Non-isomorphic Distribution Supports for Calculating Entropic Vectors

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Outline

The Region of Entropic Vectors $\overline{\Gamma^*_N}$

Enumerating Canonical k-atom Supports
- Equivalence of k-atom Supports
- Orbit data structure
- Non-isomorphic k-atom supports

Mapping the Entropy Region with k-atom Distributions
- Maximal Ingleton Violation and the Four Atom Conjecture
- Optimizing Inner Bounds to Entropy from k-atom Distribution

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The Region of Entropic Vectors $\overline{\Gamma_N^*}$

Joint entropy viewed as a vector

For N discrete random variables, there is a $2^N - 1$ dimensional vector $\bar{\boldsymbol{h}} = (h_{\mathcal{A}} | \mathcal{A} \subseteq \mathcal{N}) \in \mathbb{R}^{2^N-1}$ associated with it, we call it *entropic vectors*.

For example:

N = 2: $\bar{\boldsymbol{h}} = \{h_1 \quad h_2 \quad h_{12}\}$

N = 3: $\bar{\boldsymbol{h}} = \{h_1 \quad h_2 \quad h_{12} \quad h_3 \quad h_{13} \quad h_{23} \quad h_{123}\}$

The Region of Entropic vectors: all valid entropic vectors

$\Gamma_N^* = \{\boldsymbol{h} \in \mathbb{R}^{2^N-1} | \boldsymbol{h} \text{ is } \textit{entropic}\}$

Examples of two random variables Γ_2^*(Polyhedral cone):

$h_1 + h_2 \geqslant h_{12}$, $h_{12} \geqslant h_1$ and $h_{12} \geqslant h_2$

Motivation to study the Region of Entropic Vectors

- Calculate Network Coding capacity region
- Determine the limit of large coded distributed storage system

The Region of Entropic Vectors $\overline{\Gamma_N^*}$

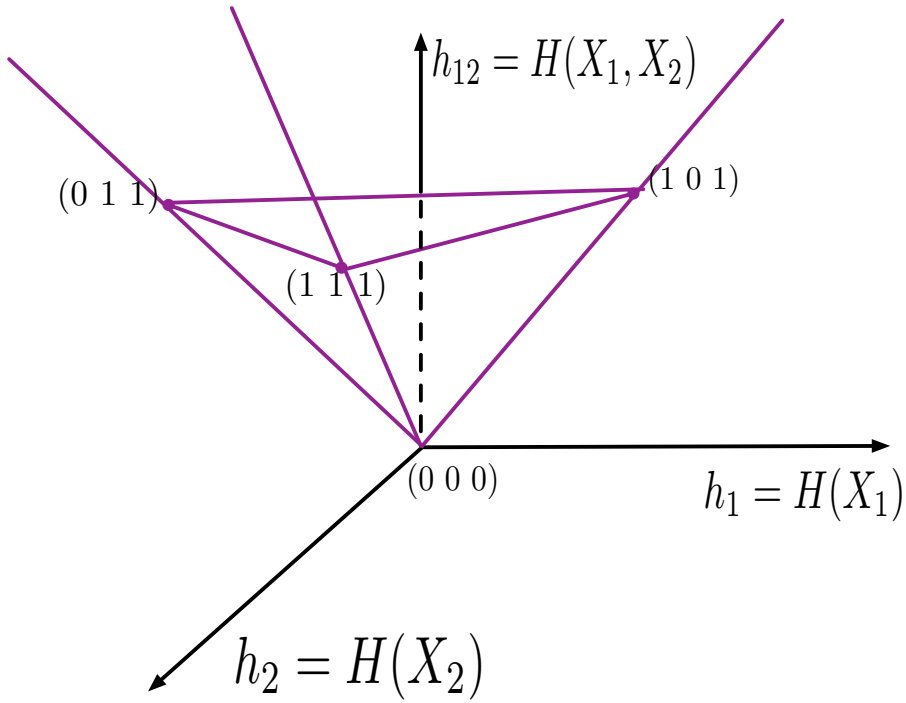
Shannon-Type Information inequality

$$\Gamma_N = \left\{ \boldsymbol{h} \in \mathbb{R}^{2^N-1} \;\middle|\; \begin{array}{c} h_{\mathcal{A}} + h_{\mathcal{B}} \geq h_{\mathcal{A}\cap\mathcal{B}} + h_{\mathcal{A}\cup\mathcal{B}} \;\forall \mathcal{A}, \mathcal{B} \subseteq \mathcal{N} \\ h_{\mathcal{P}} \geq h_{\mathcal{Q}} \geq 0 \quad \forall \mathcal{Q} \subseteq \mathcal{P} \subseteq \mathcal{N} \end{array} \right\}$$

Relationship between Γ_n^* and Γ_n

$\Gamma_2^* = \Gamma_2$ and $\overline{\Gamma_3^*} = \Gamma_3$:
All the Information inequalities on $N \leqslant 3$ variables are Shannon-Type.

