrow X \oplus Z_1 \\
U_2 \leftrightarrow X \leftrightarrow Z_1, U_1 \leftrightarrow X \end{array} \right. \right\}
\]

(1)

**Proof.** The achievability proof resembles [8], so we only present the converse proof. We define the auxiliaries $U_1(n) := \{M_1, X_1^{n-1}, X_2^n\}$, $U_2(n) := M_2$. The rate for the first round can be lower bounded as

\[
R_1 \geq H(M_1) \\
\geq I(X_1^N, X_2^N; M_1) \\
= \sum_{n=1}^N H(X_1(n)X_2(n)) - H(X_1(n)X_2(n)|M_1X_1^{n-1}, X_2^n) \\
\geq \sum_{n=1}^N H(X_1(n)) - H(X_1(n)|M_1X_1^{n-1}, X_2^n) \\
= \sum_{n=1}^N I(X_1(n); U_1(n))
\]

(2)

in $a$ we used the positivity of the conditional mutual information.
This proves the converse.

And for the second round:

\[
R_2 \geq H(M_2)
\]

\[
\geq I(X_2^n; X_1^n; M_2|M_1)
\]

\[
= \sum_{n=1}^{N} H(X_1(n)X_2(n)|M_1X_2^nX_1^n) - H(X_2(n)|M_1M_2X_2^{n-}X_1^n)
\]

\[
\geq \sum_{n=1}^{N} H(X_2(n)|M_1X_2^nX_1^n) - H(X_2(n)|M_1M_2X_2^{n-}X_1^n)
\]

\[
= \sum_{n=1}^{N} I(X_2(n); U_2(n)|U_1(n))
\]

(3)

The Markov constraints \(M_1, X_1^n, X_2^n \leftrightarrow X_1(n) \leftrightarrow X_2(n)\), and \(M_2 \leftrightarrow X_2(n)\), \(M_1, X_1^n, X_2^n \leftrightarrow X_1(n)\) can be verified in Fig. 5. To prove the single letter characterization of the distortion, we use the converse assumption, that there exists a decoding function \(\phi(M_1, M_2)\), with \(n\)th element \(\phi_n\), obeying

\[
D \geq \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}[d(Z_1(n), \phi_n(M_1, M_2))]
\]

(4)

Define the function \(g: M_1 \times M_2 \rightarrow \mathbb{Z}_1^N\) with \(n\)th element \(g_n\), to be the Bayes detector for \(Z_1(n)\) from \(M_1, M_2\):

\[
g_n(M_1, M_2) = \arg \min_{z \in \mathbb{Z}_1} \mathbb{E}[d(Z_1(n), z)|M_1, M_2].
\]

(5)

Defining \(g_n\) via (5) shows that

\[
\mathbb{E}[d(Z_1(n), \phi_n(M_1, M_2))] \geq \mathbb{E}[d(Z_1(n), g_n(M_1, M_2))]
\]

(6)

Next, define \(\tilde{g}_n\) to be the Bayes detector for \(Z_1(n)\) from \(M_1, M_2, X_1^n, X_2^n\), i.e. let \(\tilde{g}_n(M_1, M_2, X_1^n, X_2^n) = \arg \min_{z \in \mathbb{Z}_1} \mathbb{E}[d(Z_1(n), z)|M_1, M_2, X_1^n, X_2^n].
\]

(7)

The optimality (7) then shows

\[
\mathbb{E}[d(Z_1(n), g_n(M_1, M_2))] \geq \mathbb{E}[d(Z_1(n), \tilde{g}_n(M_1, M_2, X_1^n, X_2^n))]
\]

(8)

Putting together (4), (6), and (8), we have

\[
D \geq \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}[d(Z_1(n), \tilde{g}(U_1(n), U_2(n)))]
\]

(9)

This proves the converse.

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Here we evaluate and the region for the binary sources with hamming distortion.

