Source Coding and Function Computation: Optimal Rate in Zero-Error and Vanishing Zero-Error Regime

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Outline

- Introduction
  - $\Rightarrow$ Motivation
  - $\Rightarrow$ Some graph theory quantities
- Zero error source coding (Scalar vs. Block coding)
  - $\Rightarrow$ Fixed length coding
  - $\Rightarrow$ Variable length coding
- Asymptotically zero probability of error
  - $\Rightarrow$ Optimal rate for function compression
  - $\Rightarrow$ Achievable coding scheme
- Function compression in zero-error regime
- Compression of specific functions

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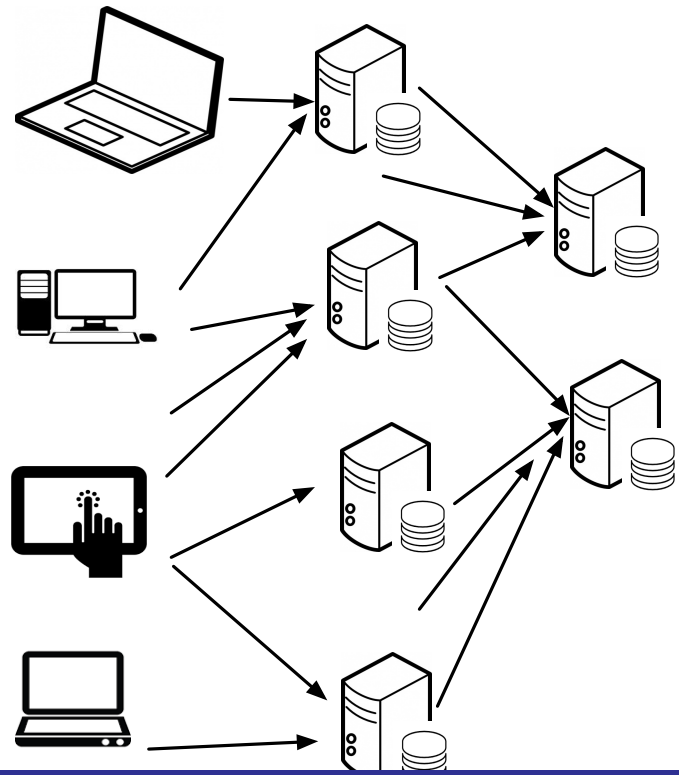
- Function compression in zero-error regime

Motivation

In big data problems

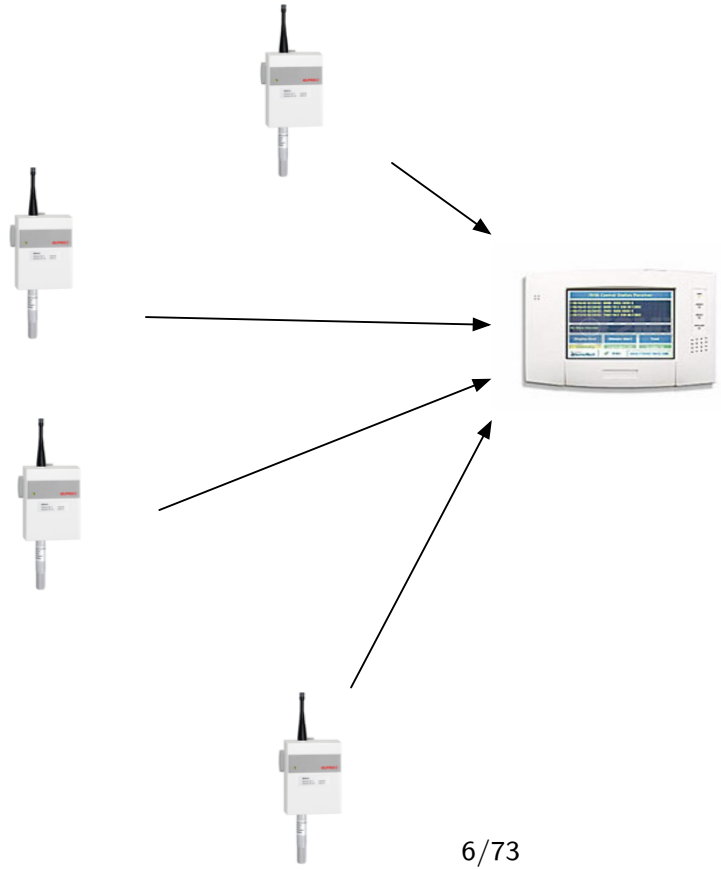
- How to design computational methodologies suited to statistical calculation
  - Scale easily as the data being processed becomes immense
- Fundamental problems in massive datasets where some statistical primitives needs to be computed for some tasks.

- Users submit machine learning tasks
- Analytics servers translate the tasks into statistical primitives calculation



Motivation

- Central receiver wishes to compute the average temperature of the building
- Bandwidth limited sensors do not communicate with each other.
- Considers the recovery not of the sources but the function of sources



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Some graph theory quantities

- Quantities to be discussed for undirected graphs:(undirected, finite and contain neither loops nor multiple edges)

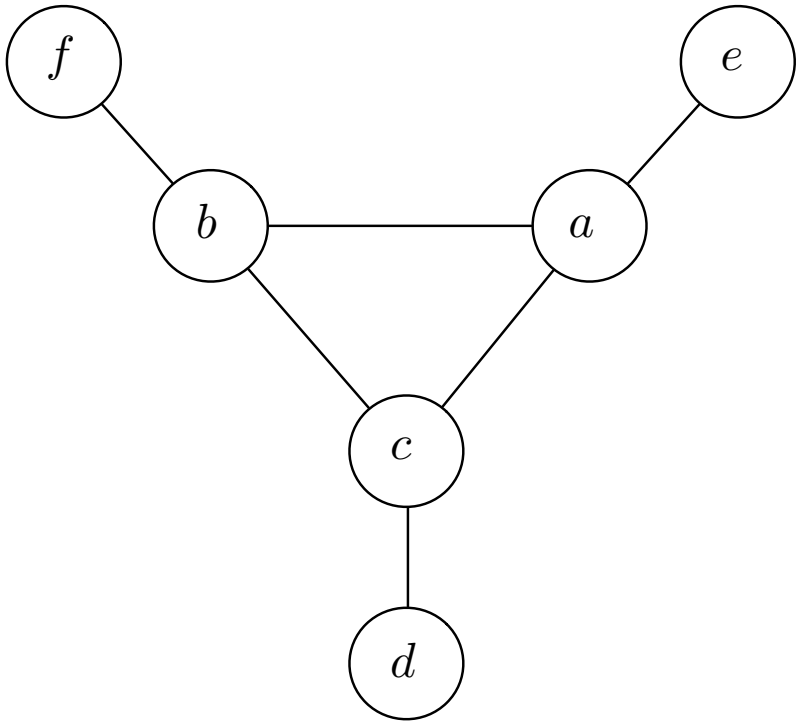
$\Rightarrow$ Independent set of the graph

$\Rightarrow$ Coloring of the graph

$\Rightarrow$ Product of the graph

Independent set

- An independent set or stable set is a set of vertices in a graph, no two of which are adjacent
- An independent set that is not the subset of another independent set is called maximal



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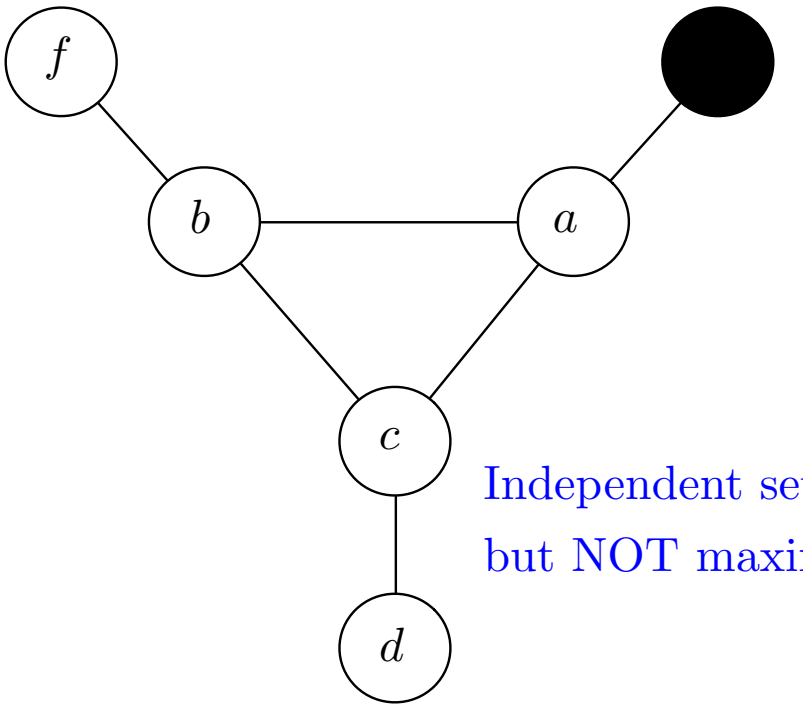

