SPACE TIME CODING

Jie Ren

ASPITRG Drexel

- B. Vucetic and J. Yuan, *Space-Time Coding*, Wiley, 2003
- Erik G. Larsson and Petre Stoica Space-Time Block Coding for Wireless Communications, Cambridge, 2005

Outline

- MIMO Wireless Communication Systems
- Space-Time Coding Performance Analysis
- Space-Time Block Codes
- Space-Time Trellis Codes

Outline

- MIMO Wireless Communication Systems
  - MIMO System Model
  - MIMO System Capacity Derivation
  - MIMO Capacity Examples
- Space-Time Coding Performance Analysis
- Space-Time Block Codes
- Space-Time Trellis Codes

Outline

- MIMO Wireless Communication Systems
  - MIMO System Model
  - MIMO System Capacity Derivation
  - MIMO Capacity Examples
- Space-Time Coding Performance Analysis
- Space-Time Block Codes
- Space-Time Trellis Codes

MIMO System Model

- Notations

  n_T transmit antennas

  n_R receive antennas

  x transmitted signals, N(0,μ) i.i.d.

  n noise

  r received signals

  R_{xx}, R_{nn}, R_{rr} covariance matrix of **x**, **n** and **r**

MIMO System Model

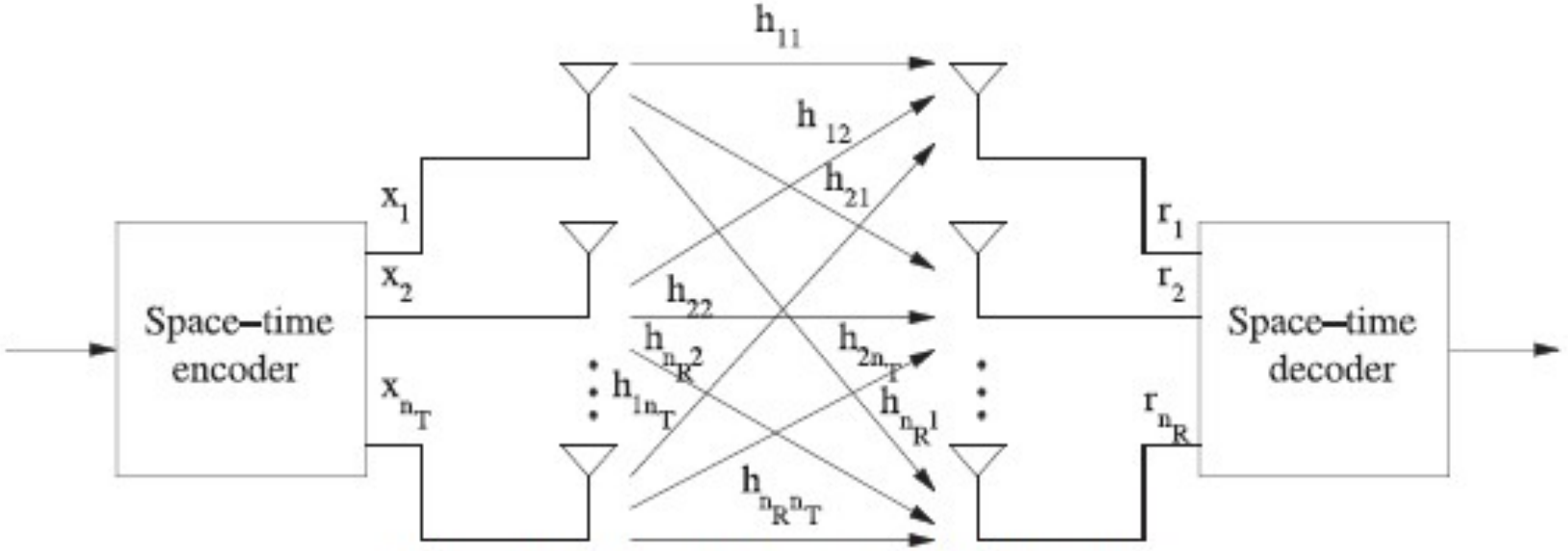


Figure 1.1 Block diagram of a MIMO system

MIMO System Model

- Covariance matrix of the transmitted signal

$$\boldsymbol{R}_{xx} = E\{\boldsymbol{x}\boldsymbol{x}^H\}$$

- Transmitted power constraint
  - Channel is unknown at the transmitter

$$P = tr(\boldsymbol{R}_{xx})$$

$$\boldsymbol{R}_{xx} = \frac{P}{n_T}\boldsymbol{I}_{n_T}$$

MIMO System Model

- Noise **n**
  - independent complex zero-mean Gaussian

$$\boldsymbol{R}_{nn} = E\{\boldsymbol{n}\boldsymbol{n}^H\}$$

- No correlation between components of **n**

$$\boldsymbol{R}_{nn} = \sigma^2 \boldsymbol{I}_{n_R}$$

MIMO System Model

- MIMO Channel **H**
  - n_R by n_T complex matrix
  - perfectly known at the receiver
  - not known at the transmitter
  - normalization:

$$P_{r,i} = P, \quad i = 1,2, \dots, n_R$$

$$\sum_{j=1}^{n_T} |h_{ij}|^2 = n_T, \quad i = 1,2, \dots, n_R$$

MIMO System Model

- Average SNR at each receive antenna