**Theorem 2.** For the interactive lossy Körner-Marton problem, \((R, D)\) is achievable if and only if

\[
R_1 \geq 1 - h(\alpha)
\]

\[
R_2 \geq h(\alpha + p_U) - h(\beta)
\]

for some \(0 \leq \beta, \alpha \leq 1/2\) such that \(\beta + \alpha = D\).

**Proof.** Achievability: Let \(U_1\) be the output of BSC(\(\alpha\)) with input \(X\). Therefore, we have \(X = U_1 \oplus N_1\) where \(N_1 \sim \text{Bern}(\alpha)\). The second user receiving \(U_1\), generates \(U_2\) which is the output of BSC(\(\beta\)) with input \(X \oplus Z_1 \oplus U_1\).

\[
X \oplus Z_1 \oplus U_1 = U_2 \oplus N_2
\]

(12)

where \(N_2 \sim \text{Bern}(\beta)\). We set the decoding function \(\tilde{g}\) to be \(\tilde{g}(U_1, U_2) = U_2\). Hence, the distortion achieved at the receiver is

\[
\mathbb{E}[d(Z_1, \tilde{Z}_1)] = p(Z_1 \oplus \tilde{Z}_1 = 1)
\]

\[
= p(Z_1 \oplus U_2 = 1)
\]

\[
= p(U_1 \oplus N_1 \oplus U_1 \oplus N_2 = 1)
\]

\[
= p(U_1 \oplus N_1 \oplus U_1 \oplus N_2 = 1) = \alpha * \beta
\]

(13)

where \(a\) follows from (12). In \(b\), we used \(X = U_1 \oplus N_1\). Note that, since \(\alpha * \beta\) is a monotonically increasing and continues function in both \(\alpha\) and \(\beta\), all the distortion \(0 \leq D \leq 1/2\) can be achieved by this scheme. For \(D \geq 1/2\) we can simply let \(\tilde{g}(U_1, U_2) = 0\). The scheme explained above that achieves distortion \(\alpha * \beta\) requires the rate \(R_1 \geq I(X; U_1)\) as in (1).

\[
I(X; U_1) = H(X) - H(X|U_1)
\]

\[
= 1 - h(\alpha)
\]

(14)

and the second rate

\[
I(X \oplus Z_1; U_2|U_1)
\]

\[
= H(X \oplus Z_1|U_1) - H(X \oplus Z_1|U_1, U_2)
\]

\[
= h(\alpha + p) - h(\beta)
\]

(15)

\[
\square
\]

4. THREE-NODE LOSSY INTERACTIVE FUNCTION COMPUTATION

In this section there is another random variable involved which is provided to the decoder to help the decoder compute a function of all of the sources. For an arbitrary number of users, an inner and outer bound for this problem is provided in [8], which is not tight in general. In this section, we give a matching outer bound will be proved to be tight for a correlated two-user and a side information scenario, where given the side information, the observations are independent. Note that the model handled in Section 3 is not included in Fig. 1, and Fig. 4. The reason is that in Fig. 3 there is no side information available to the decoder to optimally handle the randomness that is common to \(X_1\) and \(X_2\).
Theorem 3. For a two-user lossy interactive function computation with a side information, where node $i \in \{1, 2\}$ observes $X_i$ and the CEO observes $X_3$ such that $X_1 \leftrightarrow X_3 \leftrightarrow X_2$, the whole rate region can be obtained as follows:

$$\mathcal{R}_i \geq \{(R_1, D) \mid \begin{align*}
R_1 &\geq I(X_1; U_i|U_1, X_2) \quad \text{i odd} \\
R_1 &\geq I(X_2; U_i|U_1, X_1) \quad \text{i even} \\
|\mathcal{U}_i| &\leq |X_i| \sum_{i=1}^{i-1} |\mathcal{U}_i| + 1 + t - i \\
\mathbb{E}[d(X_3, \tilde{g}(U_1, X_3))] &\leq D \\
U_i \leftrightarrow X_j, U_{1:i-1} \leftrightarrow X_{(1,2,3)}(j)
\end{align*} \right\}$$

