Distributed Lossy Interactive Function Computation

Solmaz Torabi

Dept. of Electrical and Computer Engineering
Drexel University
Advisor: Dr. John M. Walsh

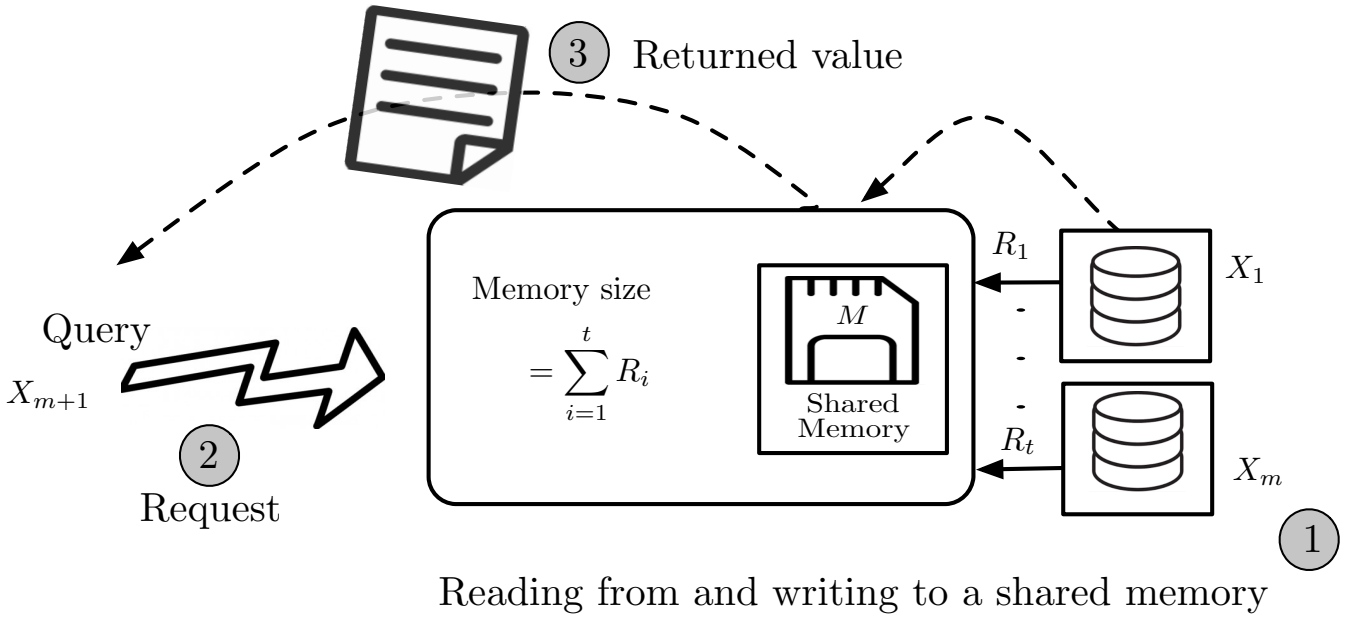
54th Annual Allerton Conference
on Communication, Control, and Computing

Outline

- Overview and problem formulation
  - Motivation and application
  - Problem setup
  - Related work
- Main Result
  - Achievability proof
  - Proof of the converse

Motivation

Approximate computing in shared memory database Systems

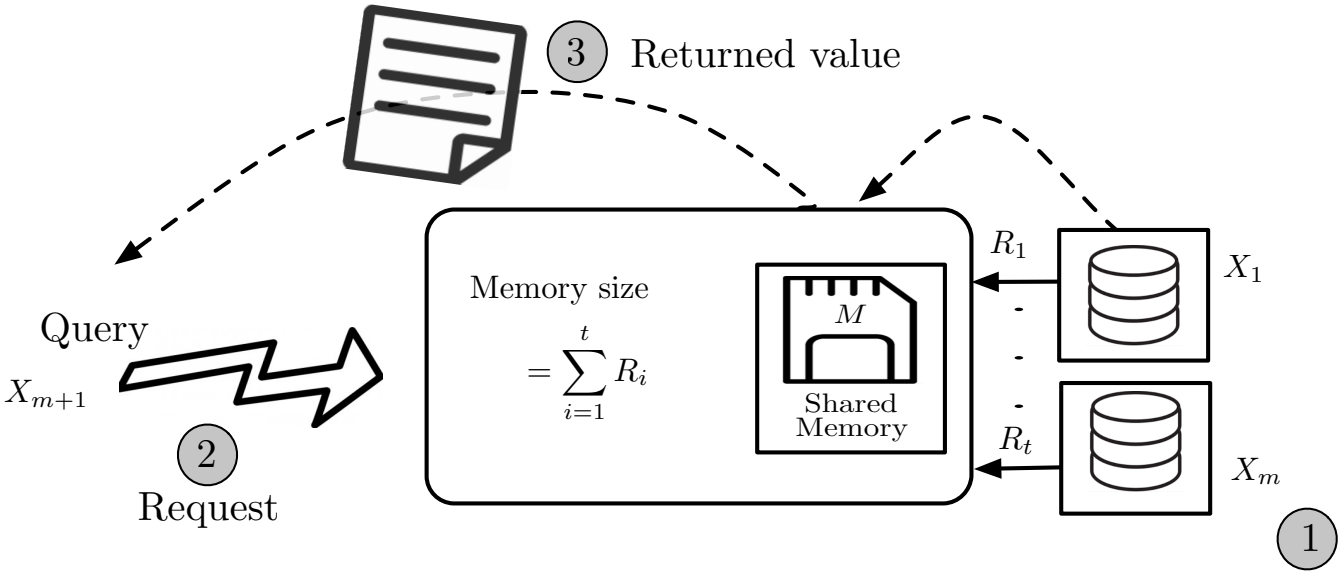


Reading from and writing to a shared memory

- Performing analysis over big bata inside main memory.
- Queries can be answered much faster if they are processed within the memory
- Goal is store relevant information to the cache such that when the query arrives, the content of the cache would be enough to approximately compute any particular query from the query set.

Motivation

Approximate computing in shared memory database Systems

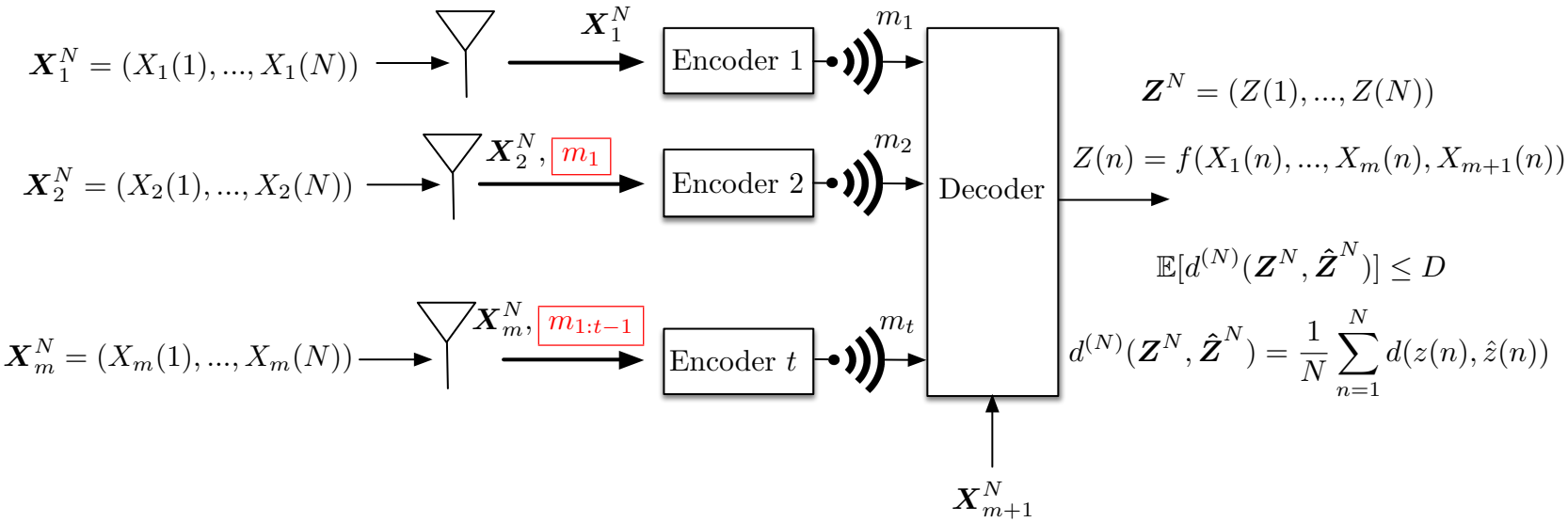


Reading from and writing to a shared memory

- Trade-off between the *memory size* and *distortion* is studied.
- In future works distortion reflects extra data rate required upon *cache miss* in shared memory database systems.

4

Overview



- We represent the query as the side information which is available at the memory but not the disks.