$$\gamma = \frac{P_r}{\sigma^2} = \frac{P}{\sigma^2}$$

- Received vector

$$\boldsymbol{r} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n}$$

$$\boldsymbol{R}_{rr} = E\{\boldsymbol{r}\boldsymbol{r}^H\} = \boldsymbol{H}\boldsymbol{R}_{xx}\boldsymbol{H}^H$$

Outline

- MIMO Wireless Communication Systems
  - MIMO System Model
  - MIMO System Capacity Derivation
  - MIMO Capacity Examples
- Space-Time Coding Performance Analysis
- Space-Time Block Codes
- Space-Time Trellis Codes

MIMO System Capacity Derivation

- Theorem: Singular value decomposition
  - Suppose M is an m×n matrix whose entries come from the field K. (either the field of real numbers or the field of complex numbers)

    Then,

$$\boldsymbol{M = U\Sigma V^{H}}$$

  - where U is an m×m unitary matrix over K, V* is the conjugate transpose of the n×n unitary matrix V over K, Σ is an m×n diagonal matrix with non-negative real numbers on the diagonal.

MIMO System Capacity Derivation

- Singular value decomposition

$$\boldsymbol{H} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^{H}$$

$$\boldsymbol{r} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^{H}\boldsymbol{x} + \boldsymbol{n}$$

$$\boldsymbol{U}^{H}\boldsymbol{r} = \boldsymbol{U}^{H}\boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^{H}\boldsymbol{x} + \boldsymbol{U}^{H}\boldsymbol{n} = \boldsymbol{D}(\boldsymbol{V}^{H}\boldsymbol{x}) + \boldsymbol{U}^{H}\boldsymbol{n}$$

MIMO System Capacity Derivation

- Singular value decomposition

$$\boldsymbol{r}' = \boldsymbol{U}^H \boldsymbol{r}$$
$$\boldsymbol{x}' = \boldsymbol{V}^H \boldsymbol{x}$$
$$\boldsymbol{n}' = \boldsymbol{U}^H \boldsymbol{n}$$

- Equivalent channel

$$\boldsymbol{r}' = \boldsymbol{H}\boldsymbol{x}' + \boldsymbol{n}'$$

MIMO System Capacity Derivation

- Singular value decomposition

$$\boldsymbol{r}' = \boldsymbol{U}^H \boldsymbol{r}$$
$$\boldsymbol{x}' = \boldsymbol{V}^H \boldsymbol{x}$$
$$\boldsymbol{n}' = \boldsymbol{U}^H \boldsymbol{n}$$