(16)

where the alphabets $\mathcal{U}_i$ satisfies

$$|\mathcal{U}_i| \leq |X_i| \sum_{i=1}^{i-1} |\mathcal{U}_i| + 1 + t - i$$

Proof. The achievability proof resembles [8], so we just prove the converse. The lower bound for the rate in the first round can be derived as follows:

$$R_1 \geq H(M_i)$$

$$= I(Z_2^n, Z_3^n; M_i|Z_1^n)$$

$$= \sum_{n=1}^{N} H(Z_2(n)|Z_3(n)|Z_1(n)) - H(Z_2(n)|Z_3(n), Z_1(n))|M_i Z_1^n - Z_3^n + Z_2^n$$

$$\geq \sum_{n=1}^{N} H(Z_1(n)|Z_2(n)) - H(Z_1(n)|M_i Z_1^n - Z_3^n + Z_2^n)$$

$$\geq \sum_{n=1}^{N} H(Z_1(n)|Z_2(n)) - H(Z_1(n)|M_i Z_1^n, Z_3^n, Z_2^n)$$

$$\geq \sum_{n=1}^{N} I(Z_1(n); U_1(n)|Z_2(n))$$

(17)

For the first term in $b_1$ we used conditioning reduces the entropy, and for the second term we have $Z_2(n) \leftrightarrow M_i Z_1^n, Z_3^n, Z_2^n \leftrightarrow Z_3^n$. In $b_2$ we have $Z_2(n) \leftrightarrow M_i Z_1^n, Z_3^n, Z_2^n \leftrightarrow Z_3^n, Z_1(n) \leftrightarrow Z_1^n$. See Figure 6 for the proof. In $b_3$ we defined the auxiliaries $U_i := \{M_i\}$ for $i \geq 2$.

In $a_1$ we have $Z_3^n \leftrightarrow M_i Z_1^n - Z_3^n + Z_2^n \leftrightarrow Z_1(n)$. See Figure 6 for the proof. In $a_2$ the auxiliary is chosen to be $U_1(n) := \{M_1, Z_1^n - Z_3^n, Z_2^n \}$.
For $i \geq 2$, and $i$ odd we have