5

Outline

- Overview and problem formulation
  - Motivation and application
  - Related work
  - Problem setup
- Main Result
  - Achievability proof
  - Proof of the converse

Related Work

- Source coding with side information by Wyner & Ziv (1976)
  - Two terminals, 1 message, reproducing X.
- Interactive source coding by Kaspi (1986)
  - Two terminals, t messages, reproducing Xs.
- Interactive function computation by Ma & Ishwar (2009)
  - Two terminals, t messages, reproducing f
- The Wyner-Ziv Problem With Multiple Sources by Gastpar (2002)
  - Multiple terminals, 1 message, reproducing Xs
- Here: Interactive function computation
  - Multiple terminals, t messages, reproducing f

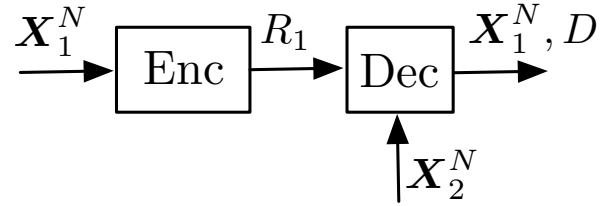
7

Related Work: Point to point models

Source coding with side information by Wyner & Ziv (1976)

$$\mathcal{RD}_{WZ} = \left\{ (\mathbf{R}, D) \;\middle|\; \begin{array}{c} R_1 \geq I(X_1; U_1|X_2) \\ \mathbb{E}\big[d(X_1, \hat{g}(U_1, X_2))\big] \leq D \\ U_1 \leftrightarrow X_1 \leftrightarrow X_2 \end{array} \right\}$$

- Two terminals
- 1 message
- Reproducing X

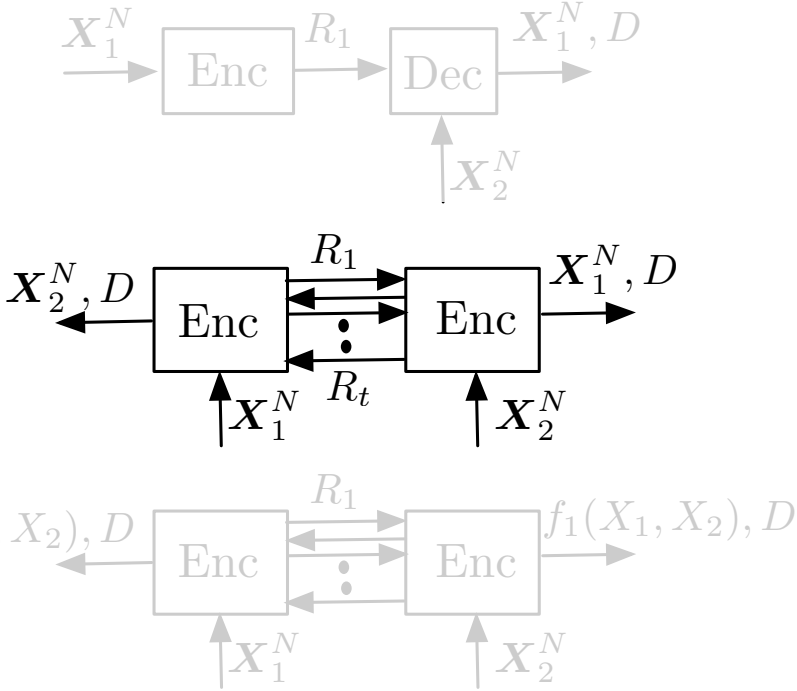


8

Related Work: Point to point models

Interactive source coding by Kaspi (1986)

$$\mathcal{RD}_K^t = \left\{ (\mathbf{R}, D) \;\middle|\; \begin{array}{c} R_i \geq I(X_1; U_i | U_{1:i-1} X_2) \\ R_i \geq I(X_2; U_j | U_{1:j-1} X_1) \\ \mathbb{E}\left[d(X_1, \hat{g}(U_{1:t}, X_2))\right] \leq D \\ \mathbb{E}\left[d(X_2, \tilde{g}(U_{1:t}, X_1))\right] \leq D \\ U_i \leftrightarrow X_1 U_{1:i-1} \leftrightarrow X_2 \\ U_j \leftrightarrow X_2 U_{1:j-1} \leftrightarrow X_1 \end{array} \right\}$$



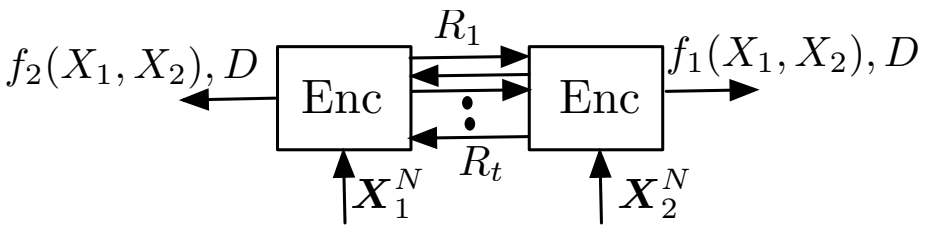
- Two terminals
- t messages
- Reproducing Xs

9

Related Work: Point to point models

Interactive function computation by Ma & Ishwar (2009)

$$\mathcal{RD}^t = \left\{ (\mathbf{R}, D) \;\middle|\; \begin{array}{c} R_i \geq I(X_1; U_i | U_{1:i-1} X_2) \\ R_i \geq I(X_2; U_j | U_{1:j-1} X_1) \\ \mathbb{E}\left[d(Z_1, \hat{g}(U_{1:t}, X_2))\right] \leq D \\ \mathbb{E}\left[d(Z_2, \tilde{g}(U_{1:t}, X_1))\right] \leq D \\ U_i \leftrightarrow X_1 U_{1:i-1} \leftrightarrow X_2 \\ U_j \leftrightarrow X_2 U_{1:j-1} \leftrightarrow X_1 \end{array} \right\}$$

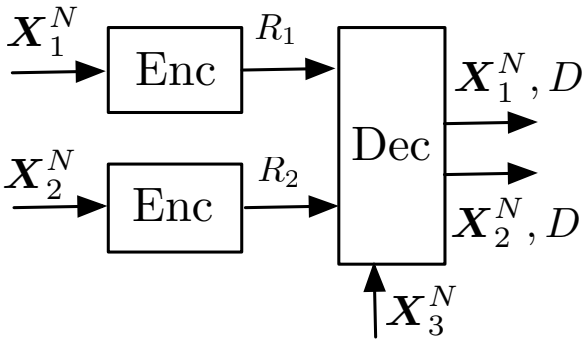


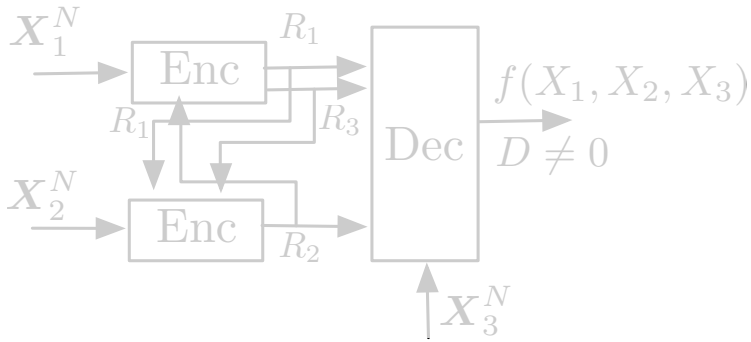
- Two terminals
- t messages
- reproducing f

Related Work: Distributed models

Wyner-Ziv Problem With Multiple Sources by Gastpar (2002)

$$\mathcal{RD} = \left\{ (\mathbf{R}, D) \;\middle|\; \begin{array}{c} R_1 \geq I(X_1; U_1|X_3) \\ R_2 \geq I(X_2; U_2|X_3) \\ \mathbb{E}\left[d(X_1, \hat{g}(U_{1,2}, X_3))\right] \leq D \\ \mathbb{E}\left[d(X_2, \tilde{g}(U_{1,2}, X_3))\right] \leq D \\ U_1 \leftrightarrow X_1 \leftrightarrow X_2, X_3 \\ U_2 \leftrightarrow X_2 \leftrightarrow X_1, X_3 \end{array} \right\}$$