Independent set

but NOT maximal independent set

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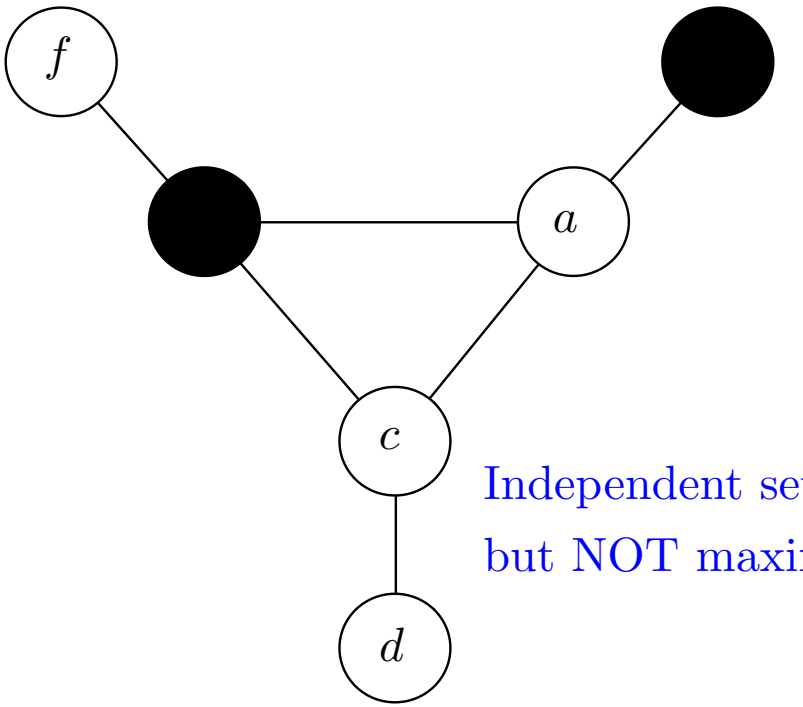

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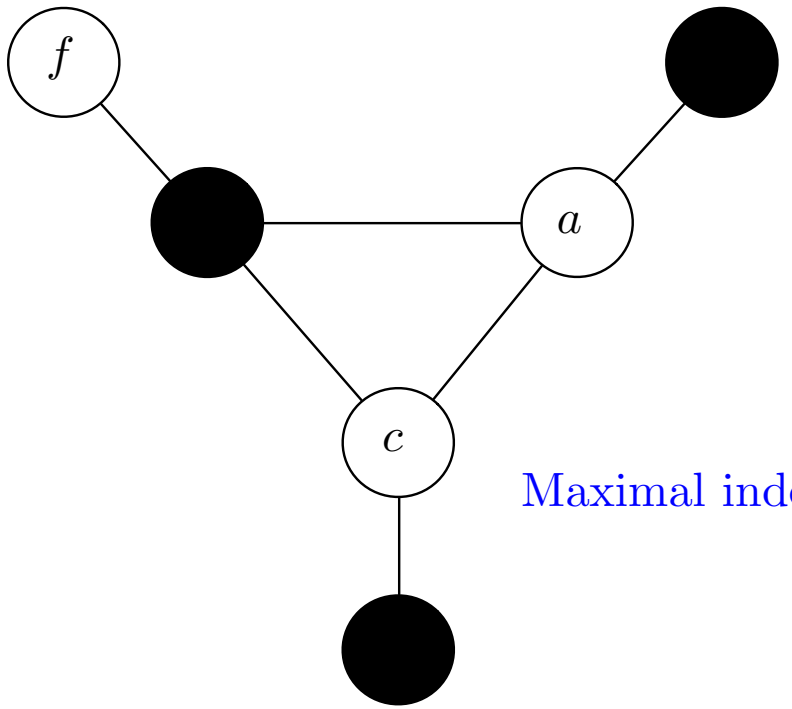

Maximal independent set

Independent set

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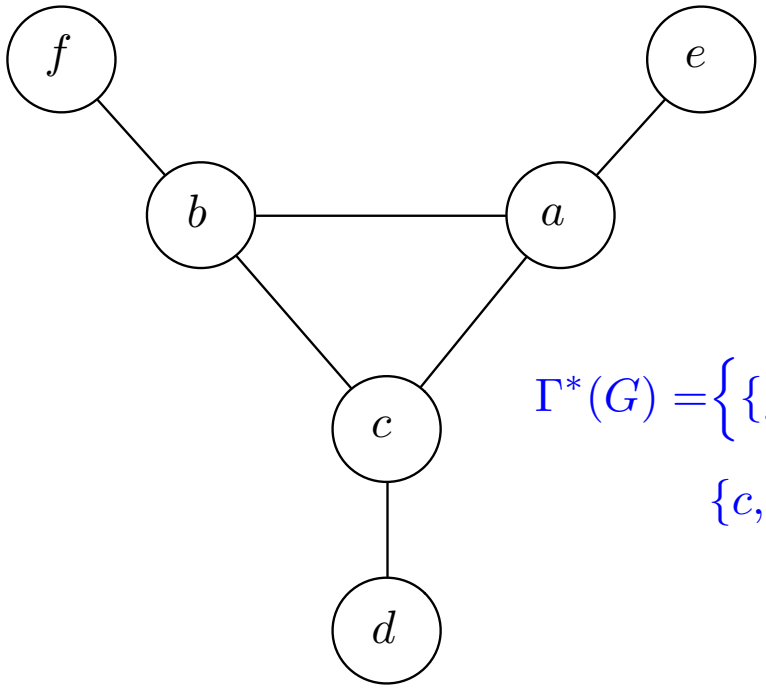
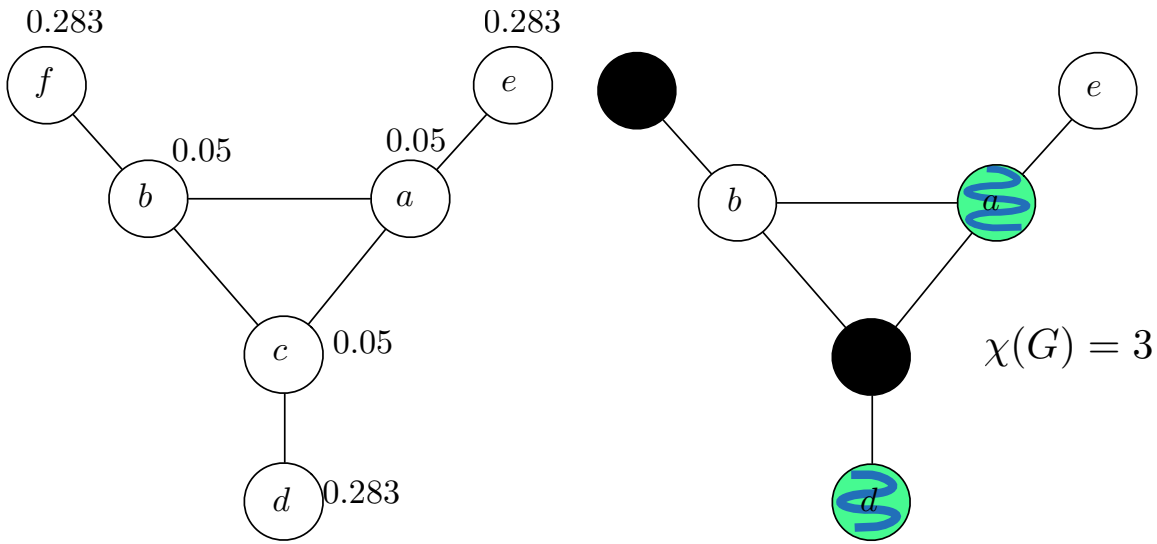


$$\Gamma^*(G) = \Big\{ \{f, d, e\}\{b, d, e\} \quad \{c, f, e\}\{a, d, f\} \Big\}$$

Probabilistic Graph-Chromatic Number

- $\chi(G)$ is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color.
- The chromatic number of a graph $\geq$ clique number(maximum clique size).



Entropy of the coloring

- Random variable $X \sim P(x)$, and function $c : \mathcal{X} \to \mathbb{N}$
- Entropy of the coloring is defined as

$$H(c(X)) = -\sum_{i \in c(\mathcal{X})} p_i \log p_i$$

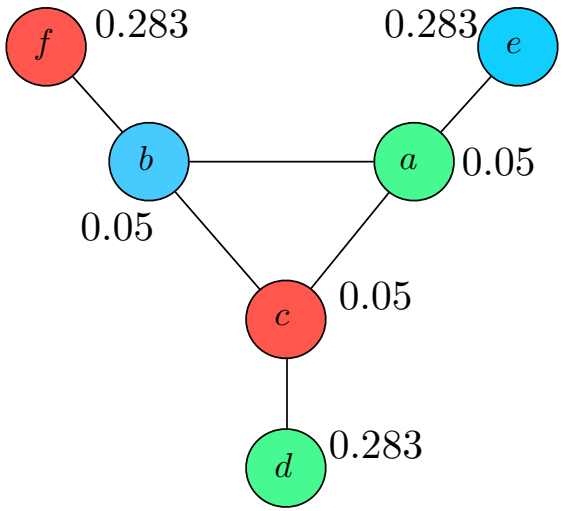
$$p_i = P(c^{-1}(i)) = \sum_{x \in V : c(x) = \gamma} P(x)$$

- $c^{-1}(i)$ is a color class (a set of vertices assigned the same color i)
- Coloring of G is any function c over V such that $c^{-1}(.)$ induced a partition of V into independent sets of G