For N = 2, Use $h_1 + h_2 \geqslant h_{12}$, $h_{12} \geqslant h_1$ and $h_{12} \geqslant h_2$, we get Γ_2^*

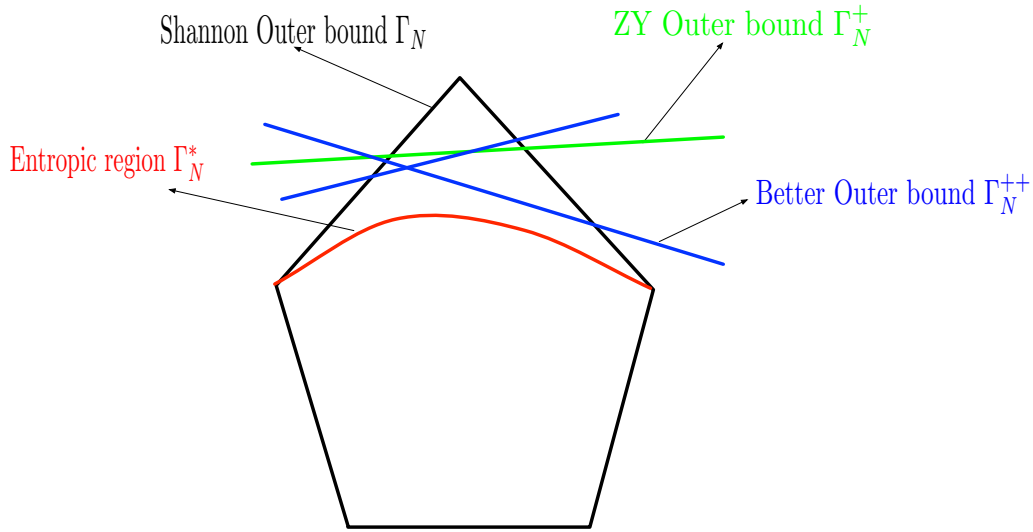


The Region of Entropic Vectors $\overline{\Gamma_N^*}$

- Γ_2^* and $\overline{\Gamma}_3^*$ are polyhedral cone and fully characterized.
- $\overline{\Gamma}_4^*$ is an unknown convex cone, we use outer bound and inner bound to approximate it.

Outer bound for $\overline{\Gamma_N^*}$: non-Shannon type Inequality

For $N \geq 4$ we have $\Gamma_N \neq \overline{\Gamma}_N^*$, indicating non-Shannon type Information inequalities exist for $N \geqslant 4$



First Non-Shannon-Type Information inequality(Zhang-Yeung[1]):

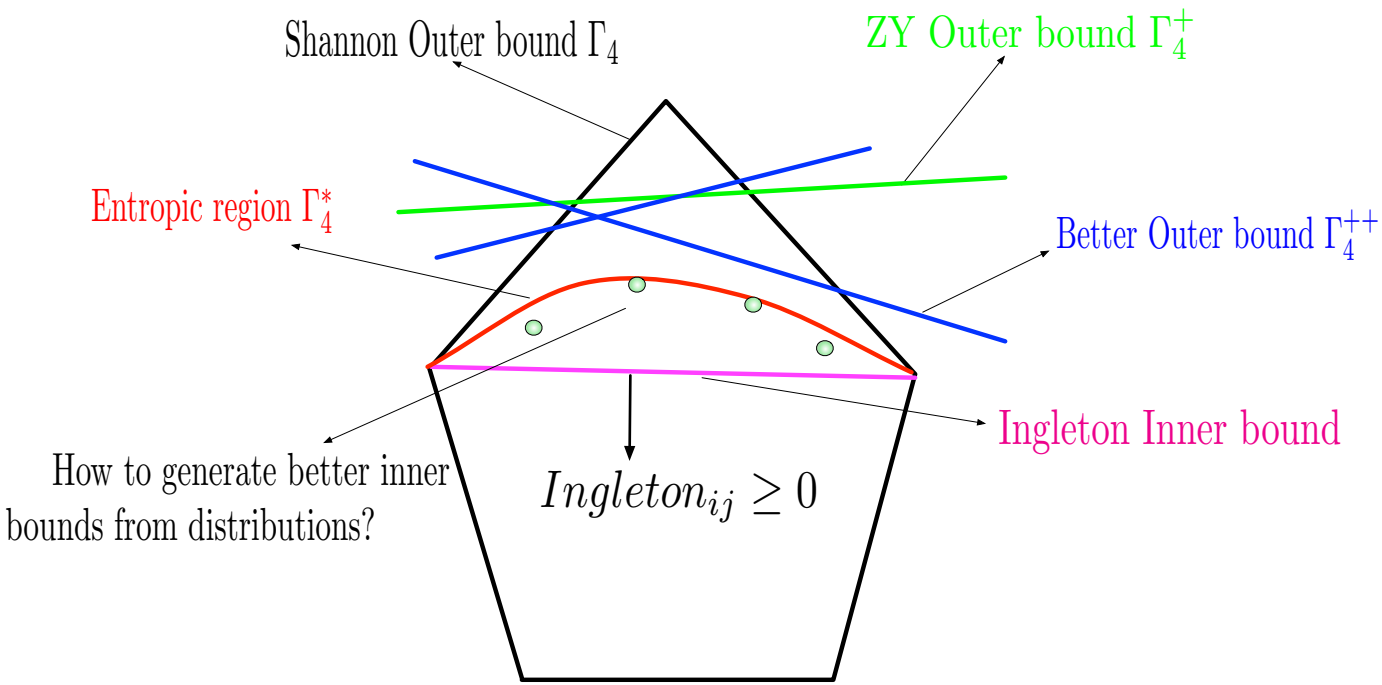
$$2I(C;D) \leq I(A;B) + I(A;C,D) + 3I(C;D|A) + I(C;D|B)$$