- Equivalent channel

$$\boldsymbol{r}' = \boldsymbol{H}\boldsymbol{x}' + \boldsymbol{n}'$$

MIMO System Capacity Derivation

- Covariance Matrix

$$\boldsymbol{R}_{r'r'} = \boldsymbol{U}^H \boldsymbol{R}_{rr} \boldsymbol{U}$$
$$\boldsymbol{R}_{x'x'} = \boldsymbol{V}^H \boldsymbol{R}_{xx} \boldsymbol{V}$$
$$\boldsymbol{R}_{n'n'} = \boldsymbol{U}^H \boldsymbol{R}_{nn} \boldsymbol{U}$$

- Power constraint

$$tr(\boldsymbol{R}_{r'r'}) = tr(\boldsymbol{R}_{rr})$$
$$tr(\boldsymbol{R}_{x'x'}) = tr(\boldsymbol{R}_{xx})$$
$$tr(\boldsymbol{R}_{n'n'}) = tr(\boldsymbol{R}_{nn})$$

MIMO System Capacity Derivation

- Capacity

$$C = W \sum_{i=1}^{r} \log_2 \left(1 + \frac{P_{r,i}}{\sigma^2}\right)$$

$$P_{r,i} = \frac{\lambda_i P}{n_T}$$

$$C = W \sum_{i=1}^{r} \log_2 \left(1 + \frac{\lambda_i P}{n_T \sigma^2}\right)$$

MIMO System Capacity Derivation

- Capacity: Relates to the channel matrix H

$$\boldsymbol{Q} = \begin{cases} \boldsymbol{H}\boldsymbol{H}^{H}, & n_R < n_T \\ \boldsymbol{H}^{H}\boldsymbol{H}, & n_R \geq n_T \end{cases}$$

$$\prod_{i=1}^{m}(\lambda - \lambda_i) = \det\left(\lambda \boldsymbol{I_m} - \boldsymbol{Q}\right)$$

$$substitute\ \lambda = -\frac{n_T \sigma^2}{P}$$

$$C = W \log_2 \det\left(\boldsymbol{I_m} + \frac{P}{n_T \sigma^2}\boldsymbol{Q}\right)$$

Outline

- MIMO Wireless Communication Systems
  - MIMO System Model
  - MIMO System Capacity Derivation
  - MIMO Capacity Examples
- Space-Time Coding Performance Analysis
- Space-Time Block Codes
- Space-Time Trellis Codes

Examples 1

- SISO channel
  - 1 receive antennas and 1 transmit antennas

$$C = Wlog_2(1 + \frac{P}{\sigma^2})$$

Examples 2

- MIMO channel with unity H

$$C = Wlog_2(1 + n_T n_R \frac{P}{\sigma^2})$$

  - Coherent combining
  - Reduces to a single effective channel

Example 3

- Receive Diversity
  - n receive antennas and 1 transmit antennas

$$C = Wlog_2(1 + \frac{P}{\sigma^2}\max_I|h_i|^2)$$

Example 4

- Transmit Diversity
  - n transmit antennas and 1 receive antennas

$$C = Wlog_2(1 + \frac{nP}{\sigma^2})$$

Outline

- MIMO Wireless Communication Systems
- Space-Time Coding Performance Analysis
- Space-Time Block Codes
- Space-Time Trellis Codes

Outline

- MIMO Wireless Communication Systems
- Space-Time Coding Performance Analysis
  - Diversity-Multiplexing Tradeoff
  - ML Detection
  - Error Analysis
  - Space-Time Code Design Criteria
- Space-Time Block Codes
- Space-Time Trellis Codes