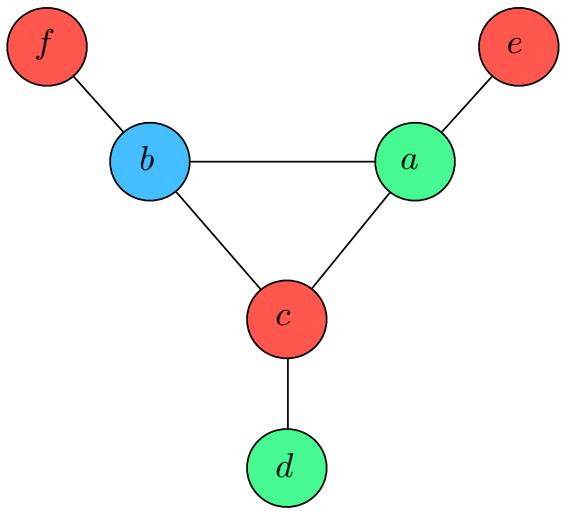
$$H_{\mathcal{X}}(G, X) = \min\{H(c(X)) \ : \ \text{c is a coloring of } G\}$$

Chromatic Entropy

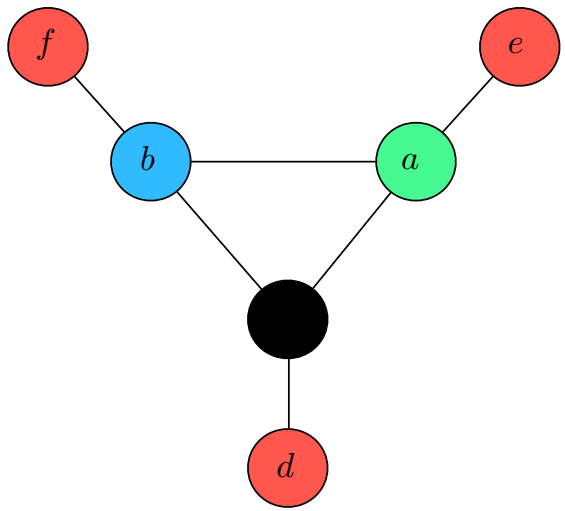
- Note that $\boxed{\chi_{H(G,P)} \geq \chi_G}$



$H(0.333, 0.333, 0.333) = 1.58$



$H(0.616, 0.333, 0.05) = 1.17$



$H(0.85, 0.05, 0.05, 0.05) = 0.84$

Chromatic Entropy

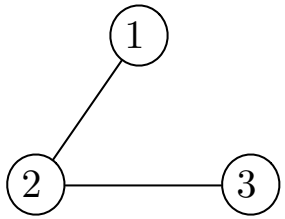
- Chromatic entropy is known to be NP-hard [1]
- Find independent sets with high probability mass and establish these as color classes,
  - Unbalance the distribution as much as possible
- Add a node to the color class with the highest probability if possible. (concavity of $f(x) = -x\log(x)$)
- If color classes C_1 and C_2 can be merged properly, $\Rightarrow C' = \{C_1 \cup C_2, C_3, ..., C_k\}$ has strictly lower entropy than the original coloring C.

[1] J. Cardinal, S. Fiorini, and G. Joret, "Minimum entropy coloring," in Algorithms and Computation

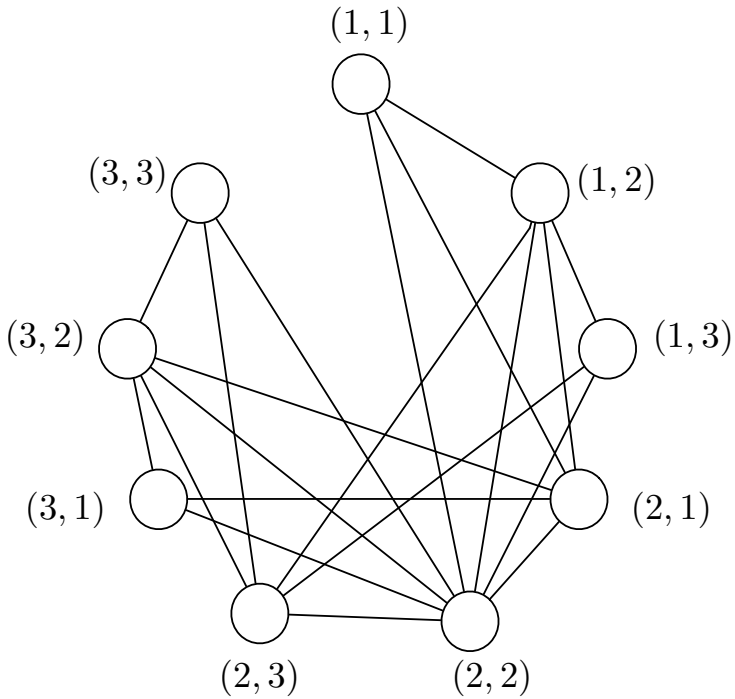
AND power graph vs OR power graph

- AND-product graph of $G_1(X_1, E_1)$ and $G_2(X_2, E_2) \Rightarrow G_1 \wedge G_2$
  - $\Rightarrow$ has vertex set $X_1 \times X_2$
  - $\Rightarrow$ (x_1, x_2) and (x_1', x_2') are connected if {either $x_i = x_i'$ or x_i is connected to x_i' in graph G_i} for each $i = 1, 2$
- OR-product graph of $G_1(X_1, E_1)$ and $G_2(X_2, E_2) \Rightarrow G_1 \vee G_2$
  - $\Rightarrow$ has vertex set $X_1 \times X_2$
  - $\Rightarrow$ (x_1, x_2) and (x_1', x_2') are connected if either x_1 is adjacent to x_1' or x_2' is adjacent to x_2' in graphs G_1 and G_2, respectively.
- $\overline{G_1 \wedge G_2} = \overline{G_1} \vee \overline{G_2} \Rightarrow \overline{G^{\vee n}} = \overline{G}^{\wedge n}$

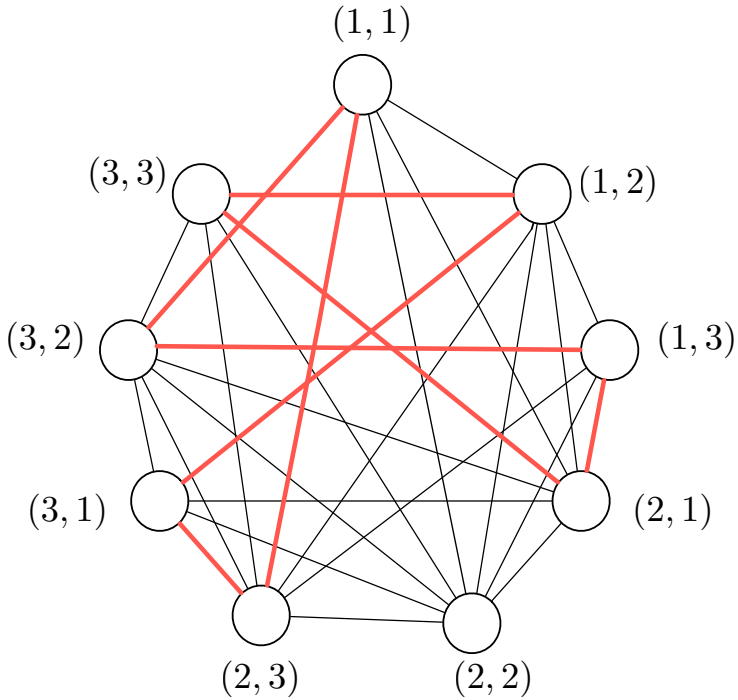
AND product graph vs OR product graph



G original graph



$2-$ AND Power graph



$2-$ OR Power graph

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Optimal Rate: Source coding, Function compression

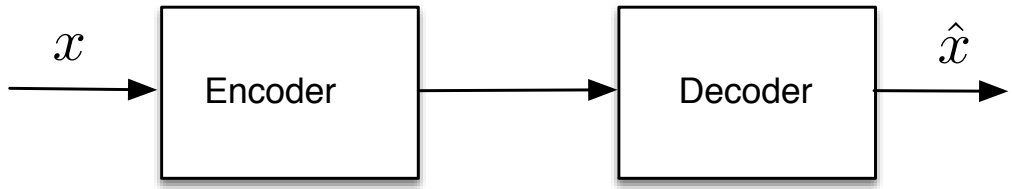
	Source, side info fixed length	Source, side info variable length	Function
scalar code Zero error			
block code Zero error			
block code $\epsilon - error$			

Zero error source coding

- In information theory: Increasing the number of repetitions decreases $p(e)$
- In some applications no error can be tolerated.
- Sometimes there are only few source instances available.
- The error prob. decreases as the no. of instances increases $\rightarrow$Not useful

Zero error- Simple point-to-point communication - Scalar

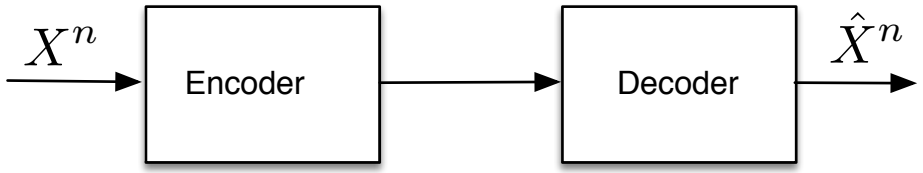
- Every source instances, a random variable X is generated according to $p(x)$ independently of all other instance.
- for a single instance, the smallest number of bits in the worst case is $\log|\mathcal{X}| \leq \lceil \log|\mathcal{X}| \rceil \leq \log|\mathcal{X}| + 1$



$$X^n = [x(1), x(2), ..., x(n)]$$
$$x \in \mathcal{X}$$
$$x \sim p(x)$$

Zero error- Simple point-to-point, Bocking

- For n independent instances, the number of bits needed in the worst case is $n \log |\mathcal{X}| \leq \lceil n \log |\mathcal{X}| \rceil \leq n \log |\mathcal{X}| + 1$
- The asymptotic per-instance
  - In worst case is between $\log |\mathcal{X}|$ and $\log |\mathcal{X}| + 1/n$ bits