[1] **Zhen Zhang and Raymond W. Yeung,**
On Characterization of Entropy Function via Information Inequalities,
IEEE Trans. on Information Theory July 1998.

The Region of Entropic Vectors $\overline{\Gamma_N^*}$

Inner bound for $\overline{\Gamma_4^*}$ (N =4): Ingleton Inequality

$$Ingleton_{ij}$$
$$= I(X_k; X_l|X_i) + I(X_k; X_l|X_j) + I(X_i; X_j|\emptyset) - I(X_k; X_l|\emptyset)$$



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Equivalence of k-atom Supports

An example of 3-variable 4-atom support

$$\begin{bmatrix} (0,0,0) & - & - > & \alpha \\ (0,2,2) & - & - > & \beta \\ (1,0,1) & - & - > & \gamma \\ (1,1,3) & - & - > & 1-\alpha-\beta-\gamma \end{bmatrix} \tag{1}$$

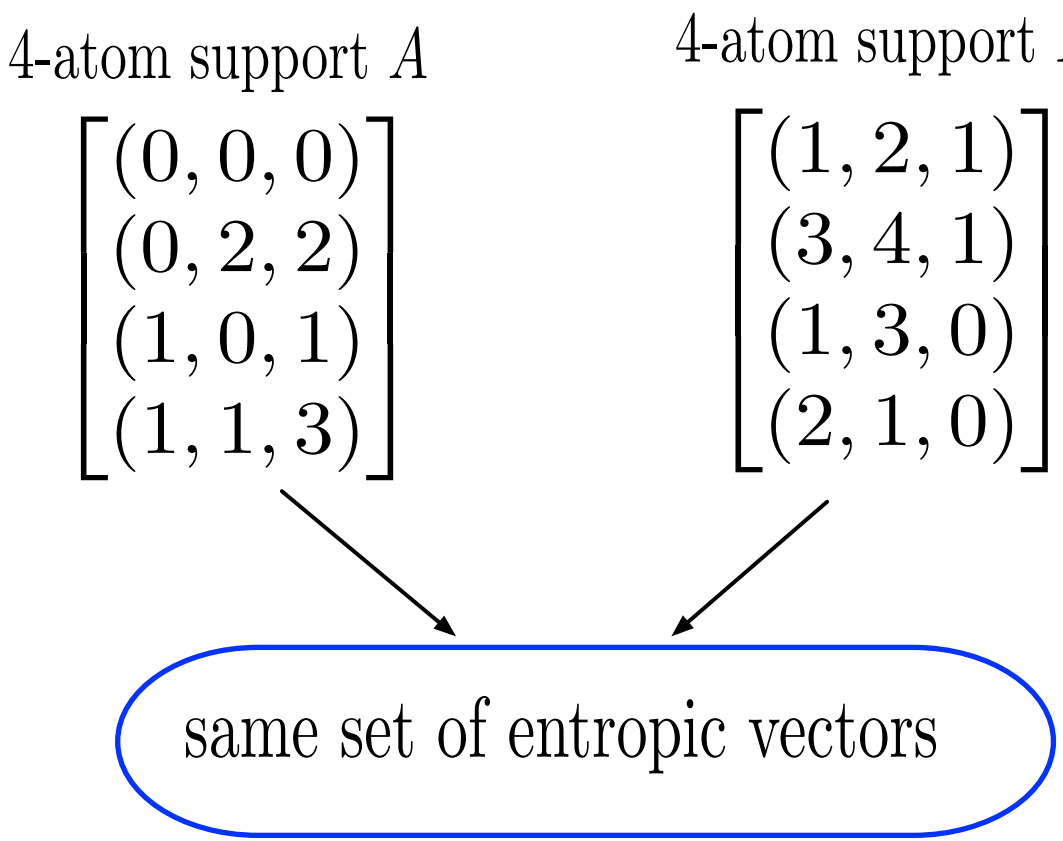
In distribution support (1), each row corresponding to one outcome/atom, each column represent a variable.

Assign variables from left to right as X_1, X_2 and X_3 such that $\boldsymbol{X} = (X_1, X_2, X_3)$. If we assume $|\mathcal{X}_1| = 2$, $|\mathcal{X}_2| = 3$ and $|\mathcal{X}_3| = 4$, then $\boldsymbol{X}$ has totally 24 outcomes, among these outcomes, only 4 of them have non-zero probability, we call it a 4-atom distribution support.