Diversity-Multiplexing Tradeoff

- Why MIMO?
  - Utilize multiple antennas to improve wireless system performance

$$C = Wlog_2\,(1 + \frac{P}{\sigma^2})|h|^2$$

  - Higher capacity

$$C = W\log_2 \det\,(\boldsymbol{I_m} + \frac{P}{n_T\sigma^2}\boldsymbol{Q})$$

  - Lower error probability

Definitions

- Diversity Gain d
  - Change in slope of the error probability

$$-d = \lim_{SNR \to \infty} \frac{\log P_e(SNR)}{\log SNR}$$

- Multiplexing Gain r
  - Change in slope of the rate

$$r = \lim_{SNR \to \infty} \frac{R(SNR)}{\log_2 SNR}$$

Beamforming

- Antennas transmit the same signal
- Pre-coding and shaping matrices (vectors): **u**, **v**

$$y = \boldsymbol{u}^H H \boldsymbol{v} x + \boldsymbol{u}^H \boldsymbol{n}$$

- Corresponding SNR

$$\gamma = \sigma_{max}^2 \rho$$

Diversity-Multiplexing Trade-offs

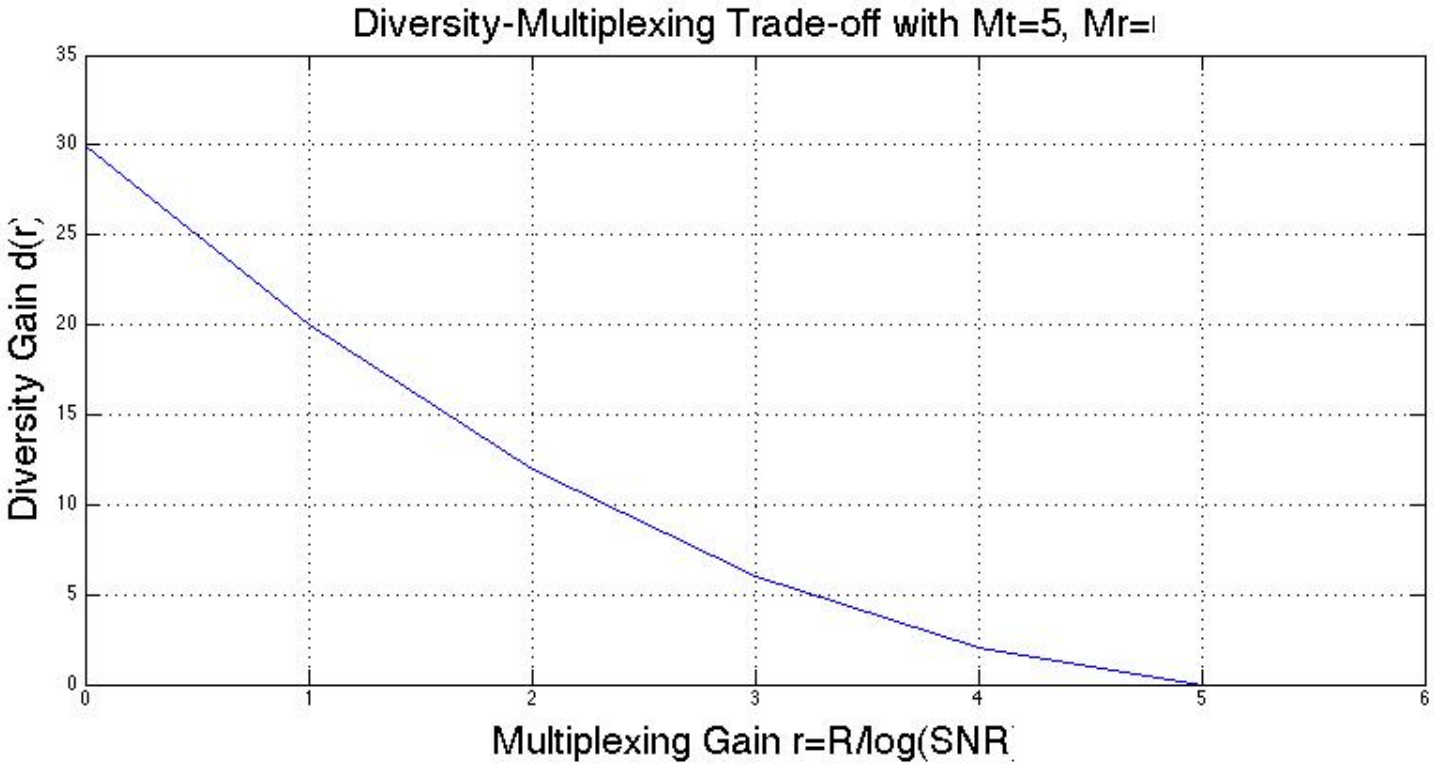
- Obtain full multiplexing gain
  - Decompose the MIMO into parallel SISO
  - multiplexing different data streams
  - each SISO quality depends on the singular values of $\mathbf{H}^H\mathbf{H}$
  - may have poor performance
- Obtain full diversity gain
  - Apply beamforming