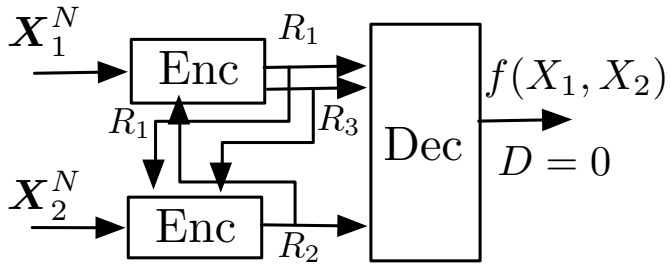


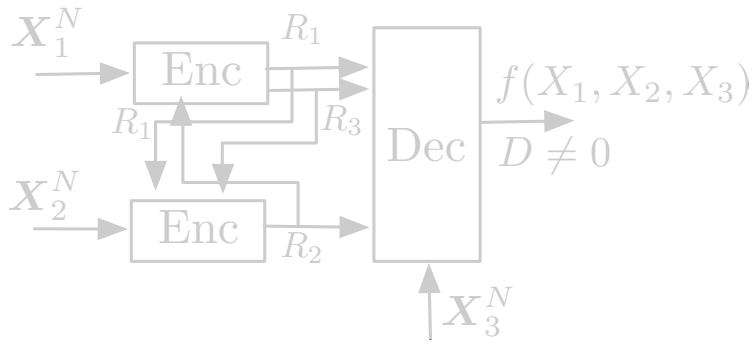
- Multiple terminals
- $X_1 \leftrightarrow X_3 \leftrightarrow X_2$
- 1 message
- reproducing Xs

Related Work: Distributed models

Interactive Lossless function computation by Ma & Ishwar (2012)

$$\mathcal{R}^t = \left\{ \mathbf{R} \;\middle|\; \begin{array}{c} R_i \geq I(X_1; U_i | U_{1:i-1}) \\ R_j \geq I(X_2; U_j | U_{1:j-1}) \\ H(f(X_1, X_2) | U_1, U_2) = 0 \\ U_i \leftrightarrow X_1 U_{1:i-1} \leftrightarrow X_2 \\ U_j \leftrightarrow X_2 U_{1:j-1} \leftrightarrow X_1 \end{array} \right\}$$





- Multiple terminals
- Independent sources: $X_1 \perp\!\!\!\perp X_2$
- t messages
- reproducing f

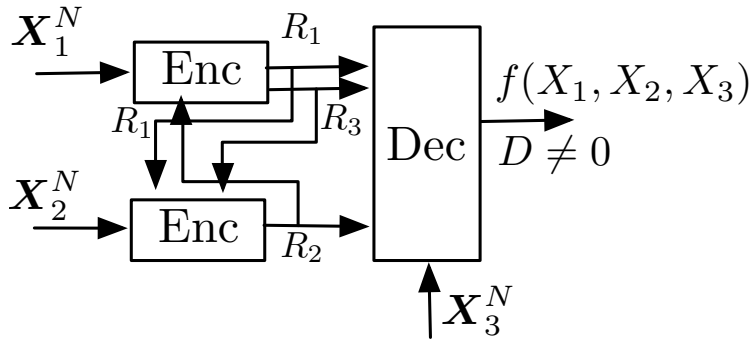
12

Distributed models

Here: Lossy Interactive function computation

$$\mathcal{RD}^t = ?$$

- Multiple terminals
- $X_1 \leftrightarrow X_3 \leftrightarrow X_2$
- t messages
- reproducing f



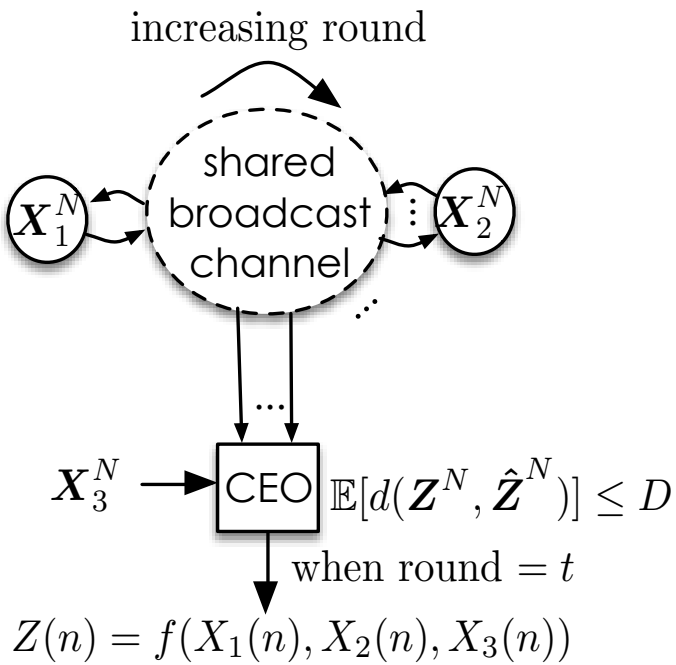
13

Outline

- Overview and problem formulation
  - Motivation and application
  - Related work
  - Problem setup
- Main Result
  - Achievability proof
  - Proof of the converse

Problem setup

Distortion measures the approximation in computing



$$\psi_i : \mathcal{X}_1^N \times \bigotimes_{l=1}^{i-1} \mathcal{M}_l \to \mathcal{M}_i \quad i \text{ odd}$$

$$\psi_i : \mathcal{X}_2^N \times \bigotimes_{l=1}^{i-1} \mathcal{M}_l \to \mathcal{M}_i \quad i \text{ even}$$

$$\phi : \mathcal{X}_3^N \times \bigotimes_{\ell=1}^{t} \mathcal{M}_l \to \mathcal{Z}$$

$$R_t = (1/N) \log |\mathcal{M}_t|$$

15

Outline

- Overview and problem formulation
  - Motivation and application
  - Related work
  - Problem setup
- Main Result
  - Achievability proof
  - Proof of the converse

Rate-Distortion Region

Theorem

For a two-user lossy *t*-round interactive function computation with a side information X_3, where node $i \in \{1,2\}$ observes X_i, and $X_1 \leftrightarrow X_3 \leftrightarrow X_2$, the rate distortion region is:

$$\mathcal{RD}^t = \left\{ (\mathbf{R}, D) \;\middle|\; \begin{array}{c} R_i \geq I(X_1; U_i | U_{1:i-1} X_2) \quad i \text{ odd} \\ R_i \geq I(X_2; U_i | U_{1:i-1} X_1) \quad i \text{ even} \\ j = ((i-1) \bmod 2) + 1 \\ \mathbb{E}\big[d(X_{1:3}, \hat{g}(U_{1:t}, X_3))\big] \leq D \\ U_i \leftrightarrow X_j, U_{1:i-1} \leftrightarrow X_{\{1,2,3\}\setminus\{j\}} \end{array} \right\}$$

where the alphabets $\mathcal{U}_i$ satisfy $|\mathcal{U}_i| \leq |\mathcal{X}_j| \prod_{r=1}^{i-1} |\mathcal{U}_r| + 1 + t - i$.

17

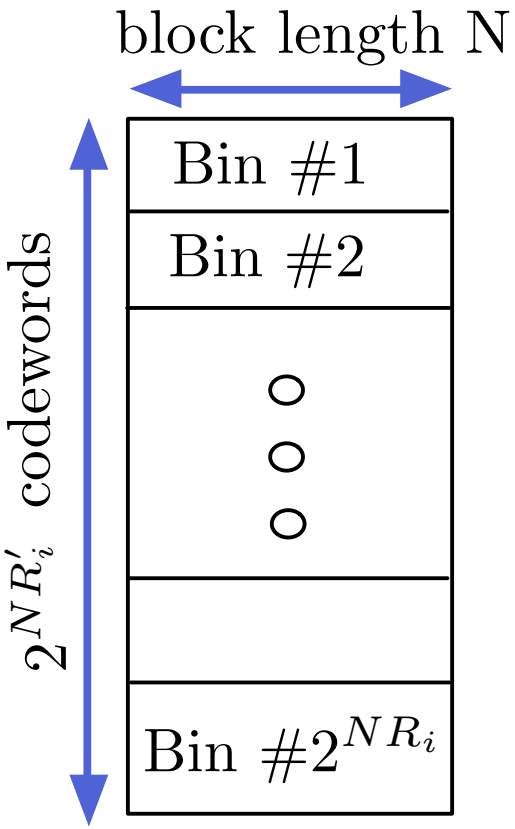
Outline

- Overview and problem formulation
  - Motivation and application
  - Related work
  - Problem setup
- Main Result
  - Achievability proof
  - Proof of the converse

Achievability

- Encoding: Given the source sequence X_1^N and the received message $\left(U_{1:i-1}^N(s_{1:i-1})\right)$, encoder i at node 1 looks for a codeword $U_i^N(s_i)$ such that $\left(X_1^N, U_{1:i}^N(s_{1:i})\right) \in \mathcal{T}_\epsilon^N$.
- The size of the generated codeword sequence should be large enough to ensure the existence of jointly typical codeword with the observation.

$$\boxed{R_i' \geq I(U_i; X_1, U_{1:i-1})} \qquad (1)$$



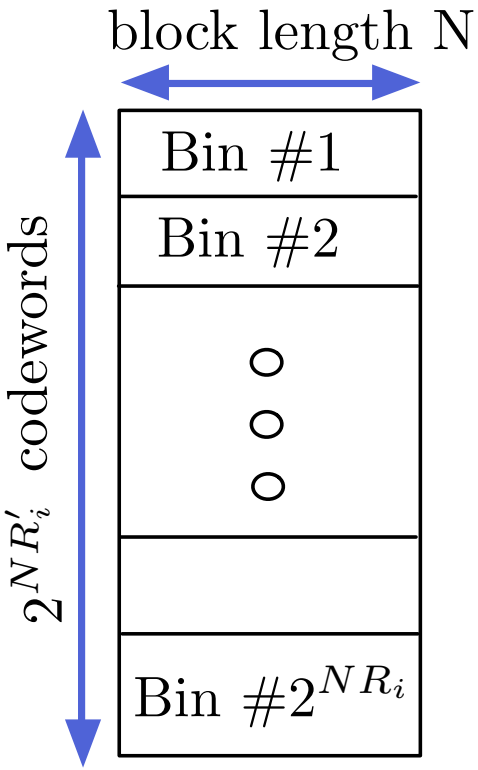
19

Achievability

- Decoding: After receiving the message, the other nodes as well as the CEO, look for a codeword with the same bin index jointly typical with their own observations. That is, among those $s_i \in \mathcal{B}_i(r_i)$, all the other nodes X_2, X_3, find an index such that $\left(X_2^N, U_{1:i}^N(s_{1:i})\right) \in \mathcal{T}_\epsilon^N$, and $\left(X_3^N, U_{1:i}^N(s_{1:i})\right) \in \mathcal{T}_\epsilon^N$.