Equivalence of k-atom Supports

Equivalence of two k-atom supports

Two k-atom supports $\boldsymbol{X}, \boldsymbol{X}'$, $|\boldsymbol{X}| = |\boldsymbol{X}'| = k$, will be said to be *equivalent*, for the purposes of tracing out the entropy region, if they yield the same set of entropic vectors, up to a permutation of the random variables.



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Orbit data structure

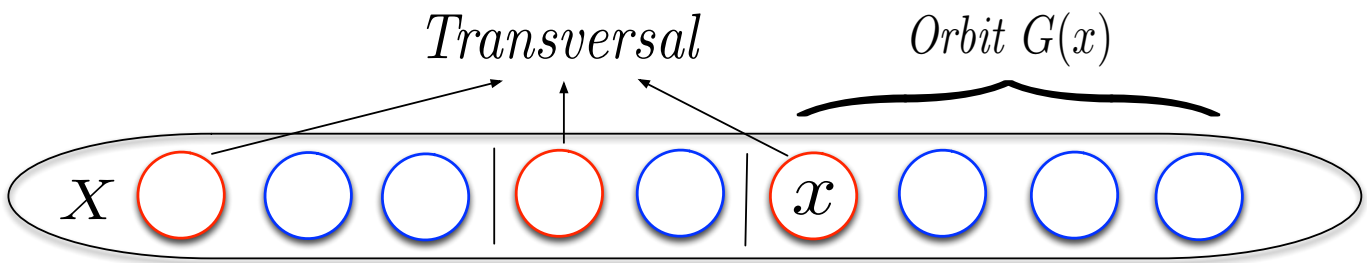
Orbit of elements under group action

Let a finite group G acting on a finite set X, For $x \in X$, the *orbit* of x under G is defined as

$$G(x) = \{gx \mid g \in G\}$$

Transversal $\mathcal{T}$ of the Orbit $G(x)$ under group G

transversal is the collection of canonical representative of the orbit $G(x)$, e.g. least under some ordering of X.



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Non-isomorphic k-atom supports

set partitions

A *set partition* of $\mathbb{N}_1^k$ is a set $\mathcal{B} = \{B_1, \ldots, B_t\}$ consisting of t subsets $B_1, \ldots, B_t$ of $\mathbb{N}_1^k$ that are pairwise disjoint $B_i \cap B_j = \emptyset, \forall i \neq j$, and whose union is $\mathbb{N}_1^k$, so that $\mathbb{N}_1^k = \bigcup_{i=1}^t B_i$. Let $\Pi(\mathbb{N}_1^k)$ denote the set of all set partitions of $\mathbb{N}_1^k$. The cardinality of $\Pi(\mathbb{N}_1^k)$ is commonly known as *Bell numbers*.