$X^n = [x(1), x(2), ..., x(n)]$

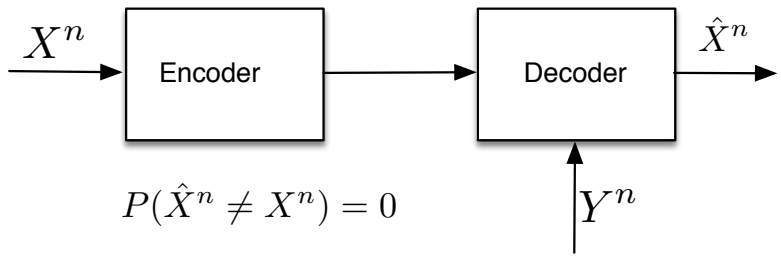
$x \in \mathcal{X}$

$x \sim p(x)$

$$p\{X^n \neq \hat{X}^n\} = 0$$

Zero-error source coding with side information

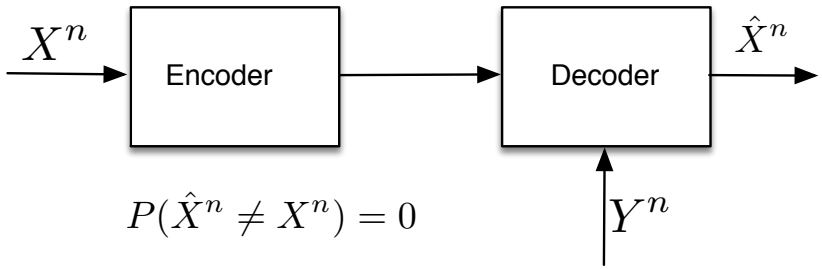
- The receiver knows some possibly related rv. Y jointly distributed with X
- Probability of error must be exactly zero.
- How many bits are needed in the worst case?
- Any advantages over block coding of any independent instances?



Zero-error source coding, Fixed length code

- Receiver wants to know X without error. (No error is tolerated!)
- $\mathcal{X} = \{1, 2, ..., 5\}$ and $\mathcal{Y} = \{1, 2, ..., 5\}$

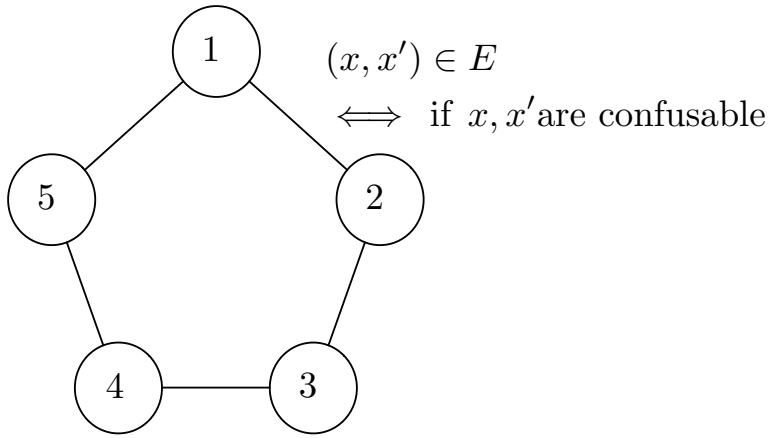
y \ x	1	2	3	4	5
1	0.1	0.1			
2		0.1	0.1		
3			0.1	0.1	
4				0.1	0.1
5	0.1				0.1



Confusability graph

- x, x' are confusable if $\exists y \in \mathcal{Y}$ such that $p(x,y)p(x',y) > 0$
- Given $y = 1$, then x can be either 1 or 2 (1 and 2 are confusable).
- This confusability relation can be captured by the characteristic graph G.

y \ x	1	2	3	4	5
1	0.1	0.1			
2		0.1	0.1		
3			0.1	0.1	
4				0.1	0.1
5	0.1				0.1



Encoding 1-instance at a time, Scaler code

- The smallest number of possible messages is $\chi(G)$
- In worst case the smallest number of bits needed is $\log(\chi(G))$

y \ x	1	2	3	4	5
1	0.1	0.1			
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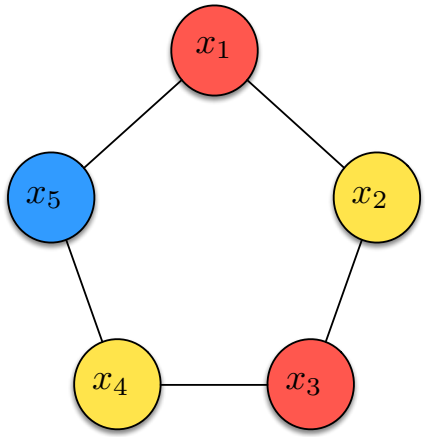


$\chi(G) = 3$

n-instance encoding

- Assign different codewords to the confusable vectors
- $x^n = (x_1, .., x_n)$ and $x'^n = (x'_1, ..., x'_n)$ are confusable if $\exists y^n \in \mathcal{Y}^n$ such that $p(x^n, y^n)p(x'^n, y^n) > 0$
- Confusability of the vectors can be captured by the AND power graph. $G^{\wedge n} = (V^n, E^n)$

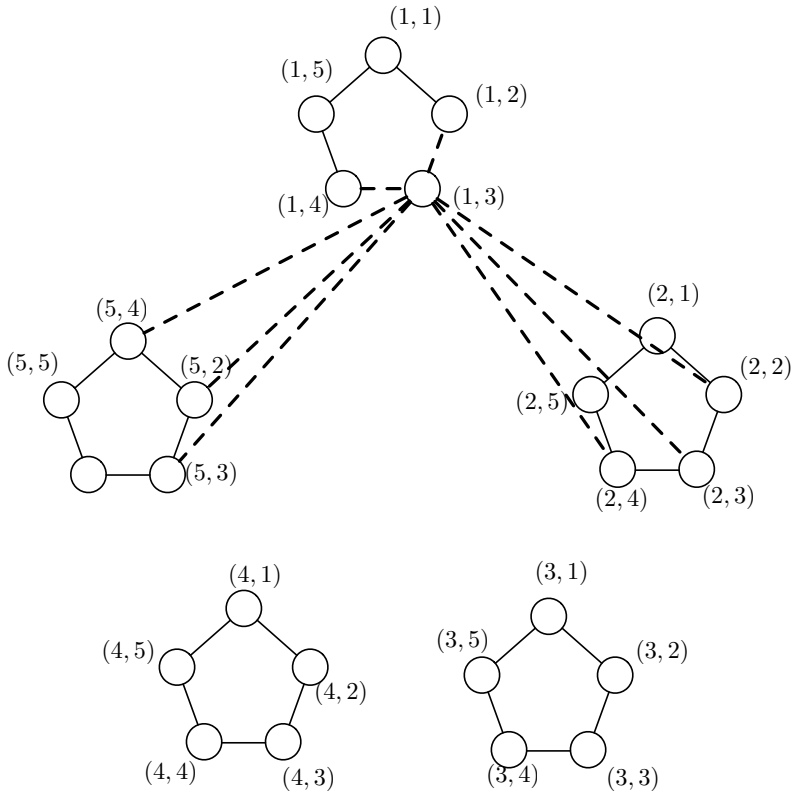
$$(x^n, x'^n) \in E^n \text{ if } (x_i, x'_i) \in E \text{ or } x_i = x'_i \text{ for all } i \in \{1, ..., n\}$$

- The number of bits needed in worst case is $\log \chi(G^{\wedge n})$
- In general $\chi(G^{\wedge n}) \leq (\chi(G))^n \Rightarrow$ benefit of block encoding

2-AND power graph of the pentagon

- The five individual pentagons represent the confusability relation for x_2, each for fixed x_1
- meta-pentagon represents the confusability relation for x_1 alone

- $\log_2 \chi(G^{\wedge 2}) = \log_2 5 < \log_2 \chi^2(G) = \log_2 9$
- Benefit from blocking can be obtained



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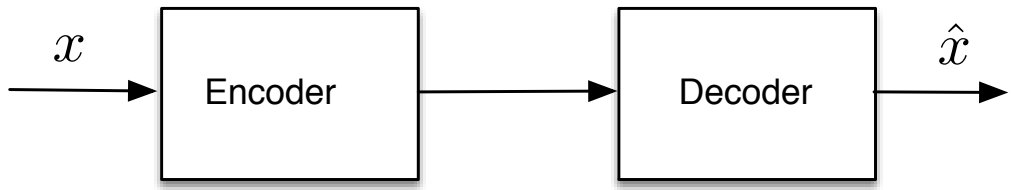
Variable length coding, Simple point-to-Point

- Scalar code: the smallest expected number of bits needed is between $H(X)$ and $H(X)+1$
- Block encoding: the number of bits needed on average is between $nH(X)$ and $nH(X)+1$
- The asymptotic per-instance on average is between $H(X)$ and $H(X)+\frac{1}{n}$ bits



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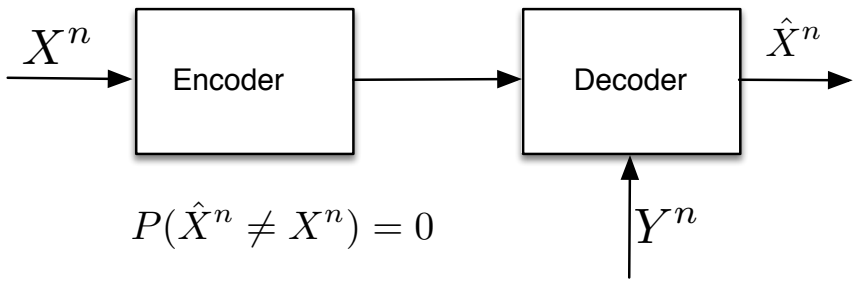
Variable length coding, Side information

- Variable length codes assigns shorter codes to the most frequent symbols and vice versa.
- Alon and Orlitsky considered two family of codes.[2]
- No single letter characterization exists for optimal variable length code $\Rightarrow$ Complementary graph entropy
- For a subclass of variable length codes the optimal rate is $\Rightarrow$Graph entropy.