- The number of codewords in each bin should be small enough so that the probability of having more jointly typical sequences in each bin is small.

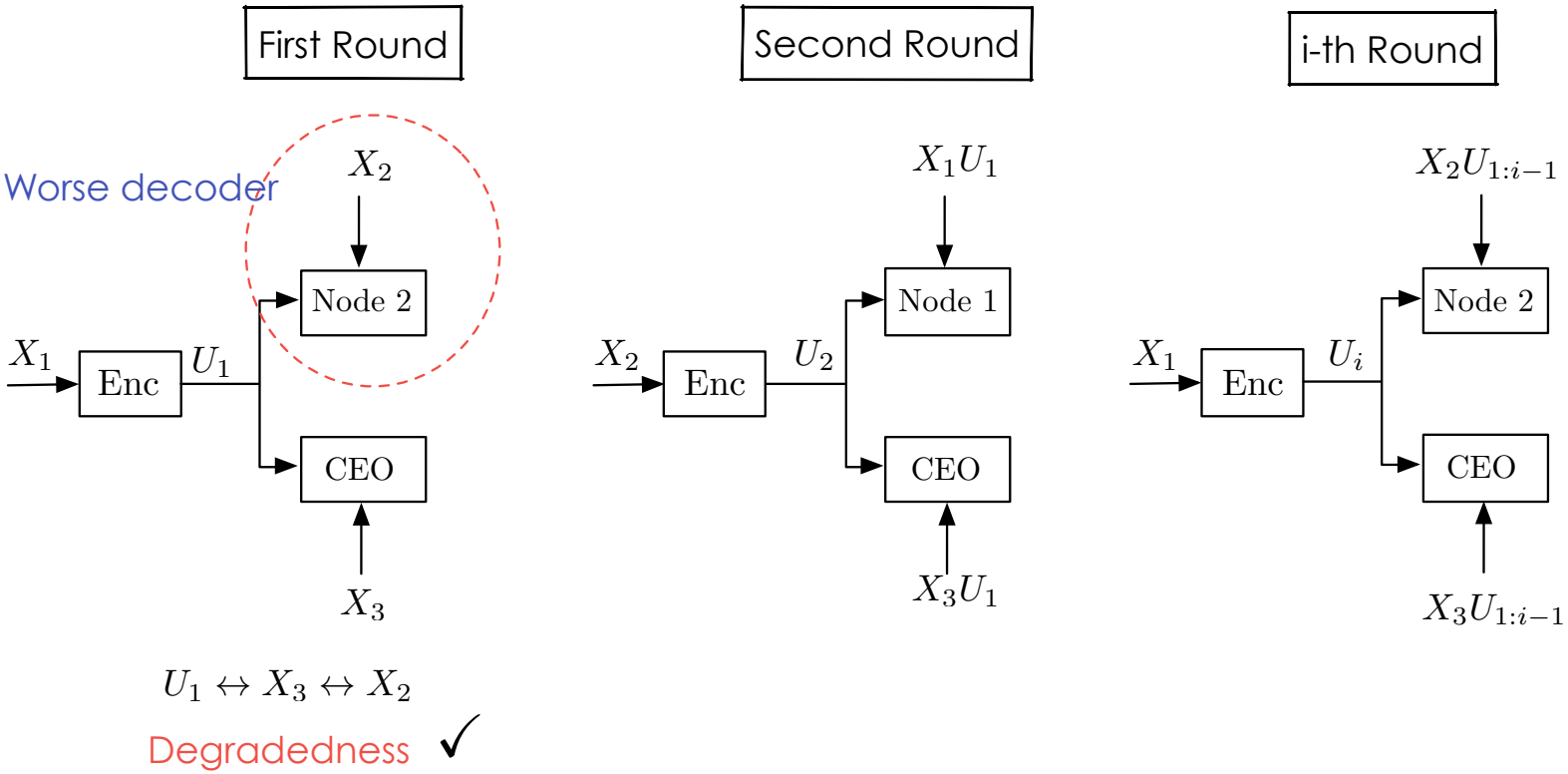
$$R_i' - R_i \leq I\left(U_i; X_2, U_{1:i-1}\right)$$
$$R_i' - R_i \leq I\left(U_i; X_3, U_{1:i-1}\right)$$



20

Achievability proof sketch

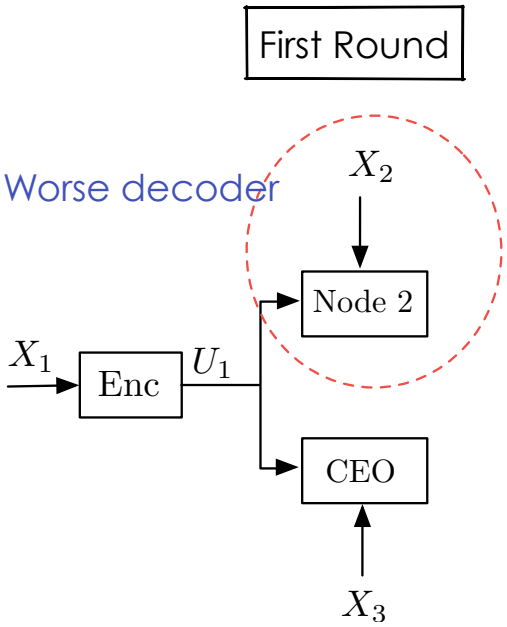
- Find the worst decoder at each round



21

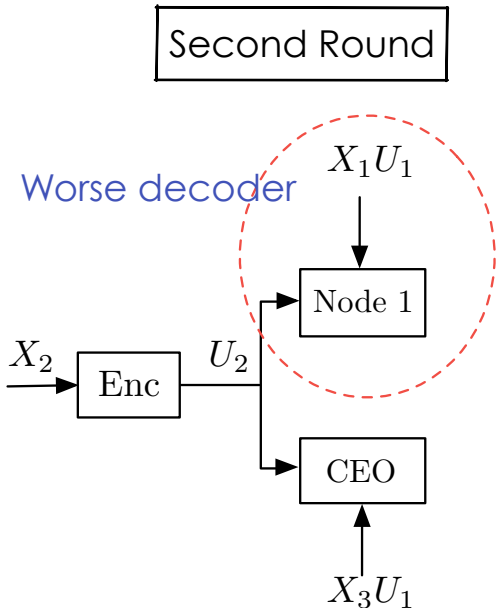
Achievability proof sketch

- Find the worst decoder at each round



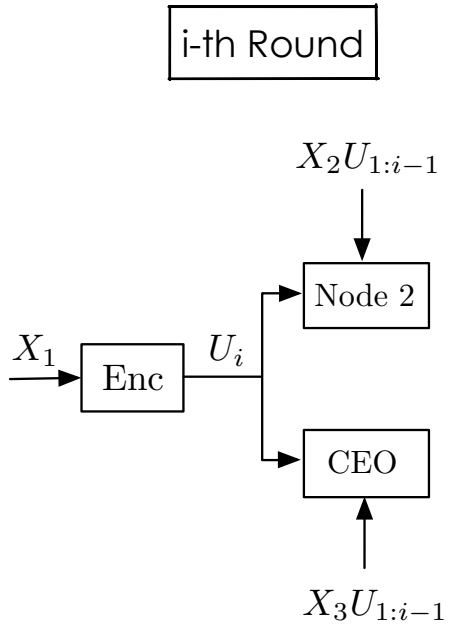
$U_1 \leftrightarrow X_3 \leftrightarrow X_2$