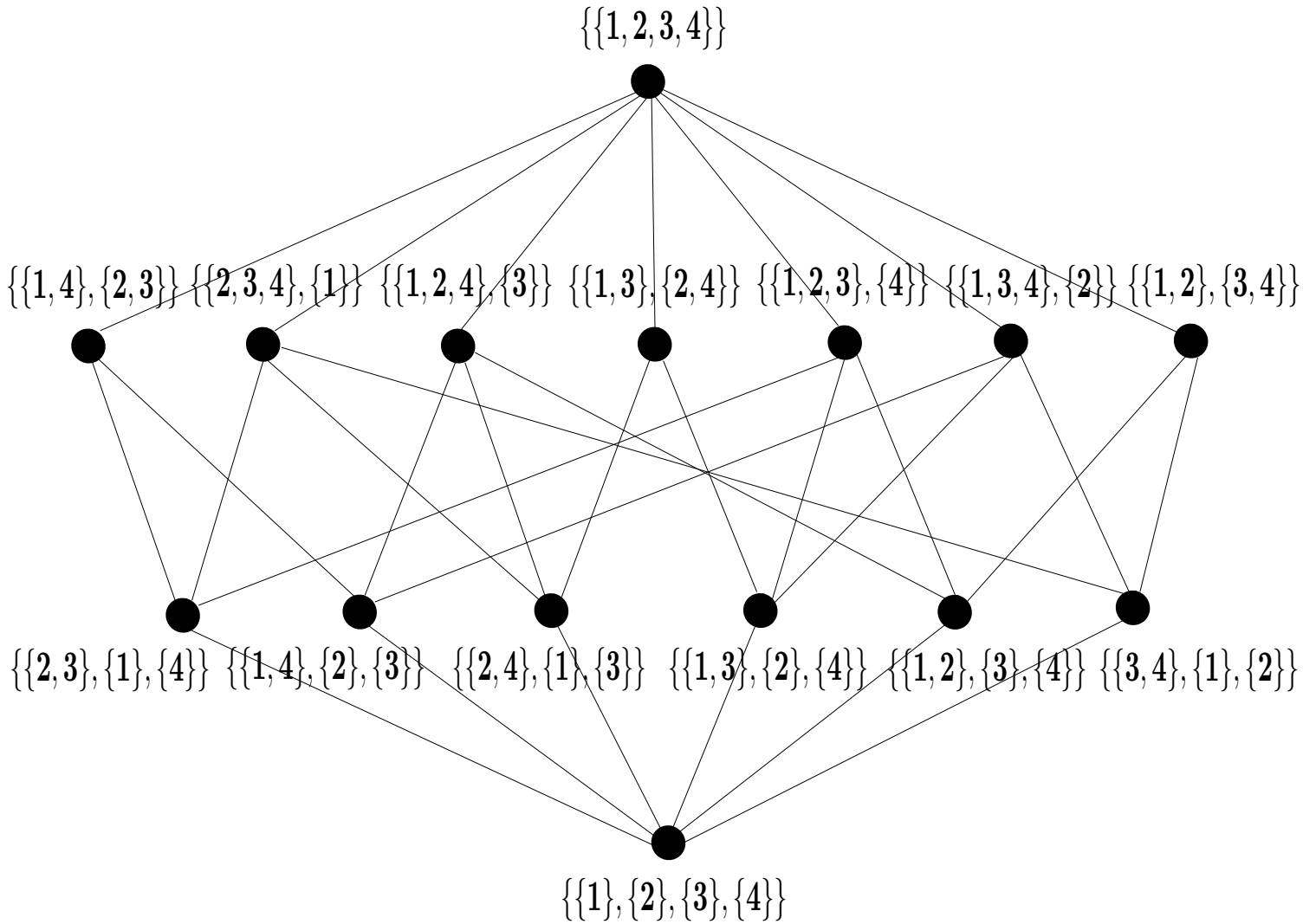
Examples of *set partitions*

$$\Pi(\mathbb{N}_1^3) = \{\{\{1,2,3\}\}, \{\{1,2\},\{3\}\}, \{\{1,3\},\{2\}\}, \{\{2,3\},\{1\}\}, \{\{1\},\{2\},\{3\}\}\},$$

$$\Pi(\mathbb{N}_1^4) = \{\{\{1,2,3,4\}\}, \{\{1,2,3\},\{4\}\}, \{\{1,2,4\},\{3\}\}, \{\{1,3,4\},\{2\}\}, \{\{2,3,4\},\{1\}\}, \{\{1,2\},\{3,4\}\}, \{\{1,3\},\{2,4\}\}, \{\{1,4\},\{2,3\}\}, \{\{1,2\},\{3\},\{4\}\}, \{\{1,3\},\{2\},\{4\}\}, \{\{1,4\},\{2\},\{3\}\}, \{\{2,3\},\{1\},\{4\}\}, \{\{2,4\},\{1\},\{3\}\}, \{\{3,4\},\{1\},\{2\}\}, \{\{1\},\{2\},\{3\},\{4\}\}\}.$$

Non-isomorphic k-atom supports

Lattice of $\Pi(\mathbb{N}_1^4)$: the set of all set partitions for $k = 4$



Non-isomorphic k-atom supports

Outcome for each variables in a k-atom support is a *set partition*, entropy remain unchanged for set partitions under symmetric group $\mathbb{S}_k$

Consider the 4-atom support in (1)

$$\begin{bmatrix} (0,0,0) \\ (0,2,2) \\ (1,0,1) \\ (1,1,3) \end{bmatrix}$$

outcome of X_1 can be encoded as $\{\{1,2\},\{3,4\}\}$
outcome of X_2 can be encoded as $\{\{1,3\},\{2\},\{4\}\}$
outcome of X_3 can be encoded as $\{\{1\},\{2\},\{3\},\{4\}\}$

The symmetric group $\mathbb{S}_k$ induces a natural group action on a set partition $\mathcal{B} \in \Pi(\mathbb{N}_1^k)$, $\mathcal{B} = \{B_1, \ldots, B_t\}$ with $\pi \in \mathbb{S}_k$

$$\pi(\mathcal{B}) := \{\pi(B_1), \ldots, \pi(B_t)\}\,. \tag{2}$$

Non-isomorphic k-atom supports

Use combinations of N set partitions to represent distribution supports for N variables

Let Ξ_N be the collection of all sets of N set partitions of $\mathbb{N}_1^k$ whose meet is the finest partition (the set of singletons),

$$\Xi_N := \left\{ \xi \,\middle|\, \xi \subseteq \Pi(\mathbb{N}_1^k),\ |\xi| = N,\ \bigwedge_{\mathcal{B}\in\xi} \mathcal{B} = \bigcup_{i=1}^{N} \{\{i\}\} \right\}. \tag{3}$$