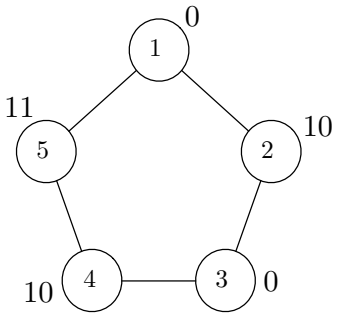


[2]N. Alon and A. Orlitsky, "Source coding and graph entropies," Information Theory, IEEE Transactions

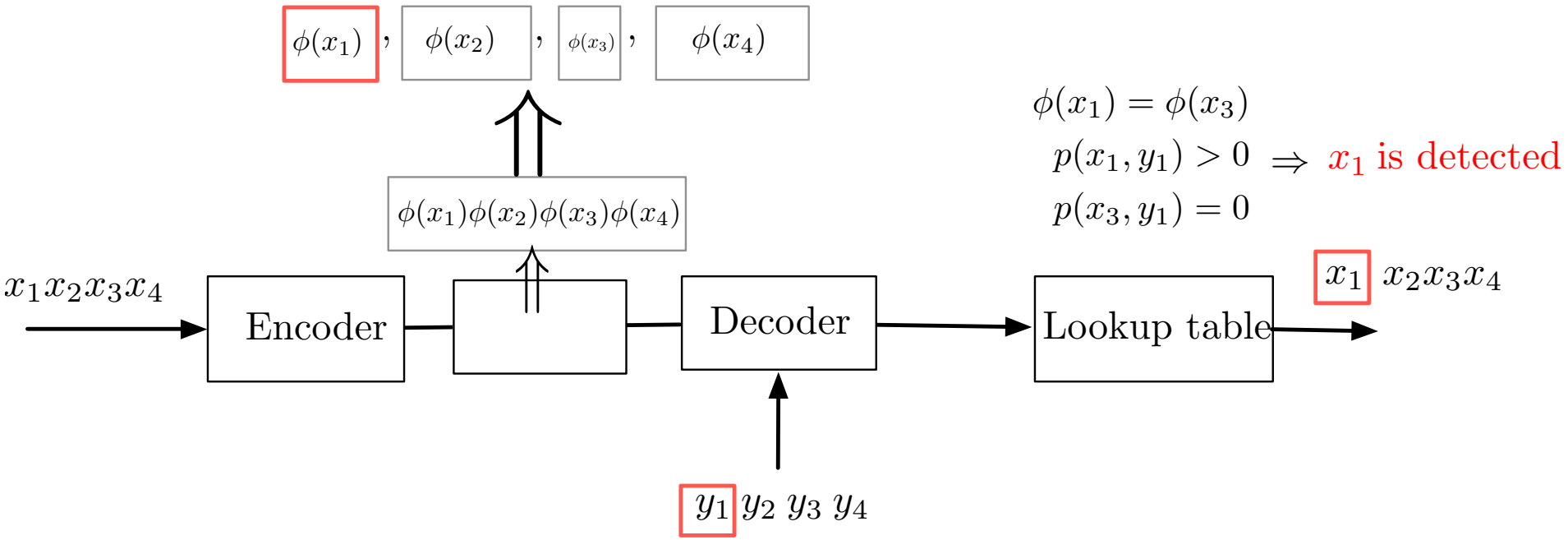
Coding method for unrestricted inputs (UI)

- Encoding function $\phi : \mathcal{X} \to \{0,1\}^*$

$$\begin{cases} \text{for all } x, x' \in V \to \phi(x) \neq_P \phi(x') \\ (x, x') \in E \Rightarrow \phi(x) \neq \phi(x') \end{cases}$$



- The decoder learns x only if $p(x, y) > 0$. o.w., the decoder may be wrong on the value of x.



Coding method for unrestricted inputs (UI)

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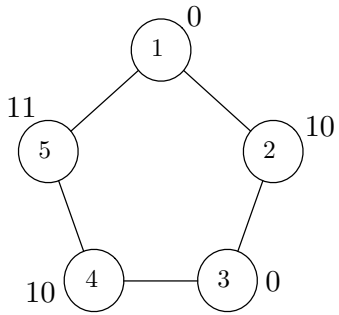
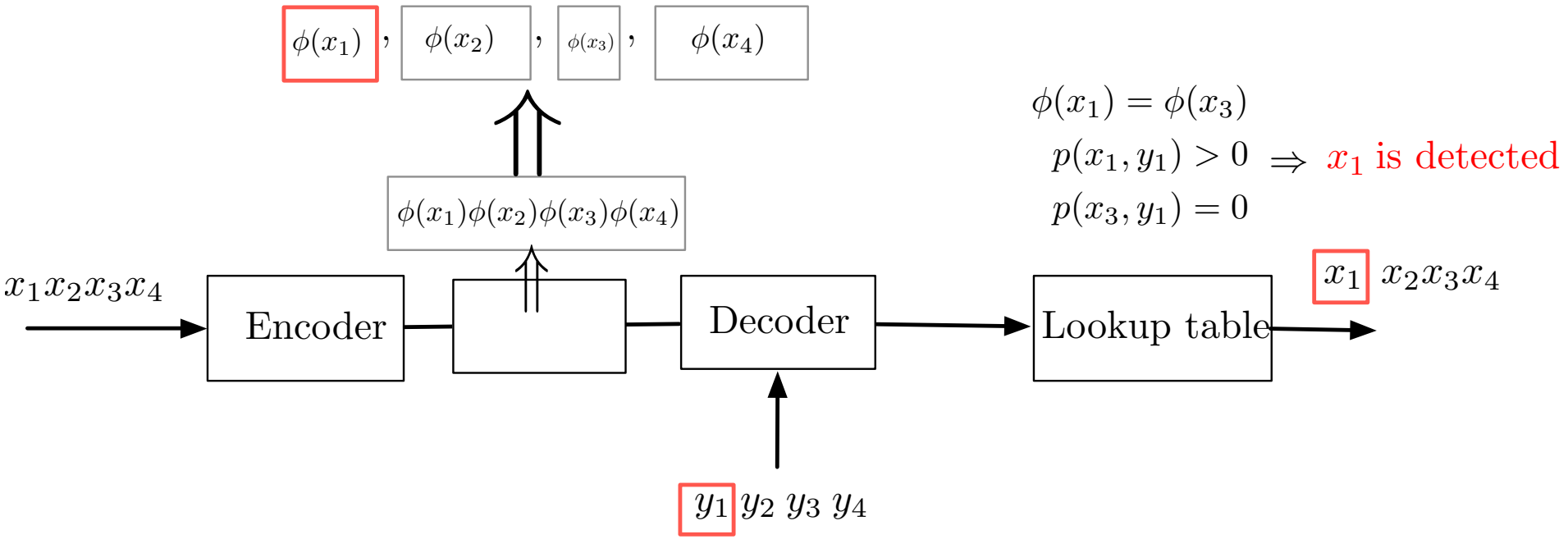


- UI code is a coloring of G followed by a prefix-free coding of the colors.



Expected number of bits

- Note The optimal code isn't necessarily induced by the optimal coloring

$\Rightarrow$ The expected number of bits transmitted is

$$\ell(\psi) = \sum_{x \in V} P(x)|\psi(x)|$$

$\Rightarrow$ The minimum Rate of UI codes are:

$$L(G) = \min\{\ell(\psi) : \psi \text{ is an UI code}\}$$

call the codes attaining these minima the optimal code

UI code-Scalar

$$\boxed{H_\chi(G,X) \leq L \leq H_\chi(G,X)+1}$$

- Upper bound: Take a coloring of G achieving $H_\chi(G,X)$.