Degradedness ✓



We prove degradedness by

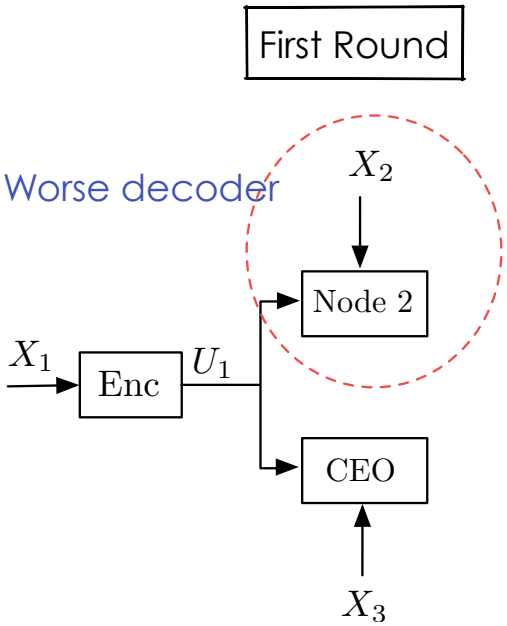
$U_2 \leftrightarrow X_3U_1 \leftrightarrow X_1U_1$



22

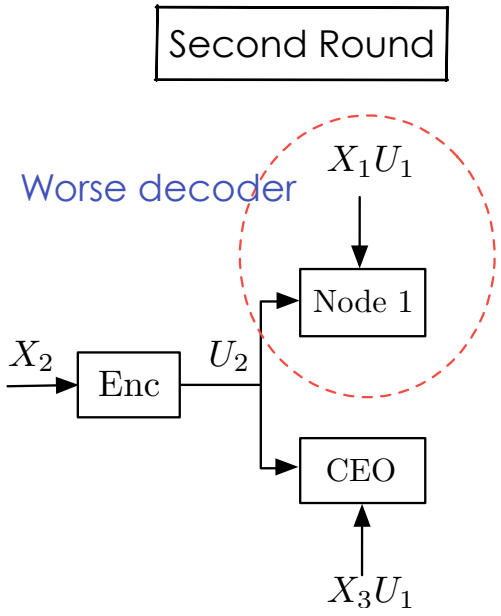
Achievability proof sketch

- Find the worst decoder at each round



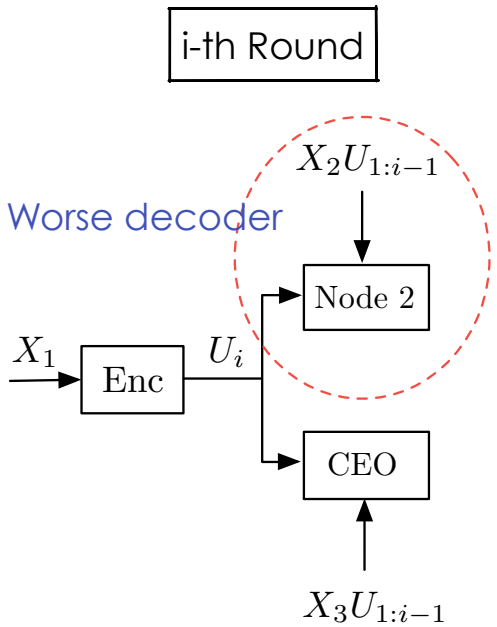
$U_1 \leftrightarrow X_3 \leftrightarrow X_2$

Degradedness ✓



We prove degradedness by

$U_2 \leftrightarrow X_3U_1 \leftrightarrow X_1U_1$



We prove degradedness by

$U_i \leftrightarrow U_{1:i-1}, X_3 \leftrightarrow U_{1:i-1}, X_2$

23

Finding the worse decoder at each round

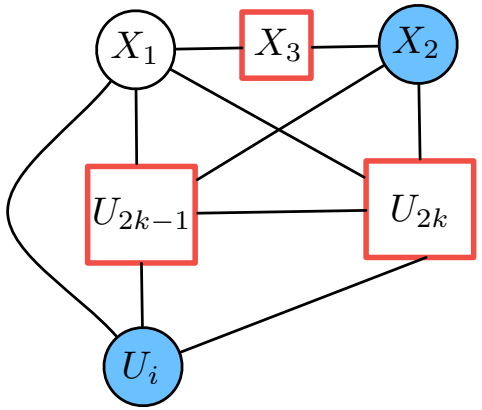
If $X_1 \leftrightarrow X_3 \leftrightarrow X_2$, we have

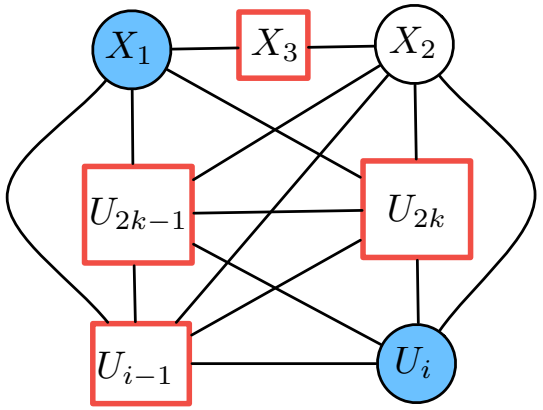
$$U_i \leftrightarrow U_{1:i-1}, X_3 \leftrightarrow U_{1:i-1}, X_2 \quad i \text{ odd}$$
$$U_i \leftrightarrow U_{1:i-1}, X_3 \leftrightarrow U_{1:i-1}, X_1 \quad i \text{ even}$$

Graphical representation of the factorized distribution:
$(A \leftrightarrow C \leftrightarrow B)$ if every path from A to B contains some node in C

$\forall k = 1 : \frac{i-1}{2}$

i odd





$\forall k = 1 : \frac{i-2}{2}$

i even

$$p(x_1, x_2, x_3, u_{1:i}) = p(x_3)p(x_1|x_3)p(x_2|x_3)p(u_1|x_1)p(u_2|x_2u_1)\times\ldots\times p(u_i|x_1u_{1:i-1})$$

24

Achievability continued

If $X_1 \leftrightarrow X_3 \leftrightarrow X_2$, we have

$$U_i \leftrightarrow U_{1:i-1}, X_3 \leftrightarrow U_{1:i-1}, X_2$$

By data processing inequality

$$I\left(U_i; X_2, U_{1:i-1}\right) \leq I\left(U_i; X_3, U_{1:i-1}\right)$$

- At round i (odd), this makes the second user with observation $U_{1:i-1}, X_2$ is the worse decoder.
- At round i (even), this makes the the first user with observation $U_{1:i-1}, X_1$ is the worse decoder.

$$\boxed{R_i' - R_i \leq I\left(U_i; X_2, U_{1:i-1}\right)} \Rightarrow \boxed{R_i \geq I(X_1; U_i | U_{1:i-1} X_2)}$$

25

Outline

- Overview and problem formulation
  - Motivation and application
  - Related work
  - Problem setup
- Main Result
  - Achievability proof
  - Proof of the converse

26

Converse prove

Converse assumption: $\exists$ an interactive distributed block source code with parameters $(t, N, |\mathcal{M}|_1, ..., |\mathcal{M}_t|)$ satisfying

$$\frac{1}{N}\log_2|\mathcal{M}_i| \leq R_i$$
$$\mathbb{E}\left[d^{(N)}(\boldsymbol{X}_1^N, \boldsymbol{X}_2^N, \boldsymbol{X}_3^N, \phi(M_{1:t}, \boldsymbol{X}_3^N)\right] \leq D$$

- Define a set of auxiliary random variables, and a function $\hat{g}$ that obey single letter distortion

$$\mathbb{E}\left[d(X_1(n), X_2(n), X_3(n), \hat{g}(U_{1:t}, X_3(n))\right] \leq D$$

27

Converse prove: Single letter distortion

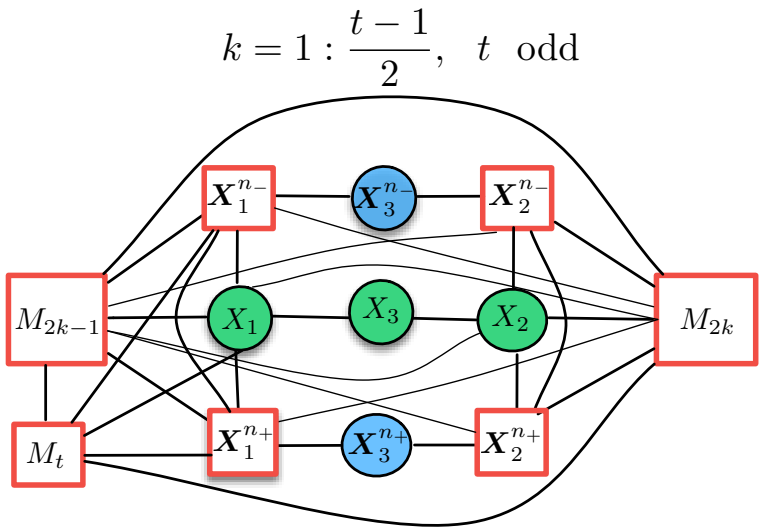
$$D \geq \frac{1}{N}\mathbb{E}\left[d\left(X_1(n), X_2(n), X_3(n), \phi_n(M_{1:t}, \boldsymbol{X}_3^{\setminus n}, X_3(n)\right)\right]$$
$$\stackrel{?}{\geq} \frac{1}{N}\mathbb{E}\left[d(X_1(n), X_2(n), X_3(n), \hat{g}(U_{1:t}, X_3(n))\right]$$