where *meet* of two partitions $\mathcal{B}, \mathcal{B}'$ is defined as

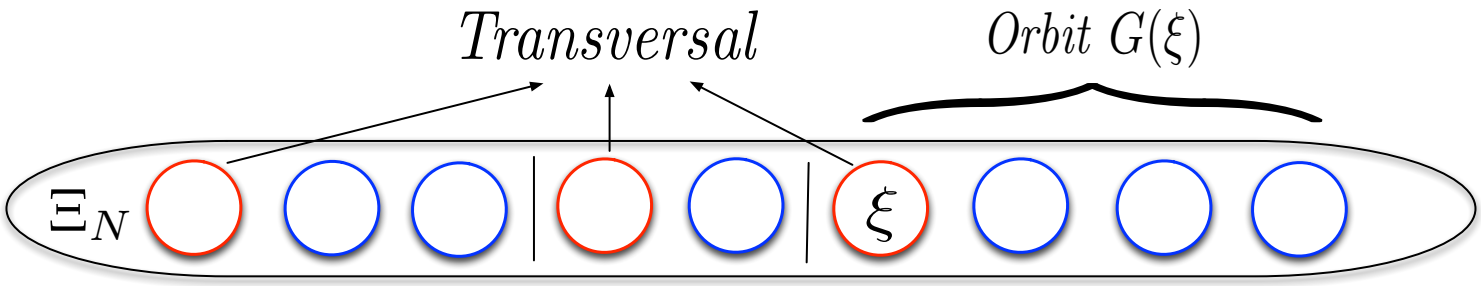
$$\mathcal{B} \wedge \mathcal{B}' = \left\{ B_i \cap B'_j \,\middle|\, B_i \in \mathcal{B}, B'_j \in \mathcal{B}', B_i \cap B'_j \neq \emptyset \right\}$$

The symmetric group $\mathbb{S}_k$ induces a natural group action on Ξ_N of subsets of N partitions from $\Pi(\mathbb{N}_1^k)$ with $\pi \in \mathbb{S}_k$

$$\pi(\xi) := \{\pi(\mathcal{B}_1), \ldots, \pi(\mathcal{B}_N)\}. \tag{4}$$

Non-isomorphic k-atom supports

Ξ_N represent all the necessary distribution supports to calculate entropic vectors for N variables and k atoms, selecting one representative from each equivalence class will lead us to a list of all non-isomorphic k-atom, N-variable supports.



The problem of generating the list of all non-isomorphic k-atom, N-variable supports is equivalent to obtaining a transversal of the orbits $\Xi_N//\mathbb{S}_k$ of $\mathbb{S}_k$ acting on Ξ_N, the set of all subsets of N set partitions of the set $\mathbb{N}_1^k$ whose meets are the set of singletons $\{\{1\},\{2\},\ldots,\{N\}\}$.

Non-isomorphic k-atom supports

Calculate transversal via the algorithm of Snakes and Ladders[2]

For given set $\mathbb{N}_1^k$, the algorithm of Snakes and Ladders first compute the transversal of Ξ_1, then it recursively increase the subsets size i, where the enumeration of Ξ_i is depending on the result of Ξ_{i-1}: $\Xi_1 \to \Xi_2 \to \cdots \to \Xi_{N-1} \to \Xi_N \cdots$

Number of non-isomorphic k-atom, N-variable supports

$N \backslash k$	3	4	5	6	7
2	2	8	18	48	112
3	2	31	256	2437	25148
4	1	75	2665	105726	5107735
5	0	132	22422	3903832	
6	0	187	161118		

[2] **Anton Betten**, **Michael Braun**, **Harald Fripertinger etc.** , **Error-Correcting Linear Codes: Classification by Isometry and Applications**, **Springer 2006 pp. 709–710**.

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Maximal Ingleton Violation and the Four Atom Conjecture

Ingleton score

Given a probability distribution, we define the Ingleton score to be

$$\text{Ingleton score} = \frac{Ingleton_{ij}}{H(X_i X_j X_k X_l)} \tag{5}$$

Four Atom Conjecture: the lowest reported Ingleton score is approximately -0.0894, it is attained by (6)

$$\begin{bmatrix} (0,0,0,0) --> 0.35 \\ (0,1,1,0) --> 0.15 \\ (1,0,1,0) --> 0.15 \\ (1,1,1,1) --> 0.35 \end{bmatrix} \tag{6}$$

Maximal Ingleton Violation and the Four Atom Conjecture

Number of non-isomorphic Ingleton violating supports for four variables

number of atoms k	3	4	5	6	7
all supports	1	75	2665	105726	5107735
Ingleton violating	0	1	29	1255	60996

The only Ingleton violating 5-atom support that is not grown from 4-atom support (6) with a Ingleton score -0.02423

$$\begin{bmatrix} (0,0,0,0) \\ (0,0,1,1) \\ (0,1,1,0) \\ (1,0,1,0) \\ (1,1,1,0) \end{bmatrix} \tag{7}$$