$\Rightarrow$ There is a prefix free encoding of the colors whose expected length is at most $H_\chi(G,X)+1$

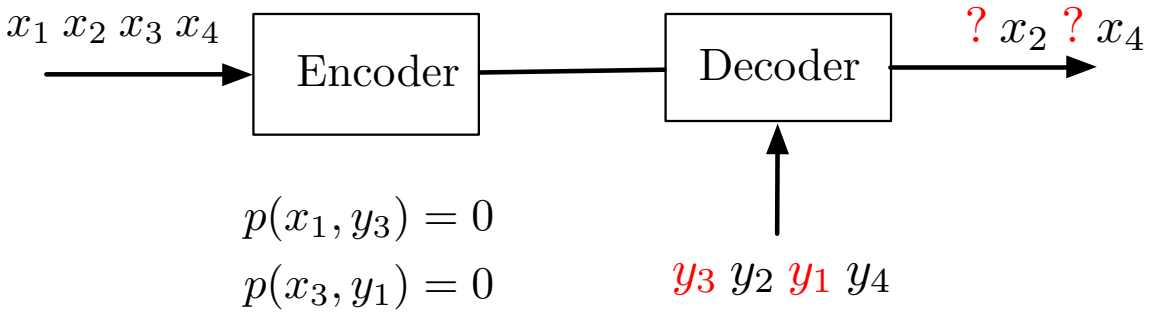
Data compression result $\Rightarrow H(X) \leq l_{p.f}(X) \leq H(X)+1$

- lower bound Every UI protocol can be viewed as a coloring of G and prefix free encoding of it's colors.

$\Rightarrow$ The entropy of a coloring is at least $H_\chi(G,X)$

$\Rightarrow$ The expected length is at least $H_\chi(G,X)$

n-independent instance



- Decoder needs to learn x_i for exactly those i's where $p(x_i, y_i) > 0$ and can be wrong about x_i for the other i's. This may require more bits.
- $x^n, x'^n \Rightarrow$ need to be distinguished if $\exists i$ such that $p(x_i, y_i)p(x'_i, y_i) > 0$
- $(x^n, x'^n) \in E^n$ if $(x_i, x'_i) \in E$ for some $i \in \{1, .., n\}$

$\Rightarrow$ This is the definition of OR-Power graph of G

Optimal rate for UI (variable length code)

Remember for one instance the optimal rate is

$$H_\chi(G, X) \leq L \leq H_\chi(G, X) + 1$$

for n-instances

$$H_\chi(G^{\vee n}, X^{(n)}) \leq L_n \leq H_\chi(G_n^{\vee n}, X^{(n)}) + 1$$

Asymptotic per instance[3] :

$$\boxed{L_{am} = \lim_{n\to\infty} \frac{1}{n} H_\chi(G^{\vee n}, X^{(n)}) = H_G(X)}$$

[3] N. Alon and A. Orlitsky, "Source coding and graph entropies," Information Theory, IEEE Transactions

Graph entropy

- Was originally defined by Körner(1973).

$$H_G(X) = \min_{X \in W \in \Gamma(G)} I(W, X)$$

- $\Gamma(G)$ is the set of independent sets of the confusability graph G
- W is an independent set of G
- The minimization can be restricted to W ranging over the independent sets.

	Source, side info fixed length	Source, side info variable length	Function
scalar code Zero error	$\log \chi(G)$	$H_\chi(G, X)$	?
block code Zero error	$\log \chi(G^{\wedge n})$	$H_G(X)$	?
block code $\epsilon - error$		?	?

$$H_G(X|Y) \leq H(X|Y) \leq H_G(X) \leq H_\chi(G, X) \leq H(X) \leq \log \chi(G)$$

Outline

- Introduction
- Zero error source coding (Scalar vs. Block coding)
  - $\Rightarrow$ Fixed Length coding
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  - $\Rightarrow$ Achievable coding scheme
- Function compression in zero-error regime
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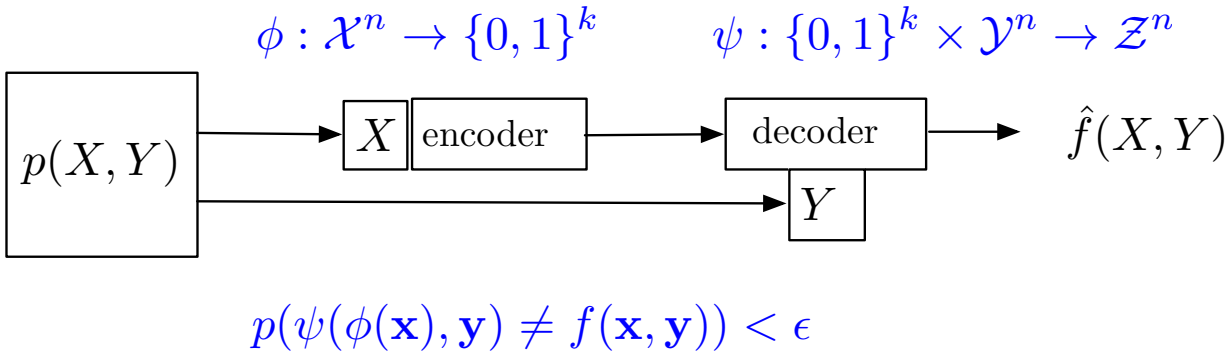
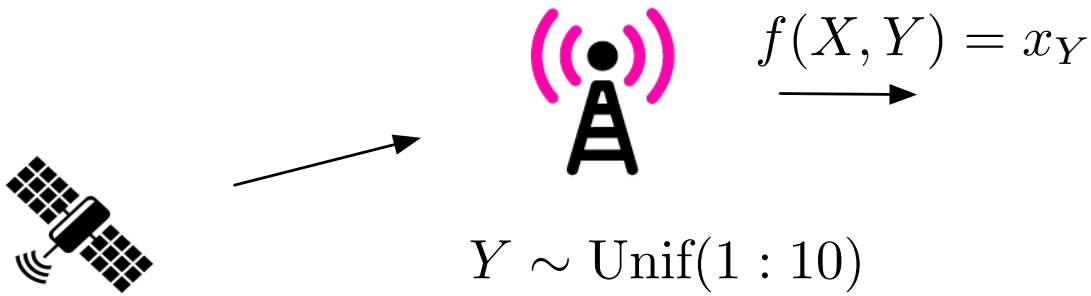
Function computing

- Compute a function $f(X, Y)$ at the decoder, and we want this encoding to be as efficient as possible
- $f(X, Y) = Z$ must be determined for a (single) block of multiple instances
- Vanishing (bit) block probability of error is allowed
- $H(f(X, Y)|Y) \leq \text{Rate} = R^* \leq H(X|Y)$



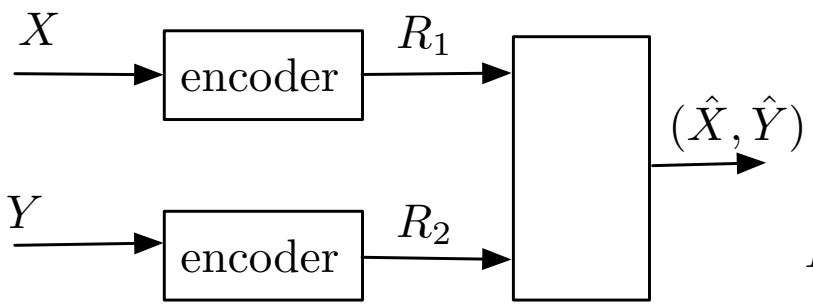
Example

- Satellite has 10 parameters to report to the base
- Base needs to learn only one parameter(only the base knows which one)
- The lower bound gives $H(x_Y|Y) = 1$ bit and the upper bound gives $H(X|Y) = 10$ bits. $R^* = 10$



Slepian Wolf

- In the case where $f(X, Y) = (X, Y)$, the well-established rate result is the Slepian-Wolf bound



$$R_1 > H(X|Y)$$

$$R_2 > H(Y|X)$$

$$R_1 + R_2 > H(X, Y)$$

- When Y is available at the decoder, the optimal rate for the transmission of X is the conditional entropy of X given Y , $H(X|Y)$.

Slepian Wolf

- To achieve this rate $H(X|Y)$, encoder use more efficient encoding by taking advantage of its correlation with Y.
- knowing that the decoder can use its knowledge of Y to disambiguate the transmitted data
- This analysis so far has ignored the fact that we are computing $f(X, Y)$ at the decoder, rather than reporting the two values directly!

Can we do better than Slepian Wolf bound?

Example: X and Y are integers and $f = X + Y \pmod 4$.

- Regardless of how large X and Y might be, just encode the last two bits of each.
- The SW bound only takes advantage of the correlation between the two rvs.
- SW doesn't take advantage of the properties of the function
- To analyze a given f, we construct f-confusability graph.

f-Confusability graph

- f-confusability graph of X given Y, f, and $p(x, y)$ is a graph $G = (V, E)$
- $V_x = \mathcal{X}$ and, $(x, x') \in E$ if $\exists y \in \mathcal{Y}$, such that, both $p(x, y) > 0$ and $p(x', y) > 0$ and $f(x, y) \neq f(x', y)$.