$$R_i \geq \mathcal{H}(M_i)$$

$$= I(Z^n; M_i; M_{i+1}, Z^N_n)$$

$$\geq \sum_{n=1}^{N} H(Z_i(n)|Z_i(n-1), Z^N_i, |i, i+1, Z^n_1, Z^n_2, Z^n_3)$$

$$- H(Z_i(n)|Z_i(n-1), Z^N_i, |i, i+1, Z^n_1, Z^n_2, Z^n_3)$$

$$\geq \sum_{n=1}^{N} H(Z_i(n)|M_{i+1}, Z^N_i, |i, i+1, Z^n_1, Z^n_2, Z^n_3)$$

$$- H(Z_i(n)|M_{i+1}, Z^N_i, |i, i+1, Z^n_1, Z^n_2, Z^n_3)$$

$$\geq \sum_{n=1}^{N} H(Z_i(n)|Z_{i+1}, Z^n_i, |i, i+1, Z^n_1, Z^n_2, Z^n_3)$$

$$- H(Z_i(n)|Z_{i+1}, Z^n_i, |i, i+1, Z^n_1, Z^n_2, Z^n_3)$$

$$\geq \sum_{n=1}^{N} H(Z_i(n)|Z_{i+1}, Z^n_i, |i, i+1, Z^n_1, Z^n_2, Z^n_3)$$

$$- H(Z_i(n)|Z_{i+1}, Z^n_i, |i, i+1, Z^n_1, Z^n_2, Z^n_3)$$

$$\geq \sum_{n=1}^{N} I(Z_i(n); U_i(1, U_{i+1}, Z_2(n))$$

For the first term in $c_1$, we used conditioning reduces the entropy, and

the second term we have $Z_1(n) \leftrightarrow M_1, Z^N_1, Z^n_1, Z^n_2 \leftrightarrow Z^n_3$. In $c_2$ we have $Z_2(n) \leftrightarrow M_{i+1}, Z^n_2, Z^n_1, Z^n_3, Z_2(n) \leftrightarrow Z^n_2$. See Figure 7 for the proof of Markov conditions. In $c_3$ the auxiliaries are chosen to be $U_i := \{M_i\}$ for $i \geq 2$.

With this choice of auxiliaries we can check that the Markov conditions in equation (16) are obeyed. The first Markov condition $U_1(n) \leftrightarrow Z_1(n) \leftrightarrow Z_2(n), Z_3(n)$ can be verified using Figure 8 (left), therefore $M_1, Z^n_1, Z^n_2, Z^n_3 \leftrightarrow Z_1(n) \leftrightarrow Z_2(n)Z_3(n)$ holds. For other rounds, the Markov can be verified in Figure 8.

This region has been derived for arbitrary $t$ number of rounds, but, in the next example, we restrict our attention to two number of rounds to discuss the effect of overhearing the first message by the second user before the second message is transmitted. The following lemma is useful in providing the explicit expression for this problem.

**Lemma 1.** $R(D) \delta$ is a convex region.

**Proof.** See section §5 for the proof.

In the next section, we evaluate and simplify the achievable region for a particular family of message distributions.

### 4.1. Interactive binary sum computation with presence of the side information

In this section we derive the rate distortion region for a binary sum computation of discrete symmetric binary sources subject to the Hamming distortion measure. In this case let $X^N_1$ be a sequence of i.i.d. Bernoulli random variables, $p(X_1 = 0) = p(X_1 = 1) = \frac{1}{2}$. Then, let the variables $X^N_1$ and $X^N_2$ be observations of $X^N_1$ through independent binary-symmetric channels with cross-over probabilities $p_1$, and $p_2$, respectively. Therefore, $(X_1, X_3) \sim$ DSBS($p_1$), and $(X_2, X_3) \sim$ DSBS($p_2$), and $X_1 \leftrightarrow X_3 \leftrightarrow X_2$, as depicted in Fig. 4. Hence the joint distributions of random variables are as follows.

$$p(x_1, x_3) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad p(x_2, x_3) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

The decoder aims to compute $Z = X_1 \oplus X_2 \oplus X_3$ subject to the Hamming distortion. That is, for $z, \tilde{z} \in \{0, 1\}$, we have $d(z, \tilde{z}) = 0$ if $z = \tilde{z}$, and $d(z, \tilde{z}) = 1$ if $z \neq \tilde{z}$. Therefore, the expected distortion is the mod 2 sum, that is, $E(d(Z, \tilde{Z})) = p(Z \neq \hat{Z}) = \mu^{-1}(H(Z \oplus \tilde{Z}))$. Next, we define the following region.

**Definition 2.** Assume $p_2 > p_1$, for each value of $D$, we define $\eta(D)$ to be set of rates constructed as follows.

$$\eta(D) = \bigcup_{\alpha, \beta, \lambda_1, \lambda_2} \left\{ R \left| \begin{array}{l}
R_1 \geq (\lambda_1 + \lambda_2)[h(p_2 * p_1 \alpha) - h(\alpha)] \\
R_2 \geq \lambda_1 h(p_2 * p_1 \beta) - h(\beta)
\end{array} \right. \right\}$$

where $h$ is a binary entropy function and the region is defined over all tuple $(\lambda_1, \lambda_2, \alpha, \beta)$, such that $0 \leq \lambda_1, \lambda_2 \leq 1$, and $0 \leq \alpha \leq p_1$, $0 \leq \beta \leq 1 - p_1$, and

$$D = \lambda_1[\alpha * \beta] + \lambda_2[p_2 * \beta] + (1 - \lambda_1 - \lambda_2)p_1 * p_2.$$  \hspace{1cm} (19)