Diversity-Multiplexing Trade-offs

- Fundamental design question:
  - Should the antennas be used for diversity gain, multiplexing gain or both?
  - Assume block fading channels with receiver CSI only
  - Maximum d for fixed r:

$$d_{opt}(r) = (M_t - r)(M_r - r), 0 \leq r \leq \min(M_t, M_r)$$

Diversity-Multiplexing Trade-offs



Outline

- MIMO Wireless Communication Systems
- Space-Time Coding Performance Analysis
  - Diversity-Multiplexing Tradeoff
  - ML Detection
  - Error Analysis
  - Space-Time Code Design Criteria
- Space-Time Block Codes
- Space-Time Trellis Codes

Space-Time Coded Systems

- Information symbols

$$\boldsymbol{c}_t = (c_t^1, c_t^2, \ldots, c_t^m)$$

- Input vector

$$\boldsymbol{x}_t = \left(x_t^1, x_t^2, \ldots x_t^{n_T}\right)^T$$

- Received vector

$$\boldsymbol{r}_t = \boldsymbol{H}_t \boldsymbol{x}_t + \boldsymbol{n}_t$$

ML Detection

$$\hat{X} = \underset{X \in \chi^{M_t \times T}}{\operatorname{argmin}} ||\boldsymbol{Y} - \boldsymbol{H}\boldsymbol{X}||^2$$

$$= \arg\min \sum_{t=1}^{T} ||\boldsymbol{y}_t - H\boldsymbol{x}_t||^2$$

Space-Time Coded Systems

- Decision Metrics

$$\sum_{t}\sum_{j=1}^{n_R}\left|r_t^j-\sum_{i=1}^{n_T}h_{j,i}^t x_t^i\right|^2$$

- Selects a code word with the minimum decision metric

Outline

- MIMO Wireless Communication Systems
- Space-Time Coding Performance Analysis
  - Diversity-Multiplexing Tradeoff
  - ML Detection
  - Error Analysis
  - Space-Time Code Design Criteria
- Space-Time Block Codes
- Space-Time Trellis Codes

Error Analysis

- AWGN fading channel
- General error probability

$$P_0 = c \cdot Q\left(\sqrt{g \cdot \frac{\gamma^2}{\sigma^2}}\right)$$

$$\gamma^2 = \boldsymbol{h}^H \boldsymbol{h} = |\boldsymbol{h}|^2$$

$$h \sim N_C(\mathbf{0}, \boldsymbol{r})$$

Error Analysis

- Theorem: The error probability, averaged over **h**, is bounded by:

$$E_h[P_0] \leq c \cdot \left(\frac{g}{2\sigma^2}\right)^{-n} \cdot \prod_{k=1}^{n} \varepsilon_k^{-1}$$

Error Analysis

- Diversity gain: G_d
- Coding gain: G_c

$$P(error) \sim G_c \cdot SNR^{-G_d}$$

Outline

- MIMO Wireless Communication Systems
- Space-Time Coding Performance Analysis
  - Diversity-Mutiplexing Tradeoff
  - ML Detection
  - Error Analysis
  - Space-Time Code Design Criteria
- Space-Time Block Codes
- Space-Time Trellis Codes