Two possibilities:

- $U_1 = \{M_1, \boldsymbol{X}_3^{\setminus n}, ...\}$

OR

- $U_1 = \{M_1, \boldsymbol{X}_1^{\setminus n}, \boldsymbol{X}_2^{\setminus n}\}$
- $\boldsymbol{X}_3^{\setminus n} \leftrightarrow M_{1:t}, \boldsymbol{X}_1^{\setminus n}, \boldsymbol{X}_2^{\setminus n} X_3(n) \leftrightarrow X_1(n)X_2(n)X_3(n)$



28

Converse prove: Single letter distortion

$$D \geq \frac{1}{N}\mathbb{E}\left[d\left(X_1(n), X_2(n), X_3(n), \phi_n(M_{1:t}, \boldsymbol{X}_3^{\setminus n}, X_3(n))\right)\right]$$
$$\overset{?}{\geq} \frac{1}{N}\mathbb{E}\left[d(X_1(n), X_2(n), X_3(n), \hat{g}(U_{1:t}, X_3(n))\right]$$

Two possibilities:

- $\boxed{U_1 = \{M_1, \boldsymbol{X}_3^{\setminus n}, ...\}}$ ✓ Now check if it satisfies the Markovs

OR

- $U_1 = \{M_1, \boldsymbol{X}_1^{\setminus n}, \boldsymbol{X}_2^{\setminus n}\}$ ✗ Does not satisfy the rate constraints ✗

29

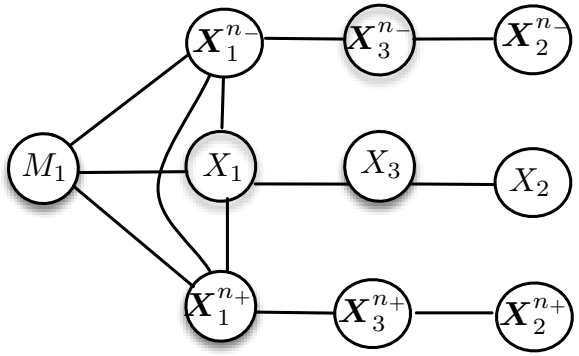
Converse: Single letter Markov chains

- Factorized distribution:

$$p(\boldsymbol{X}_1^N, \boldsymbol{X}_2^N, \boldsymbol{X}_3^N, M_1)$$
$$= p(\boldsymbol{X}_3^N) \prod_{i=1}^{2} p(\boldsymbol{X}_i^{n-} | \boldsymbol{X}_3^{n-}) \prod_{j=1}^{2} p(\boldsymbol{X}_j^{n+} | \boldsymbol{X}_3^{n+}) \prod_{k=1}^{2} p(X_k(n) | X_3(n)) p(M_1 | \boldsymbol{X}_1^N)$$

First Round

$$U_1 \leftrightarrow X_1(n) \leftrightarrow X_2(n) X_3(n)$$
$$U_1(n) = \{....$$



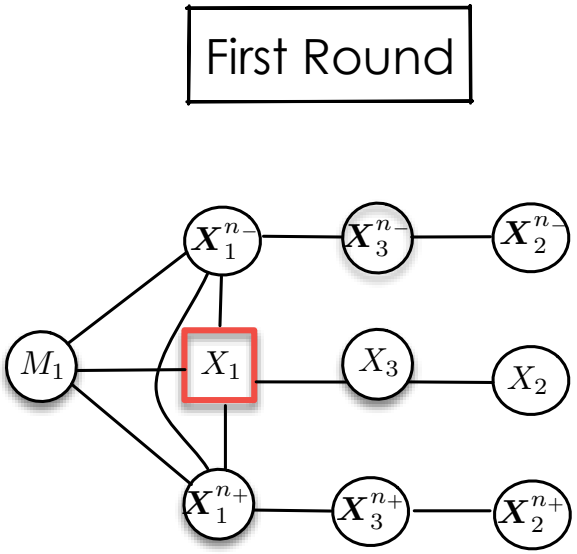
30

Converse: Single letter Markov chains

- To obey the first Markov chain, which nodes can be grouped together to construct the first auxiliary U_1?
- check if $\boxed{U_1 = \{M_1, \boldsymbol{X}_3^{\setminus n}, ...\}}$ or any possible add to can obey the Markovs.

$$U_1 \leftrightarrow X_1(n) \leftrightarrow X_2(n)X_3(n)$$

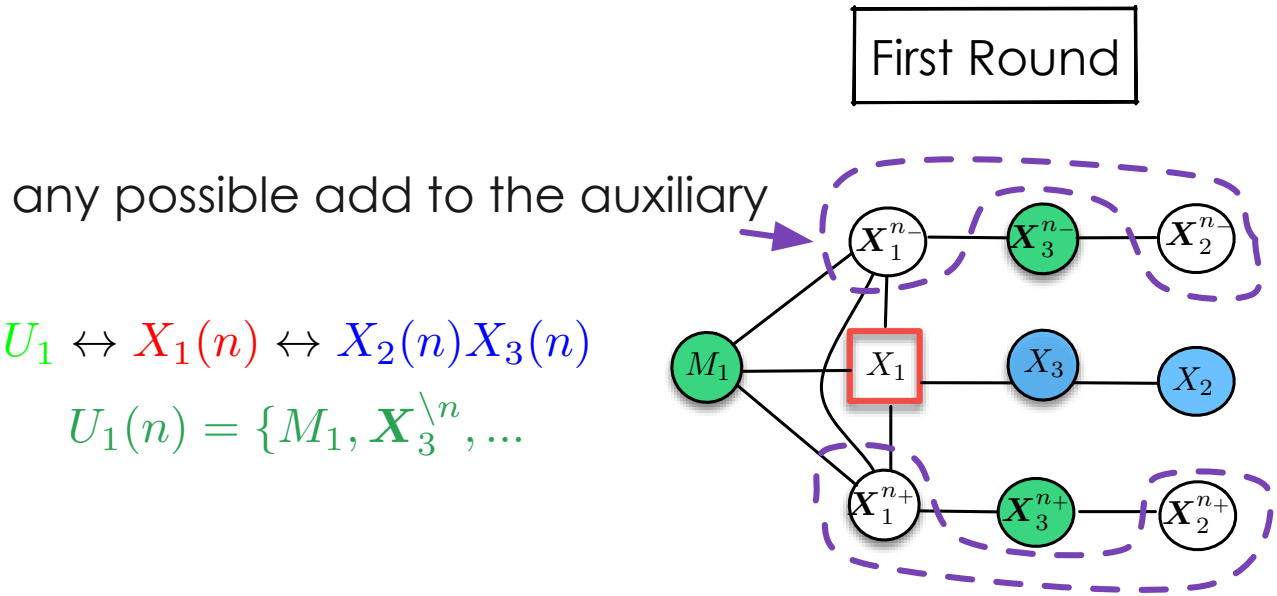
$$U_1(n) = \{M_1, \boldsymbol{X}_3^{\setminus n}, ...$$



31

Converse: Single letter Markov chains

- To obey the first Markov chain, which nodes can be grouped together to construct the first auxiliary U_1?
- No restriction in the first round
- This choice of auxiliary $U_1(n) = \{M_1, \boldsymbol{X}_3^{\setminus n}, ...\}$ so far obeys the first Markov.



32

Converse: Single letter Markov chains

- Factorized distribution:

$$p(\boldsymbol{X}_1^N, \boldsymbol{X}_2^N, \boldsymbol{X}_3^N, M_1, M_2) = p(\boldsymbol{X}_3^N) \prod_{i=1}^{2} p(\boldsymbol{X}_i^{n-} | \boldsymbol{X}_3^{n-}) \prod_{j=1}^{2} p(\boldsymbol{X}_j^{n+} | \boldsymbol{X}_3^{n+})$$

$$\times \prod_{k=1}^{2} p(X_k(n) | X_3(n)) p(M_1 | \boldsymbol{X}_1^N) p(M_2 | \boldsymbol{X}_2^N, M_1)$$