**Theorem 4.** For computing the binary sum of three doubly symmetric binary sources $(X_1, X_3) \sim$ DSBS($p_1$), and $(X_2, X_3) \sim$ DSBS($p_2$) subject to the Hamming distortion, we have $R(D) \subseteq \eta(D)$.

**Proof.** To prove the upper bound, we consider a particular joint distribution of the sources $X_1, X_2, X_3$, and the auxiliary random variables $U_1, U_2$, and we evaluate the rate distortion expression for this particular distribution. For $D \geq 1/2$ we simply let
The proof is as follows. We define $E$ for any distortion $D$. Let $\hat{p} = X$. To derive the first term in the right hand side of (20), we show that $\hat{p} = X$. Then, we get distortion $E[d(Z, \hat{p} X_2, X_3)] = E[d(X_1 \oplus X_2 \oplus X_3, X_3)] = p_1 \cdot p_2$. The proof is as follows.

$$E[d(X_1 \oplus X_2 \oplus X_3, X_3)] = p(X_1 \oplus X_2 \oplus X_3 \neq X_3)$$

$$= p(X_1 \oplus X_2 \oplus X_3 = 1) = p(X_1 \oplus X_2 = 1)$$

$$= p(Z_1 \oplus Z_2 = 1) = p_1 \cdot p_2$$

Hence, any distortion $p_1 \cdot p_2 \leq D \leq 1/2$ is achievable by this scheme. Second, for $D \leq p_1 \cdot p_2$, let $U_1$ be the output of a BSC($\alpha$), $0 \leq \alpha \leq p_1$, with input $X_1$ while $U_2 = \emptyset$. We define $\hat{g}(U_1, U_2, X_3) = U_1$. Thus, we get the distortion $E[d(Z, \hat{g}(U_1, U_2, X_3))] = E[d(X_1, U_1)] = E[d(X_1 \oplus X_2 \oplus X_3, U_1)] = p_2 \cdot \alpha$. Since $X_1 \oplus X_2 \oplus X_3 \leftrightarrow X_1 \leftrightarrow U_1$, this can be proved using Mrs. Gerber’s Lemma

$$H(X_1 \oplus X_2 \oplus X_3|U_1)$$

$$= h(h^{-1}(H(X_1 \oplus X_2 \oplus X_3|X_1))) = h^{-1}(H(X_1|U_1)).$$

To derive the first term in the right hand side of (20), we show that $X_1 \oplus X_2 \oplus X_3$ and $X_1$ are related via a BSC($p_2$).

$$p(X_1 \oplus X_2 \oplus X_3 \neq X_1) = p(X_1 \oplus X_2 \oplus X_3 = X_1)$$

$$= p(X_1 \oplus X_3 = 1) = p(Z_2 = 1) = p_2$$

Substituting (21) in (20) we have,

$$H(X_1 \oplus X_2 \oplus X_3|U_1) = h(p_2 \cdot \alpha)$$

With this scheme, any distortion $p_2 \leq D < p_1 \cdot p_2$ can be achieved. Hence, the minimum rate for the first user is

$$R_1(p_2 \cdot \alpha) = I(X_1; U_1|X_2) = H(U_1|X_2) - H(U_1|X_1)$$

$$= h(p_1 \cdot p_2 \cdot \alpha) - h(\alpha)$$

(23)