Space-Time Code Design Criteria

- Pair-wise error probability for STC

$$p(X \to \hat{X}) \leq \left(\prod_{k=1}^{R_\Delta} \lambda_k(\Delta) \right)^{-M_r} (.25\gamma)^{-R_\Delta M_r}$$

- Rank criterion: the difference matrix must be full rank to obtain the maximum diversity gain $M_r M_t$
- Determinant criterion: maximize the minimum of the Det(Δ) to obtain a high coding gain

Outline

- MIMO Wireless Communication Systems
- Space-Time Coding Performance Analysis
- Space-Time Block Codes
  - Alamouti's Space-Time Code
  - STBC
- Space-Time Trellis Codes

Alamouti Space-Time Code

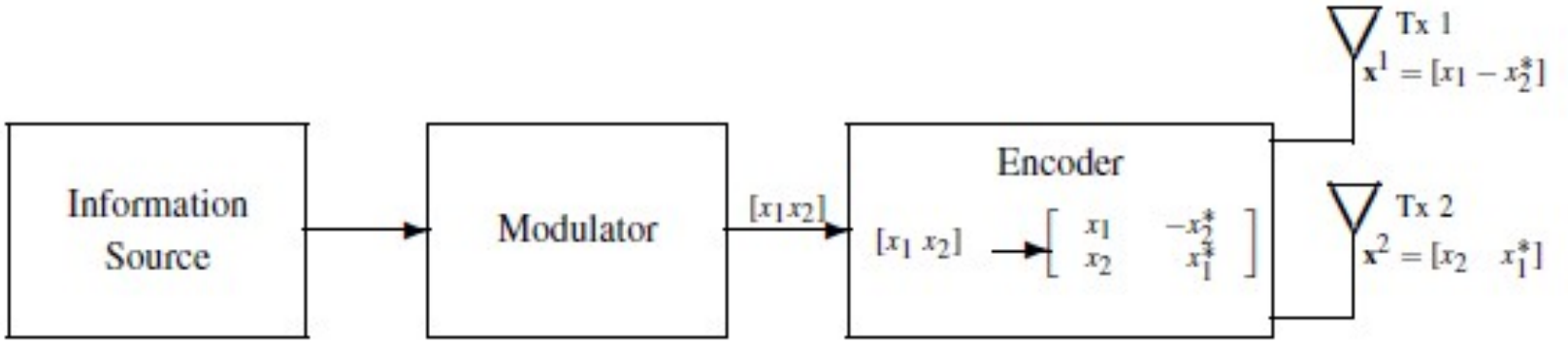


Figure 3.1 A block diagram of the Alamouti space-time encoder

Alamouti Space-Time Code

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x^1} \\ \boldsymbol{x^2} \end{bmatrix} = \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix}$$

- Orthogonal Property

$$\boldsymbol{x^1} \cdot \boldsymbol{x^2} = x_1 x_2^* - x_2^* x_1 = 0$$

- Received Signal

$$r_1 = h_1 x_1 + h_2 x_2 + n_1$$
$$r_2 = -h_1 x_2^* + h_2 x_1^* + n_2$$

Alamouti Space-Time Code

- Define

$$\boldsymbol{F} = \begin{bmatrix} h_1 & h_2 \\ -h_2^* & h_1^* \end{bmatrix}$$

$$\boldsymbol{y} = \boldsymbol{F}\boldsymbol{X} + \boldsymbol{e}$$

- where **e** is white noise

$$\boldsymbol{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \sim N_C(\mathbf{0}, \sigma^2 \boldsymbol{I})$$

Alamouti Space-Time Code

- ML detection

$$d^2(r_1, h_1\hat{x}_1 + h_1\hat{x}_2) + d^2(r_2, -h_1\hat{x}_1^* + h_2\hat{x}_1^*)$$
$$= |r_1 - h_1\hat{x}_1 - h_2\hat{x}_2|^2 + |r_2 - h_1\hat{x}_2^* - h_1\hat{x}_1^*|^2$$