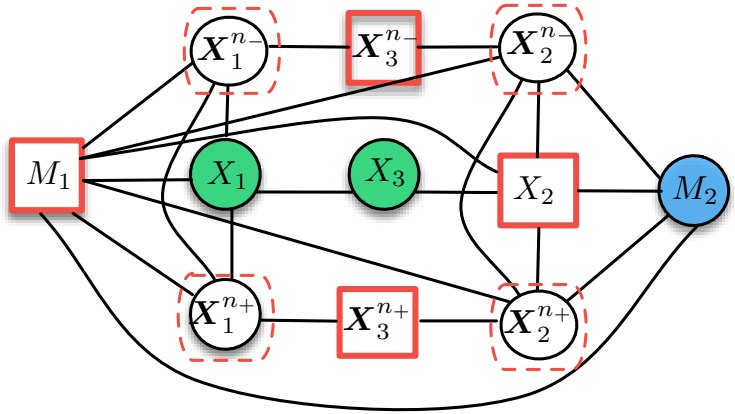
Second Round

observing other rvs doesn't violate the Markovs

$U_2 \leftrightarrow X_2(n)U_1(n) \leftrightarrow X_1(n)X_3(n)$

$U_1(n) = \{M_1, \boldsymbol{X}_3^{n-}, ...\}$

$U_2(n) = M_2$



33

Converse: Single letter rate expression

$$R_1 \geq H(M_1)$$

34

Converse: Single letter rate expression

$$
\begin{aligned}
R_1 &\geq H(M_1) \\
&\geq H(M_1|\boldsymbol{X}_2^N)
\end{aligned}
$$

Converse: Single letter rate expression

$$
\begin{aligned}
R_1 &\geq H(M_1) \\
&\geq H(M_1|\boldsymbol{X}_2^N) \\
&= H(M_1|\boldsymbol{X}_2^N) - H(M_1|\boldsymbol{X}_1^N, \boldsymbol{X}_2^N, \boldsymbol{X}_3^N)
\end{aligned}
$$

Converse: Single letter rate expression

$$
\begin{aligned}
R_1 &\geq H(M_1) \\
&\geq H(M_1|\boldsymbol{X}_2^N) \\
&= H(M_1|\boldsymbol{X}_2^N) - H(M_1|\boldsymbol{X}_1^N, \boldsymbol{X}_2^N, \boldsymbol{X}_3^N) \\
&= I(\boldsymbol{X}_1^N \boldsymbol{X}_3^N; M_1|\boldsymbol{X}_2^N)
\end{aligned}
$$

Converse: Single letter rate expression

$$
\begin{aligned}
R_1 &\geq H(M_1) \\
&\geq H(M_1|\boldsymbol{X}_2^N) \\
&= H(M_1|\boldsymbol{X}_2^N) - H(M_1|\boldsymbol{X}_1^N, \boldsymbol{X}_2^N, \boldsymbol{X}_3^N) \\
&= I(\boldsymbol{X}_1^N \boldsymbol{X}_3^N; M_1|\boldsymbol{X}_2^N) \\
&= \sum_{n=1}^{N} H(X_1(n)X_3(n)|X_2(n)) - H(X_1(n)X_3(n)|M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{n+} \boldsymbol{X}_2^N)
\end{aligned}
$$

Converse: Single letter rate expression

$$
\begin{aligned}
R_1 &\geq H(M_1) \\
&\geq H(M_1|\boldsymbol{X}_2^N) \\
&= H(M_1|\boldsymbol{X}_2^N) - H(M_1|\boldsymbol{X}_1^N, \boldsymbol{X}_2^N, \boldsymbol{X}_3^N) \\
&= I(\boldsymbol{X}_1^N \boldsymbol{X}_3^N; M_1|\boldsymbol{X}_2^N) \\
&= \sum_{n=1}^{N} H(X_1(n)X_3(n)|X_2(n)) - H(X_1(n)X_3(n)|M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{n+} \boldsymbol{X}_2^N) \\
&\geq \sum_{n=1}^{N} H(X_1(n)|X_2(n)) - H(X_1(n)|M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{n+} \boldsymbol{X}_2^N)
\end{aligned}
$$

Converse: Single letter rate expression

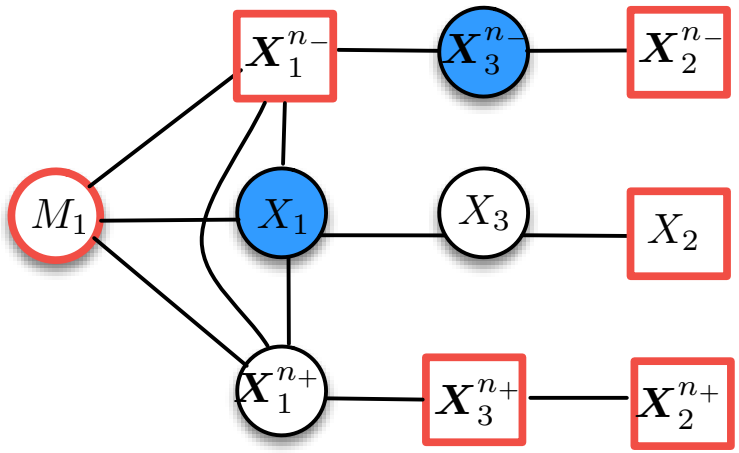
$$
\begin{aligned}
R_1 &\geq H(M_1) \\
&\geq H(M_1|\boldsymbol{X}_2^N) \\
&= H(M_1|\boldsymbol{X}_2^N) - H(M_1|\boldsymbol{X}_1^N, \boldsymbol{X}_2^N, \boldsymbol{X}_3^N) \\
&= I(\boldsymbol{X}_1^N \boldsymbol{X}_3^N; M_1|\boldsymbol{X}_2^N) \\
&= \sum_{n=1}^{N} H(X_1(n)X_3(n)|X_2(n)) - H(X_1(n)X_3(n)|M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{n+} \boldsymbol{X}_2^N) \\
&\geq \sum_{n=1}^{N} H(X_1(n)|X_2(n)) - H(X_1(n)|M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{n+} \boldsymbol{X}_2^N) \\
&\overset{a_1}{=} \sum_{n=1}^{N} H(X_1(n)|X_2(n)) - H(X_1(n)|M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{n+}, \boldsymbol{X}_2^N, \boldsymbol{X}_3^{n-})
\end{aligned}
$$

Converse: Single letter rate expression

$$R_1 \geq \sum_{n=1}^{N} H(X_1(n)|X_2(n)) - H(X_1(n)|M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{n+} \boldsymbol{X}_2^{N})$$

$$= \sum_{n=1}^{N} H(X_1(n)|X_2(n)) - H(X_1(n)|M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{n+}, \boldsymbol{X}_2^{N}, \boldsymbol{X}_3^{n-})$$

$$\Longrightarrow \boxed{X_1(n) \leftrightarrow M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{n+} \boldsymbol{X}_2^{N} \leftrightarrow \boldsymbol{X}_3^{n-}}$$



35

Converse: Single letter rate expression

$$
\begin{aligned}
R_1 &\geq \sum_{n=1}^{N} H(X_1(n)|X_2(n)) - H(X_1(n)|M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{n+} \boldsymbol{X}_2^{N}) \\
&= \sum_{n=1}^{N} H(X_1(n)|X_2(n)) - H(X_1(n)|M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{n+}, \boldsymbol{X}_2^{N}, \boldsymbol{X}_3^{n-})
\end{aligned}
$$

Converse: Single letter rate expression

$$
\begin{aligned}
R_1 &\geq \sum_{n=1}^{N} H(X_1(n)|X_2(n)) - H(X_1(n)|M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{n+} \boldsymbol{X}_2^{N}) \\
&= \sum_{n=1}^{N} H(X_1(n)|X_2(n)) - H(X_1(n)|M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{n+}, \boldsymbol{X}_2^{N}, \boldsymbol{X}_3^{n-}) \\
&\geq \sum_{n=1}^{N} H(X_1(n)|X_2(n)) - H(X_1(n)|M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{\setminus n}, \boldsymbol{X}_2^{n-}, X_2(n))
\end{aligned}
$$

Converse: Single letter rate expression

$$
\begin{aligned}
R_1 &\geq \sum_{n=1}^{N} H(X_1(n)|X_2(n)) - H(X_1(n)|M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{n+} \boldsymbol{X}_2^{N}) \\
&= \sum_{n=1}^{N} H(X_1(n)|X_2(n)) - H(X_1(n)|M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{n+}, \boldsymbol{X}_2^{N}, \boldsymbol{X}_3^{n-}) \\
&\geq \sum_{n=1}^{N} H(X_1(n)|X_2(n)) - H(X_1(n)|M_1 \boldsymbol{X}_1^{n-} \boldsymbol{X}_3^{\setminus n}, \boldsymbol{X}_2^{n-}, X_2(n)) \\
&= \sum_{n=1}^{N} I(X_1(n); U_1(n)|X_2(n))
\end{aligned}
$$

Converse: Rate expression for $i \geq 2$ and i even

$$R_i \geq H(M_i)$$

Converse: Rate expression for $i \geq 2$ and i even

$$R_i \geq H(M_i) \geq H(M_i|M_{1:i-1}, \boldsymbol{X}_1^N) - \underbrace{H(M_i|M_{1:i-1}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^N, \boldsymbol{X}_3^N)}_{=0}$$

Converse: Rate expression for $i \geq 2$ and i even

$$R_i \geq H(M_i)$$

$$\geq I(\boldsymbol{X}_2^N, \boldsymbol{X}_3^N; M_i | M_{1:i-1}, \boldsymbol{X}_1^N)$$