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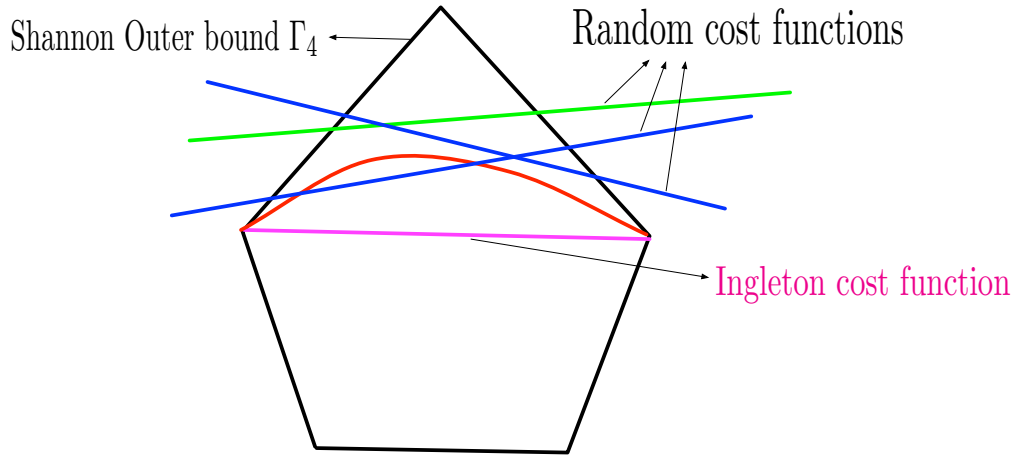


k-atom inner bound generation for four variables

- Randomly generate enough cost functions from the gap between Shannon outer bound and Ingleton inner bound
- Given a k-atom supports, find the distribution that optimize the each of the cost function, save it if it violate Ingleton
- Taking the convex hull of all the entropic vectors from the k-atom distributions that violate $Ingleton_{kl}$ to get the k-atom inner bound

Optimizing Inner Bounds to Entropy from k-atom Distribution

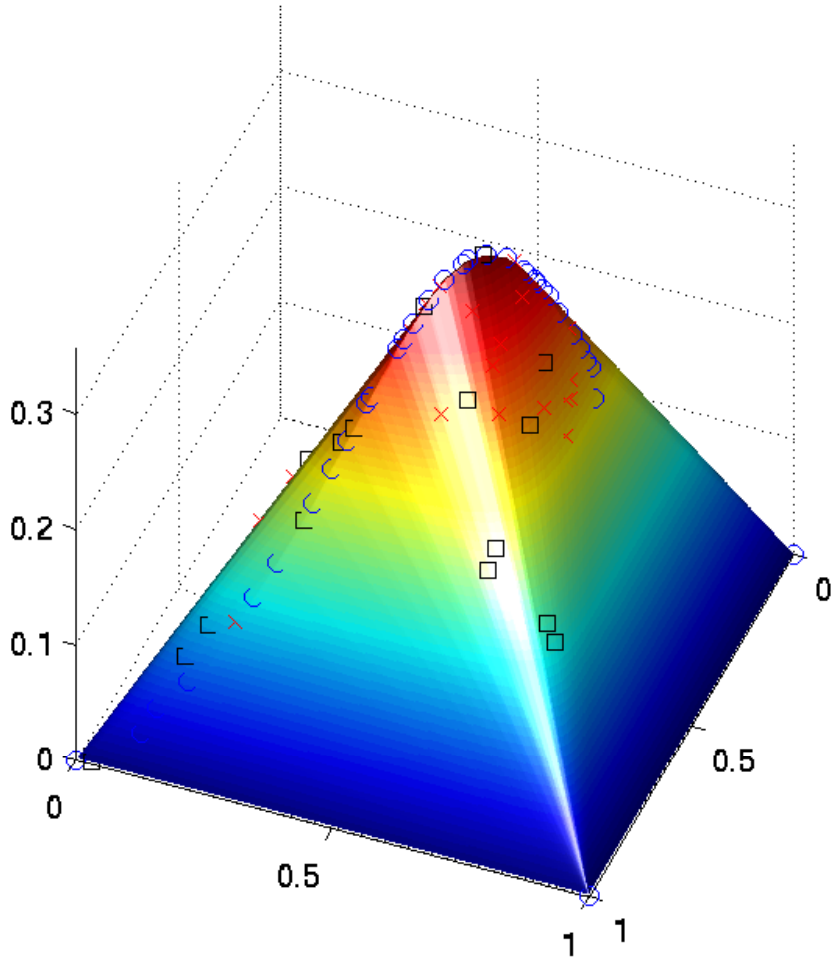
The volume increase within pyramid G_4^{34} as more atoms are included

inner and outer bounds	percent of pyramid
Shannon	100
Outer bound from Dougherty etc.[3]	96.5
4,5,6 atoms inner bound	57.8
4,5 atoms inner bound	57.1
4 atoms inner bound	55.9
4 atom conjecture point only	43.5
3 atoms inner bound	0

[3] **Randall Dougherty**, **Chris Freiling**, **Kenneth Zeger**, **Non-Shannon Information Inequalities in Four Random Variables**, **arXiv:1104.3602v1**.