Alamouti Space-Time Code

- Decision Statistics

$$\tilde{x}_1 = h_1^* r_1 + h_2 r_2^*$$
$$\tilde{x}_2 = h_2^* r_1 - h_1 r_2^*$$

- Decision Rules

$$\hat{x}_1 = arg \min_{\hat{x}_1 \in S} d^2(\tilde{x}_1, \hat{x}_1)$$
$$\hat{x}_2 = arg \min_{\hat{x}_2 \in S} d^2(\tilde{x}_2, \hat{x}_2)$$

Alamouti Space-Time Code

- Achieve a full diversity gain

$$B\left(X,\hat{X}\right)=\begin{bmatrix} x_1-\hat{x}_1 & -x_2^*+\hat{x}_2^* \\ x_2-\hat{x}_2 & x_1^*-\hat{x}_1^* \end{bmatrix}$$

$$A\left(X,\hat{X}\right)=B\left(X,\hat{X}\right)B^H\left(X,\hat{X}\right)$$

Alamouti Space-Time Code

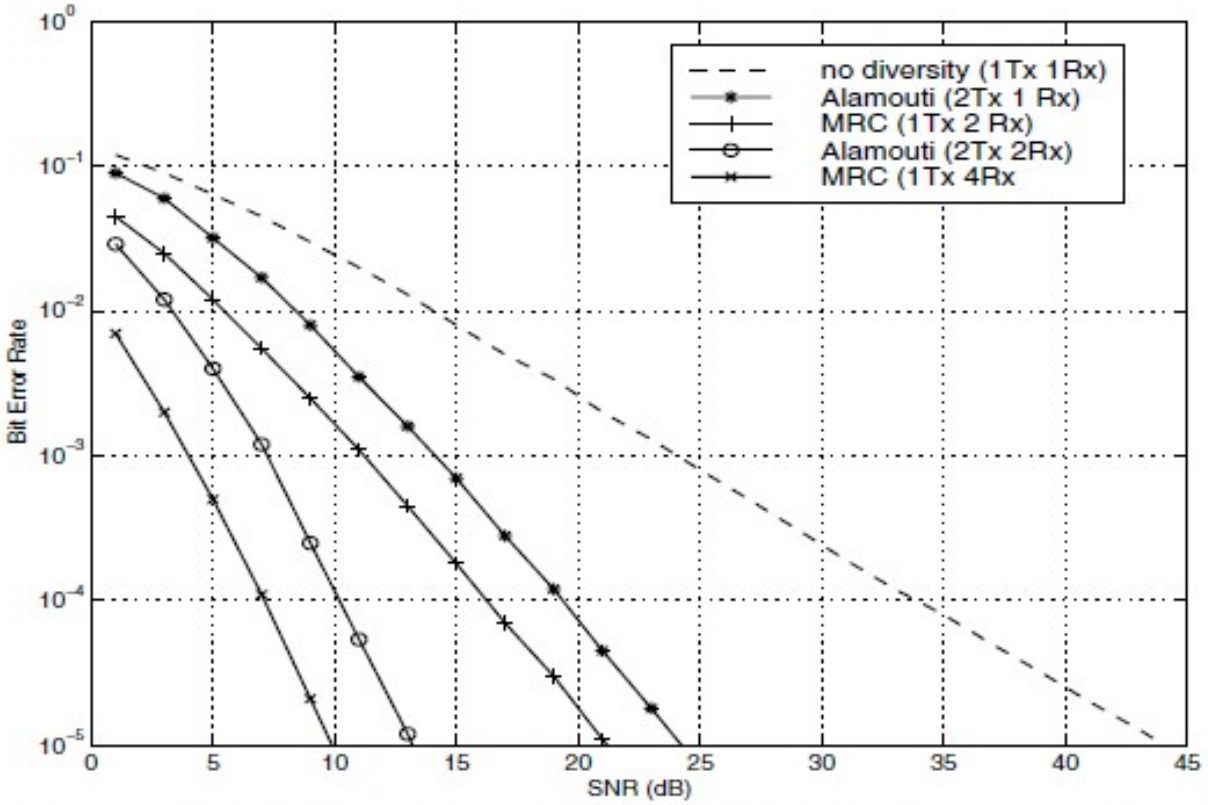


Figure 3.3 The BER performance of the BPSK Alamouti scheme with one and two receive antennas on slow Rayleigh fading channels