Finally, to achieve any distortion in $0 \leq D < p_2$, let $U_2$ be the output of a BSC(\beta), $0 \leq \beta \leq \frac{p_2 - p_1}{1 - p_1}$, with input $X_2 \oplus U_1$, while $U_1$ is the output of BSC(\alpha) with input $X_1$. We define $\hat{g}(U_1, U_2, X_3) = U_2 \oplus X_3$. Thus, the distortion will be $E[d(Z, \hat{g}(U_1, U_2, X_3)] = E[d(X_1 \oplus X_2 \oplus X_3, U_2 \oplus X_3)]$.

$$= E[d(X_1 \oplus X_2, U_2)] = \alpha \cdot \beta.$$ This can also be seen by Mrs. Gerber’s Lemma, since $X_1 \oplus X_2 \leftrightarrow X_2 \oplus U_1 \leftrightarrow U_2$.

$$H(X_1 \oplus X_2 \oplus X_3|U_2)$$

$$= h(h^{-1}(H(X_1 \oplus X_2 | X_2 \oplus U_1)) = h^{-1}(H(X_2 \oplus U_1|U_2))$$

$$\leq h(\alpha \cdot \beta)$$

(24)

where in $a$ we used the fact that $H(X_1 \oplus X_2 | X_2 \oplus U_1) = H(X_1|U_1)$.

Note that the upper bound for parameter $\beta$ is derived such that $p_1 \cdot p_2 - p_1 = p_2$. As a result, this scheme is achievable for any distortion $0 \leq D \leq p_2$. Therefore, the minimum rate for the second user that can be achieved with the distortion $\alpha \cdot \beta$ is

$$R_2(\alpha \cdot \beta) = I(X_2; U_2|U_1) = H(U_2|U_1, X_1) - H(U_2|X_2, U_1)$$

$$\leq h(p_1 \cdot p_2 \cdot \beta) - h(\beta)$$

(25)

Where in $b$ we used the Markov chain, $U_2 \leftrightarrow X_2 U_1 \leftrightarrow U_1 X_1$ and
applied Mrs. Gerber's lemma.

\[ \begin{align*}
H(U_2|U_1,X_1) &= h \left( \frac{1}{(H(U_2|X_2,U_1))} + \frac{1}{(H(X_2,U_1|U_1,X_1))} \right) \\
&= h(p_1*p_2*\beta) \\
\end{align*} \]

Now, we consider the convex combination of the three scenarios described above.

\[ D = \lambda_1[\alpha*\beta] + \lambda_2[p_2*\alpha] + (1 - \lambda_1 - \lambda_2)[p_1*p_2] \]

Since we proved that \( RD^f \) is convex

\[ R_1(D) = R_1(\lambda_1[\alpha*\beta] + \lambda_2[p_2*\alpha] + (1 - \lambda_1 - \lambda_2)[p_1*p_2]) \]

\[ \leq \lambda_1R_1(\alpha*\beta) + \lambda_2R_1(p_2*\alpha) + (1 - \lambda_1 - \lambda_2)R_1(p_1*p_2) \]

\[ = (\lambda_1 + \lambda_2)[h(p_1*p_2*\alpha) - h(\alpha)] \]

(27)

And,

\[ R_2(D) = R_2(\lambda_1[\alpha*\beta] + \lambda_2[p_2*\alpha] + (1 - \lambda_1 - \lambda_2)[p_1*p_2]) \]

\[ \leq \lambda_1R_2(\alpha*\beta) + \lambda_2R_2(p_2*\alpha) + (1 - \lambda_1 - \lambda_2)R_2(p_1*p_2) \]

\[ = \lambda_1[h(p_1*p_2*\beta) - h(\beta)] \]

(28)

Therefore, a point inside \( \eta(D) \) also lies inside \( RD^2(D) \). This proves the achievability \( RD^2(D) \subseteq \eta^*(D) \).