Converse: Rate expression for $i \geq 2$ and i even

$$R_i \geq H(M_i)$$

$$\geq I(\boldsymbol{X}_2^N, \boldsymbol{X}_3^N; M_i | M_{1:i-1}, \boldsymbol{X}_1^N)$$

$$= \sum_{n=1}^{N} H(X_2(n)X_3(n) | M_{1:i-1}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+}) - H(X_2(n)X_3(n) | M_{1:i}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+})$$

Converse: Rate expression for $i \geq 2$ and i even

$$R_i \geq H(M_i)$$

$$\geq I(\boldsymbol{X}_2^N, \boldsymbol{X}_3^N; M_i | M_{1:i-1}, \boldsymbol{X}_1^N)$$

$$= \sum_{n=1}^{N} H(X_2(n)X_3(n) | M_{1:i-1}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+}) - H(X_2(n)X_3(n) | M_{1:i}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+})$$

$$\geq \sum_{n=1}^{N} H(X_2(n) | M_{1:i-1}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+}) - H(X_2(n) | M_{1:i}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+})$$

Converse: Rate expression for $i \geq 2$ and i even

$$R_i \geq H(M_i)$$

$$\geq I(\boldsymbol{X}_2^N, \boldsymbol{X}_3^N; M_i | M_{1:i-1}, \boldsymbol{X}_1^N)$$

$$= \sum_{n=1}^{N} H(X_2(n)X_3(n) | M_{1:i-1}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+}) - H(X_2(n)X_3(n) | M_{1:i}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+})$$

$$\geq \sum_{n=1}^{N} H(X_2(n) | M_{1:i-1}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+}) - H(X_2(n) | M_{1:i}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+})$$

- Positivity of the conditional mutual information

$$I(X_2(n)X_3(n); M_i | M_{1:i-1}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+}) \geq I(X_2(n); M_i | M_{1:i-1}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+})$$

Converse: Rate expression for $i \geq 2$ and i even

$$R_i \geq \sum_{n=1}^{N} H(X_2(n)|M_{1:i-1}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+}, \boldsymbol{X}_3^{n-}) - H(X_2(n)|M_{1:i}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+}, \boldsymbol{X}_3^{n-})$$

Converse: Rate expression for $i \geq 2$ and i even

$$R_i \geq \sum_{n=1}^{N} H(X_2(n)|M_{1:i-1}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+}, \boldsymbol{X}_3^{n-}) - H(X_2(n)|M_{1:i}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+}, \boldsymbol{X}_3^{n-})$$

$$\Longrightarrow \boxed{X_2(n) \leftrightarrow M_{1:i}\boldsymbol{X}_1^N\boldsymbol{X}_2^{n-}\boldsymbol{X}_3^{n-} \leftrightarrow \boldsymbol{X}_3^{n-}}$$

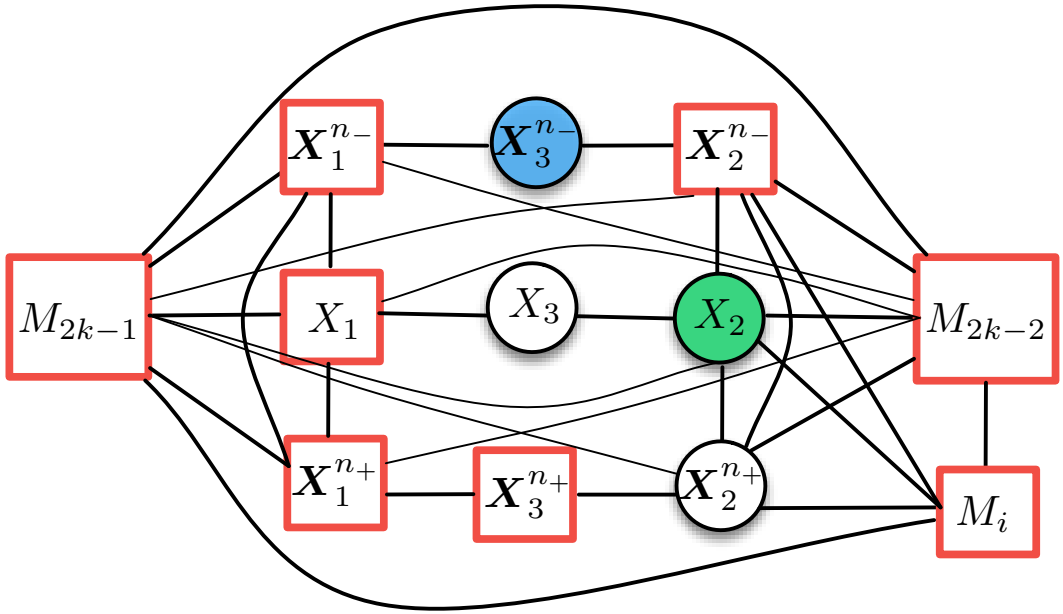
Converse: Rate expression for $i \geq 2$ and i even

$$R_i \geq \sum_{n=1}^{N} H(X_2(n)|M_{1:i-1}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+}, \boldsymbol{X}_3^{n-}) - H(X_2(n)|M_{1:i}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{n+}, \boldsymbol{X}_3^{n-})$$

$$\Longrightarrow \boxed{X_2(n) \leftrightarrow M_{1:i}\boldsymbol{X}_1^N\boldsymbol{X}_2^{n-}\boldsymbol{X}_3^{n-} \leftrightarrow \boldsymbol{X}_3^{n-}}$$

$k = 1 : \frac{i}{2}$

i even



Converse: Rate expression for $i \geq 2$ and i even

$$
\begin{aligned}
R_i &\geq \sum_{n=1}^{N} H(X_2(n)|M_{1:i-1}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{\setminus n}) \\
&\qquad - H(X_2(n)|M_{1:i}, \boldsymbol{X}_1^N, \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{\setminus n}) \\
&\geq \sum_{n=1}^{N} H(X_2(n)|M_{1:i-1}, \boldsymbol{X}_1^{n-}, X_1(n), \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{\setminus n}) \\
&\qquad - H(X_2(n)|M_{1:i}, \boldsymbol{X}_1^{n-}, X_1(n), \boldsymbol{X}_2^{n-}, \boldsymbol{X}_3^{\setminus n}) \\
&\overset{b_3}{=} \sum_{n=1}^{N} I(X_2(n); U_i|U_{1:i-1}X_1(n))
\end{aligned}
$$

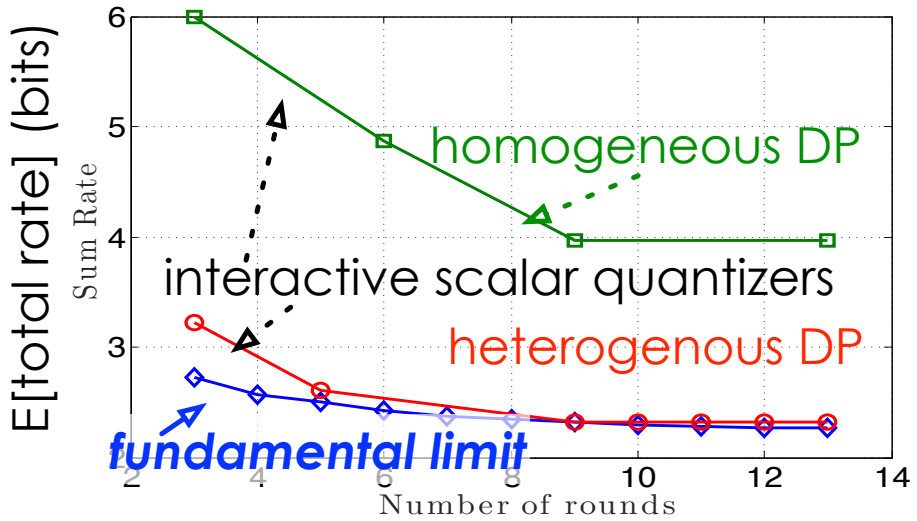
Conclusion

Lossy interactive function computation in a collocated network

- Inner bound is derived using joint typicality encoding and decoding.
- Matching outer bound is derived for conditional independence case
- If $f = \text{argmax}$

Can interaction help?

- fundamental rate for computing arg max over t number of rounds is computed for independent sources.
- Rate-Distortion region for computing arg max in one-shot is computed via Blahut-Arimoto algorithm type problem.
- Comparing the rate required for computing $D = 0$ in one-shot, and over t rounds show that interaction can help reducing the sum rate



Thank You!

Question?

Markov Lemma To ensure that the each index received by the CEO and node 2, is jointly typical with the their observation we should have For a fixed $(X_1, X_2, X_3, U_{1:i-1}) \in \mathcal{T}_\epsilon^N$. Draw U_i according to $p(U_i|X_1 U_{1:i-1})$.

$$U_i \leftrightarrow X_1 U_{1:i-1} \leftrightarrow X_2, X_3$$

This implies that, with high probability $(X_1, X_2, X_3, U_{1:i}) \in \mathcal{T}_\epsilon^N$