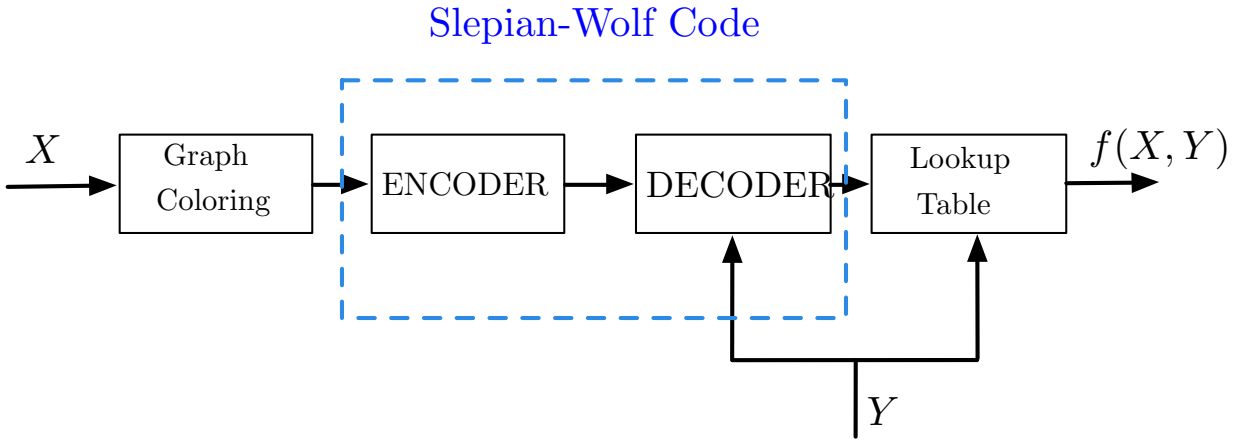
Negation

- If two nodes are not confusable, for all possible values of y

$$(x, x') \notin E \quad \begin{cases} \text{either } p(x, y) = 0 \text{ or } p(x', y) = 0 \\ \text{or} \\ f(x, y) = f(x', y) \end{cases}$$

Encoding scheme 1-instance

- $(x, x') \notin E$ can safely be given the same encoding
- Color the f-confusibility graph G_f with minimum entropy
- Transmit the name of the color class that contains each value
- SW: for asymptotically zero-error, it suffices to send the conditional entropy $H(X|Y)$ bits.
- Use SW encoding on the distribution over the color classes.
- This rate is $H_\chi(G_f, X|Y) = \min_c \sum_{y \in \mathcal{Y}} p(y) H(c(x)|Y = y)$



Encoding scheme n-instance

- How to define the notion of f-confusability for vector?

$\Rightarrow$ (x^n, x'^n) are f-confusable, if $\exists y^n \in \mathcal{Y}^n$ such that $p(x^n, y^n)p(x'^n, y^n) > 0$ and $f(x^n, y^n) \neq f(x'^n, y^n)$

$\Rightarrow$ $(x^n, x'^n) \in E^n$ if $(x_i, x'_i) \in E$ for some $i \in \{1, 2, ..., n\}$

- This is the definition of OR power graph

Optimal rate-Achievable scheme

For 1-instance

$$H_\chi(G_f, X|Y)$$

Asymptotic per instance

$$\lim_{n\to\infty} \frac{1}{n} H_\chi(G_f^{\vee_n}, X^{(n)}|Y^{(n)}) = H_{G_f}(X|Y)$$

- we can compute the function with vanishing probability of error by first coloring a sufficiently large power graph of the characteristic graph
- encode the colors using any code that achieves the entropy bound on the colored source

Optimal rate

optimal rate for the zero distortion functional compression problem with side information equals $R^* = H_G(X|Y)$ [4]

$$H_G(X|Y) = \min_{\substack{W-X-Y \\ X \in W \in \Gamma(G)}} I(W, X|Y)$$

- where $W - X - Y$ indicates that W, X, Y is a Markov chain.
- W should not contain any information about Y that is not available through X
- The minimization can be restricted to W ranging over the maximal independent sets.

[4] A. Orlitsky and J. Roche, "Coding for computing," Information Theory, IEEE Transactions

Summary

	Source, side info fixed length	Source, side info variable length	Function
scalar code Zero error	$\log \chi(G)$	$H_\chi(G, X)$	?
block code Zero error	$\log \chi(G^{\wedge n})$	$H_G(X)$	?
block code $\epsilon - error$		$H(X\|Y)$	$H_{G_f}(X\|Y)$

$$H_{G_f}(X|Y) \leq H(X|Y) \leq H_G(X) \leq H_\chi(G, X) \leq H(X) \leq \log \chi(G)$$

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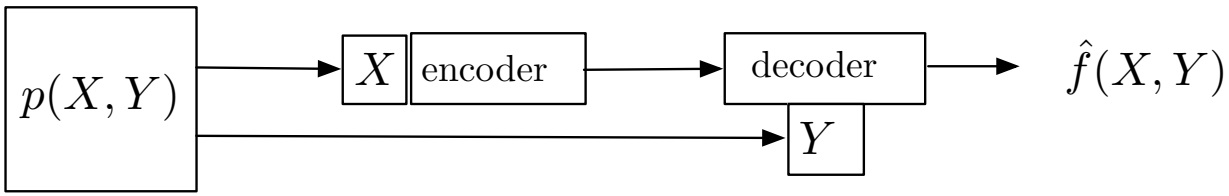
Function Compression, Scalar code, Zero-error

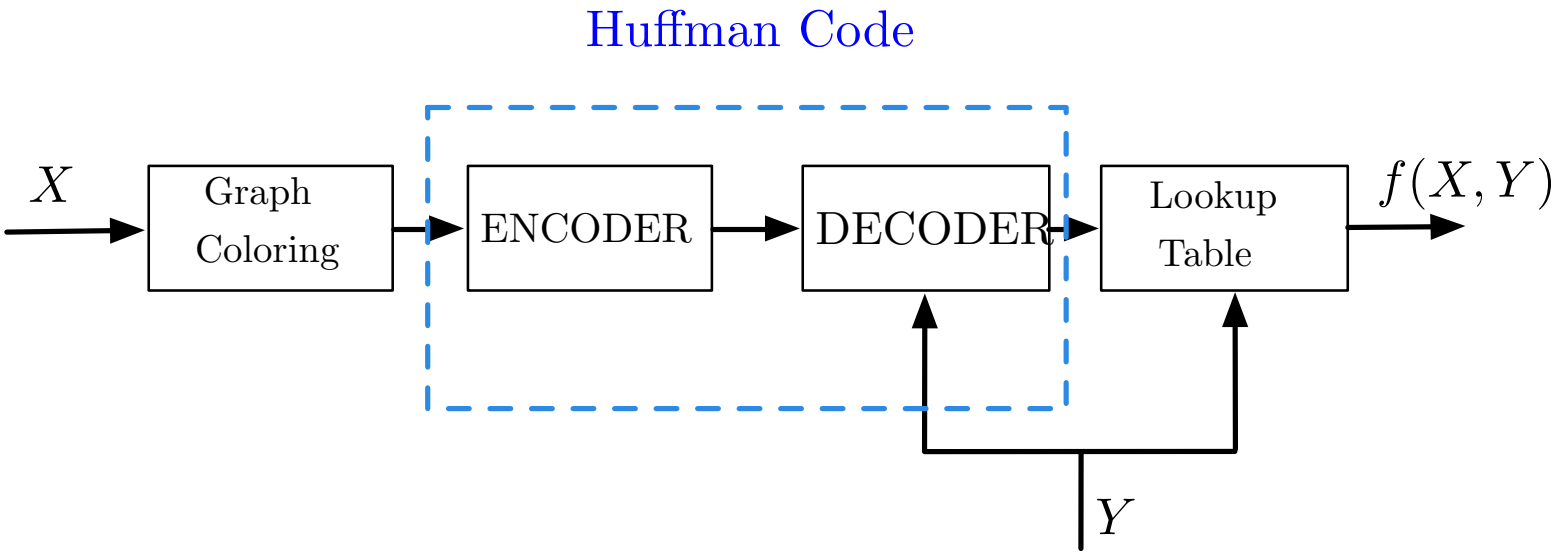
- Compute a function $f(X, Y)$ at the decoder.
- $f(X, Y)$ must be determined for a (single) block instance.
- No error is allowed (zero-error regime)
- Fundamental limit of computing a function with exactly zero probability of error?
- Is motivated by Alon, and Orlitsky (source coding with side information)



Achievable scheme

- The protocol that guarantees the zero probability of error, and to compute $f(x, y)$ only when $p(x, y) > 0$
- $(x, x') \notin E_f$ can safely be given the same encoding
  - Either $p(x, y) = 0$ or $p(x', y) = 0$
  - $f(x, y) = f(x', y)$
- Color the f-confusibility graph G_f with minimum entropy
- Transmit the name of the color class that contains each value
- Using a prefix free encoding on the distribution over the color classes yields a valid encoding scheme for the problem

Achievable coding scheme



Optimal scalar code for Function computing

Theorem

Optimal rate for computing a function $f(X, Y)$ with zero probability of error, using scalar variable length code is

$$H_\chi(G_f, X) \leq R \leq H_\chi(G_f, X) + 1$$

Proof.

- Upper bound: Take a coloring of G achieving $H_\chi(G_f, X)$.

$\Rightarrow$ There is a prefix free encoding of the colors whose expected length is at most $H_\chi(G_f, X) + 1$

- lower bound This protocol can be viewed as a coloring of G_f and prefix free encoding of it's colors.