Optimizing Inner Bounds to Entropy from k-atom Distribution

Three dimensional projection of 4-atom inner bound(3-D projection introduced by F. Matúš and L. Csirmaz[3])



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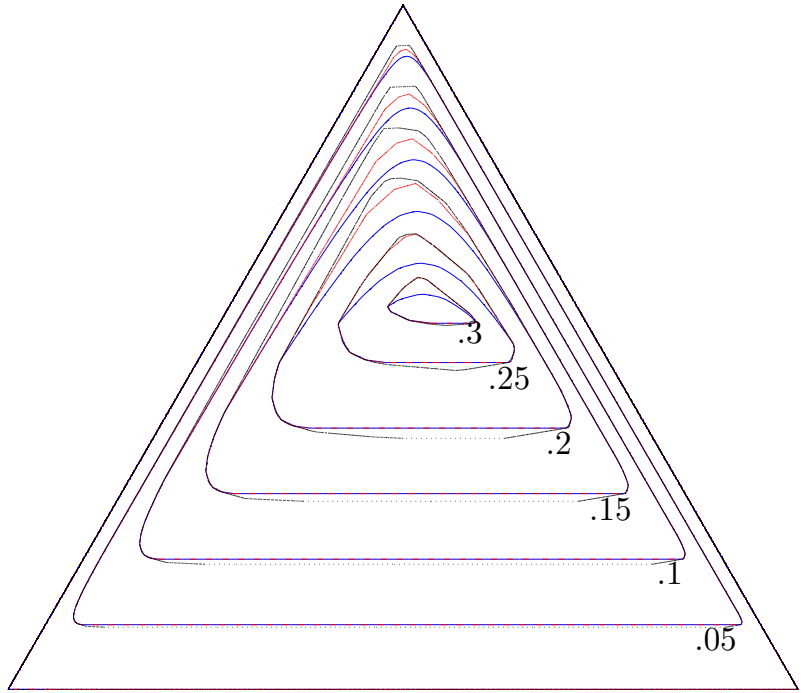

Figure : Comparison of contour plots of inner bound created from $\leq k$ atom distributions for $k \in \{4, 5, 6\}$. For each contour value, the inner most line is $k = 4$ while the outermost line is $k = 6$. The numerical inner bounds generated from only four atom distributions are quite good in this space.

Conclusion

Results summary

- Formulation and algorithms for listing non-isomorphic distribution supports for Entropic Vectors
- Ingleton violating non-isomorphic distribution supports
- Numerical inner bound generated by optimizing k-atom distributions

Thanks!

Questions?

Appendix: the algorithm of Snake and Ladder

Algorithm (orbits on subsets)

Input: $\mathrm{orbit}(G, \mathcal{P}_i^{(f)}(X)) = (\mathcal{T}_i, \sigma_i, \varphi_i)$

Output: $\mathrm{orbit}(G, \mathcal{P}_{i+1}^{(f)}(X)) = (\mathcal{T}_{i+1}, \sigma_{i+1}, \varphi_{i+1})$

(0) **for** $R \in \mathcal{T}_i$ **do**
(1) compute $\mathrm{orbit}(G_R, X \setminus R) := (\mathcal{T}_R, \sigma_R, \varphi_R)$
(2) **end for**
(3) $\mathcal{T}_{i+1} := \emptyset$

(4) **for** $R \in \mathcal{T}_i$ (in increasing order) **do**
(5) **for** $x \in \mathcal{T}_R$ (in increasing order) with $f(R \cup \{x\}) = 1$
and for which $\psi_R(x)$ has not yet been defined **do**
(6) $G_{R,x} := \sigma_R(x)$
(7) $H := G_{R,x}$
(8) **for** all $r \in R$ which are least in their H-orbit **do**
(9) $t := \varphi_i((R \setminus \{r\}) \cup \{x\})$
(10) $S := ((R \setminus \{r\}) \cup \{x\})t$
(11) $h := \varphi_S(rt)$
(12) $y := rth$
(now: $(R \cup \{x\})th = S \cup \{y\}$, $S \in \mathcal{T}_i$, $y \in \mathcal{T}_S$)
(13) **if** $S = R$ **and** $y = x$ **then** (case 1 of 9.6.1)
(14) $H := \langle H, th \rangle$
(th is an automorphism of $R \cup \{x\}$)
(15) **else** (case 2 of 9.6.1)
(16) $\psi_S(y) := (th)^{-1}$
(th is an isomorphism from $R \cup \{x\}$ to $S \cup \{y\}$)
(17) **end if**
(18) **end for**
(19) append $R \cup \{x\}$ to $\mathcal{T}_{i+1}$
(20) $\sigma_{i+1}(R \cup \{x\}) := H\ (= G_{R \cup \{x\}})$
(21) **end for**
(22) **end for**
(23) **return** $(\mathcal{T}_{i+1}, \sigma_{i+1}, \varphi_{i+1})$

Appendix: the algorithm of Snake and Ladder

Where the function φ_{i+1} is defined as follows.

(24) **function** $\varphi_{i+1}(F)$
(25) $z := \max F$, $Z := F \setminus \{z\}$ (a set of size i)
(26) $t := \varphi_i(Z)$
(27) $S := Zt$
(28) $h := \varphi_S(zt)$, $y := zth$
(29) **if** $\psi_S(y)$ has been defined **then**
(30) **return** $th\psi_S(y)$
(31) **else**
(32) **return** th
(33) **end if**
(34) **end function**