5. PROOF OF LEMMA 1, CONVEXITY

Let \( D_a \) and \( D_b \) be two distortion values, and let \( \{(U_{1,a},U_{2,a}),g_a\} \), and \( \{(U_{1,b},U_{2,b}),g_b\} \) be the variables that achieve the point \( (R_a,D_a) \in RD_f \) and \( (R_b,D_b) \in RD_f \), in the closure of the rate distortion region, respectively. Let \( Q \) be an independent time sharing random variable such that \( p(Q = a) = \lambda \). Define \( U_1 = (Q,U_{1,Q}) \), and \( U_2 = (Q,U_{2,Q}) \), and \( \hat{g}(U_1,U_2,X_3) = \hat{g}_Q(U_1,U_2,X_3) \)

\[ D = E[d(Z,Z')] = E[d(Z,\hat{g}(U_1,U_2,X_3)) = p(Q = a)E[d(Z,\hat{g}(U_1,U_2,X_3)|Q = a] + p(Q = b)E[d(Z,\hat{g}(U_1,U_2,X_3)|Q = b] = p(Q = a)E[d(Z,\hat{g}_a(U_{1,a},U_{2,a},X_3)] + p(Q = b)E[d(Z,\hat{g}_b(U_{1,b},U_{2,b},X_3)] = \lambda D_a + (1 - \lambda)D_b \]

(29)

The rate for the first round:

\[ I(X_1;U_1|X_2) = I(X_1;U_1) - I(X_2;U_1) \]

\[ = H(X_1) - H(X_1|Q,U_{1,Q}) - H(X_2) + H(X_2|U_{1,Q}) \]

\[ = H(X_1) - \lambda H(X_1|U_{1,a}) + (1 - \lambda)H(X_1|U_{1,b}) \]

\[ = \lambda(I(X_1;U_{1,a}) - I(X_2;U_{1,a})) + (1 - \lambda)(I(X_1;U_{1,b}) - I(X_2;U_{1,b})) \]

(30)

The rate at the second round becomes,

\[ I(X_2;U_2|U_1,X_1) = H(X_2|U_1,X_1) - H(X_2|U_2,X_1) \]

\[ = H(X_2|U_{1,a},X_1) - H(X_2|U_{2,a},X_1) \]

\[ = \lambda H(X_2|U_{1,a},X_1) + (1 - \lambda)H(X_2|U_{2,a}) \]

\[ = \lambda I(X_2;U_{1,a}|X_1) + (1 - \lambda)I(X_2;U_{2,a}|U_{1,a},X_1) \]

(31)

We define \( W_1 \) and \( W_2 \) to be the auxiliaries that achieve the the rate tuple \( R = (R_1,R_2) \) at the closure of the the rate distortion region for distortion \( D \) defined in (29).

\[ R_1(D) = I(X_1;W_1|X_2) \]

\[ \leq \lambda I(X_1;U_{1,a}|X_2) + (1 - \lambda)(I(X_1;U_{1,b}|X_2)) \]

(32)

In step \( a_1 \), we substitute \( W_1 \) with auxiliary \( U_1 \), because \( (U_1,U_2) \) achieve distortion \( D \) but not necessarily achieve the rates in the closure of the rate distortion region (minimum rate). Step \( a_2 \) follows from (30).

\[ R_2(D) = I(X_2;W_2|W_1) \]

\[ \leq \lambda I(X_2;U_{2,a}|U_{1,a},X_1) + (1 - \lambda)(I(X_2;U_{2,b}|U_{1,b},X_1)) \]

(33)

In step \( b_1 \) we use the the same argument as in \( a_1 \), and step \( b_2 \) follows from (31). Recall that \( (U_{1,a},U_{2,a}) \) were defined to achieve distortion \( D_a \), and \( RD(D_a) \), and \( (U_{1,b},U_{2,b}) \) were defined to achieve distortion \( D_b \), and rate tuple \( RD(D_b) \). Therefore, any point that achieves any distortion in between \( (D) \), should have its rate tuple greater than \( (R_1(D), R_2(D)) \). This proves the convexity of the rate distortion region (16).
6. REFERENCES