$\Rightarrow$ The entropy of a coloring is at least $H_\chi(G_f, X)$

$\Rightarrow$ The expected length is at least $H_\chi(G_f, X)$

Block encoding

- To protect against loss of synchronization if the side information at the decoder is occasionally wrong:
- For a sequence $x_1 x_2, ..x_n$, the function needs to be computed correctly only for those coordinates that $p(x_i, y_i) > 0$.
- For other positions decoder may not be able to compute $f(x_i, y_i)$ correctly.
- $(x^n, x'^n) \in E^{(n)}$ if $(x_i, x_i') \in E$ for some $i \in \{1, .., n\}$

$$(x^n, x'^n) \in E^{(n)} \Leftrightarrow \phi(x^n) \neq \phi(x'^n) \tag{1}$$

$$x^n, x'^n \in V^{(n)} \Leftrightarrow \phi(x^n) \neq_P \phi(x'^n) \tag{2}$$

Optimal rate for function compression in zero-error regime

One instance

$$H_\chi(G_f, X) \leq R \leq H_\chi(G_f, X) + 1$$

n-instance

$$H_\chi(G_f^{\vee_n}, X^{(n)}) \leq R^n \leq H_\chi(G_f^{\vee_n}, X^{(n)}) + 1$$

Asymptotic per instance

$$\lim_{n\to\infty} \frac{1}{n} H_\chi(G_f^{\vee_n}, X^{(n)}) = H_{G_f}(X)$$

- We can compute the function with zero-error by first coloring a sufficiently large power graph of the characteristic graph
- Encode the colors using Huffman codes

function compression in zero-error regime

Proposition

In exactly zero-error regime, using block code, the optimal rate for computing $f(X, Y)$ is less than recovering the source X.

Proof.

- $G_f = (V, E_f)$ and $G = (V, E)$, and $E_f \subseteq E$
- Let (X, W') be random variable achieving $H_G(X)$
- W' is an independent set containing X, and $G_f \subseteq G \Rightarrow$ it's also an independent set of G_f.

$$H_{G_f}(X) = \min_{X \in W \in \Gamma(G_f)} I(X; W) \leq I(X; W') = \min_{X \in W \in \Gamma(G)} I(X; W) = H_G(X)$$

□

function compression in zero-error regime

Proposition

In exactly zero-error regime, using scalar code, the optimal rate for computing $f(X, Y)$ is less than recovering the source X.

- $G_f \subseteq G$
- Any coloring of G is a coloring of G_f
- The coloring with minimum entropy of G is also a coloring of G_f.

$$H_\chi(G_f, X) \leq H(c'(G_f)) = H_\chi(G, x) \leq H(c(G))$$

Summary

Theorem

We can get saving over the block encoding in zero-error function compression regime.

$$H_{G_f}(X) \leq H_{\chi}(G_f, X)$$

Proof.

if c is a coloring of G, then $c^{-1}(c(X))$, the color class of X, induced a partition of V into independent sets of G, and always contains X.

$$\begin{aligned} H_{\chi}(G_f, X) &= \min\{H(c(X)|c^{-1}(c(X)) \text{ are the color classes}\} \\ &\overset{a}{=} \min_{X \in W \in \Gamma(G_f)} \{H(W)|\text{disjoint union of W's are the color classes}\} \\ &\overset{b}{=} \min_{X \in W \in \Gamma(G_f)} \{I(W;X)|\text{disjoint union of W's are the color classes}\} \\ &\overset{c}{\geq} \min_{X \in W \in \Gamma(G_f)} \{I(W;X)\} = H_{G_f}(X) \end{aligned}$$

Summary

	Source, side info fixed length	Source, side info variable length	Function
scalar code Zero error	$\log \chi(G)$	$H_\chi(G, X)$	$H_\chi(G_f, X)$
block code Zero error	$\log \chi(G^{\wedge n})$	$H_G(X)$	$H_{G_f}(X)$
block code $\epsilon - error$		$H(X\|Y)$	$H_{G_f}(X\|Y)$

$$H_G(X|Y) \leq H(X|Y) \leq H_G(X) \leq H_\chi(G, X) \leq H(X) \leq \log \chi(G)$$

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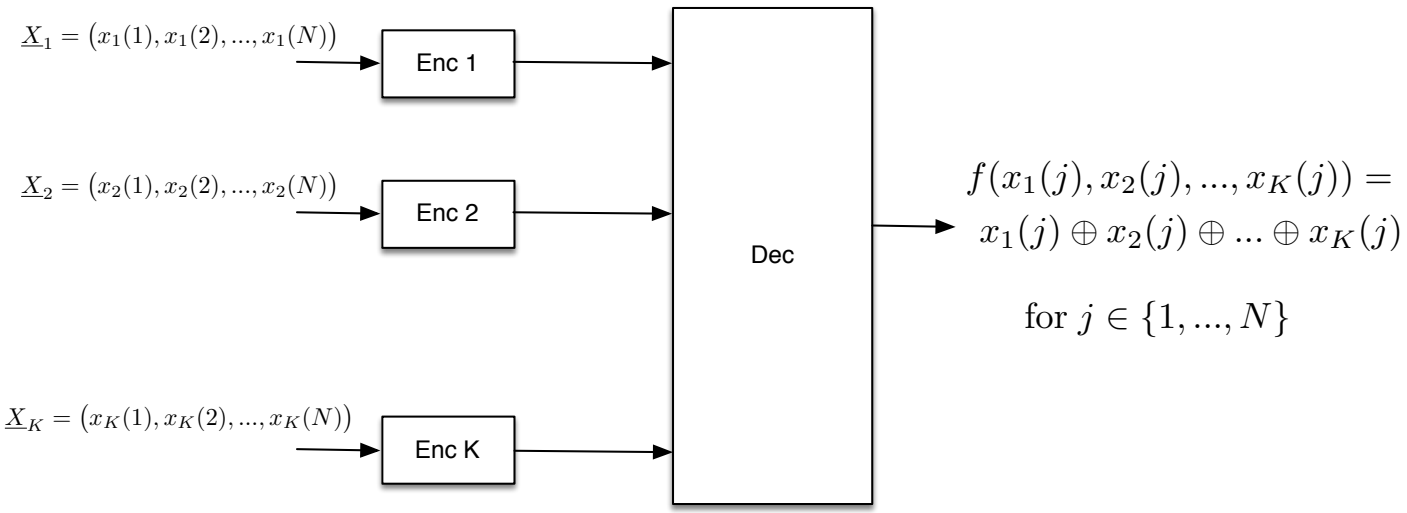
Compression of specific functions

- In environmental monitoring, a relevant statistic of temperature sensor readings $x_1, x_2, ..., x_n$ may be the mean temperature, or median, or mode.
- In "alarm" networks, the quantity of interest might be the maximum, $\max_{1 \leq i \leq n}(x_i)$ of n temperature readings.
- Frequency histogram function counts the number of occurrences of each argument among measurements. It yields many good results such as mean, variance, maximum, minimum, median and other important statistics
- Function compression on specific functions by graph coloring which provide benefit over compressing each source separately

mod r-sum

- We have K users which has access to independent sequences
- Each node (user) periodically makes measurements, which belongs to a fixed finite set $\mathcal{X} = \{1, 2, ..., |\mathcal{X}|\}$.

$$f : \mathcal{X}^K \to \{0, ..., r-1\}$$



The characteristic graph of the i-th user

Theorem

$(p, q) \notin E_i$ $(p, q \in \mathcal{X})$, *if and only if,* $p \equiv q \bmod r$

Proof.

$\Rightarrow$: Since $(p, q) \notin E_i$, we obtain that

$$\text{for all } \underline{x}_{\setminus i} \in \mathcal{X}^{N-1}, \qquad f(\underline{X}) = f(p, \underline{x}_{\setminus i}) = f(q, \underline{x}_{\setminus i}).$$

$$p + \sum_{j \neq i} x_j \equiv q + \sum_{j \neq i} x_j \bmod r \Rightarrow p \equiv q \bmod r.$$

$\Leftarrow$: If $p \equiv q \bmod r$, and $\sum_{j \neq i} x_j \equiv \sum_{j \neq i} x_j \bmod r$, then for all $\underline{x}_{\setminus i} \in \mathcal{X}^{N-1}$ we have

$$p + \sum_{j \neq i} x_j \equiv q + \sum_{j \neq i} x_j \bmod r.$$

Hence, $p, q \notin E_i$ □

The characteristic graph of the i-th user

- Note that maximal independent sets form a partition.
  - $W_0 = \{x \in \mathcal{X} | x = 0 \,(\mathrm{mod} r)\}$
  - $W_1 = \{x \in \mathcal{X} | x = 1 \,(\mathrm{mod} r)\}$
  - ...
  - $W_{r-1} = \{x \in \mathcal{X} | x = r-1 \,(\mathrm{mod} r)\}$

$\Rightarrow$ $H_\chi(G, X)$ can be achieved by assigning different colors to different maximal independent sets.

$\Rightarrow$ The minimum entropy coloring can be achieved by optimal coloring $\chi(G)$

Optimal rate for Mod r sum

$$\begin{aligned} H_G(X|Y) &= \min I(W, X|Y) \\ &= \min\{H(W|Y) - H(W|XY)\} \\ &= \min\{H(W|Y) - H(W|X)\} \\ &= H(W|Y) \\ &= H(W) = -(P_0 \log P_0 + ... + P_{r-1} \log P_{r-1}) \end{aligned}$$

- where $P_i = \sum_{x_i \in W_i} p(x_i)$
- If the distribution over the vertices is uniform, and $|\mathcal{X}| = Cr$, and $C \in \mathbb{Z}^+$, the total benefit is

$$H(X|Y) - H_G(X|Y) = \log |\mathcal{X}| - \log r = \log C \geq 0$$

Summary

- Source coding problem with side information in zero-error regime is discussed.
- The optimal rate for function compression in zero-error regime is derived
- Function computation in asymptoticly zero error regime is discussed.
- The optimal rate for computing mod r sum is derived.

Thank you

Questions?