Outline

- MIMO Wireless Communication Systems
- Space-Time Coding Performance Analysis
- Space-Time Block Codes
  - Alamouti's Space-Time Code
  - STBC
- Space-Time Trellis Codes

Space-Time Block Codes

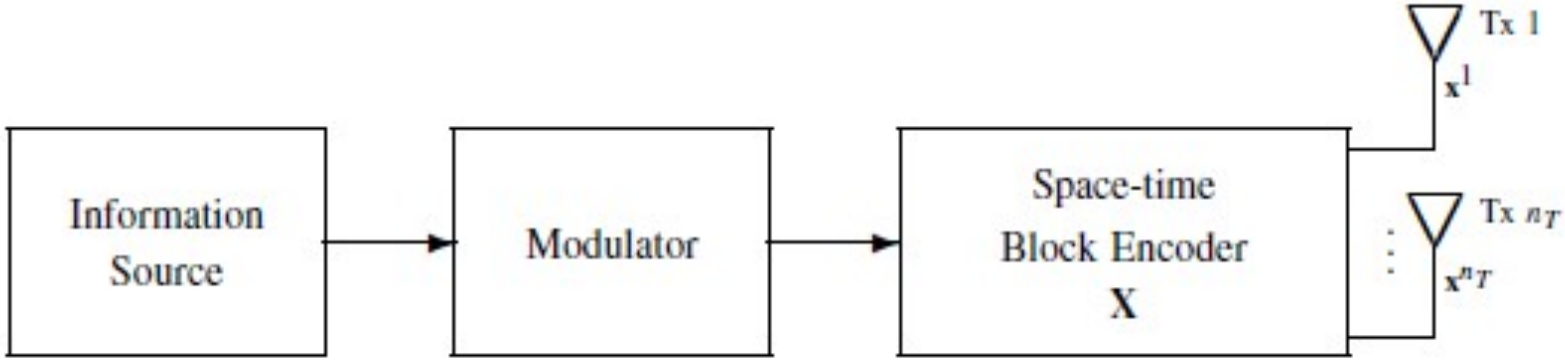


Figure 3.6 Encoder for STBC

Space-Time Block Codes

- Code Matrix

$$\boldsymbol{X}\boldsymbol{X}^{H} = c(|x_1|^2 + \cdots + |x_k|^2)\boldsymbol{I}_{n_T}$$

- Orthogonal Property

$$\boldsymbol{x^i} \cdot \boldsymbol{x^j} = \sum_{t=1}^{p} x_{i,t} \cdot x_{j,t}^{*} = 0, \qquad i \neq j$$

Decoding of STBC

- Decision Statistics

$$\tilde{x}_i = \sum_{t=1}^{n_T} \sum_{j=1}^{n_R} sgn_t(i) \cdot r_t^j \cdot h^*_{j,\epsilon_t(i)}$$

- Decision Rules

$$|\tilde{x}_i - x_i|^2 + \left(\sum_{t=1}^{n_T} \sum_{j=1}^{n_R} |h_{j,t}|^2 - 1 \right) |x_i|^2$$

Outline

- MIMO Wireless Communication Systems
- Space-Time Coding Performance Analysis
- Space-Time Block Codes
- Space-Time Trellis Codes
  - Delay Diversity Code
  - General STTC

Delay Diversity

- STTC: a steam of data is encoded via N_t convolutional encoders
- Delay Diversity for N_t=2
  - First convolutional encoder: absent
  - Second convolutional encoder: replace by time delay

$$x_1(n) = s(n)$$
$$x_2(n) = s(n - D)$$

Delay Diversity

- covariance matrix of **h**e full rank

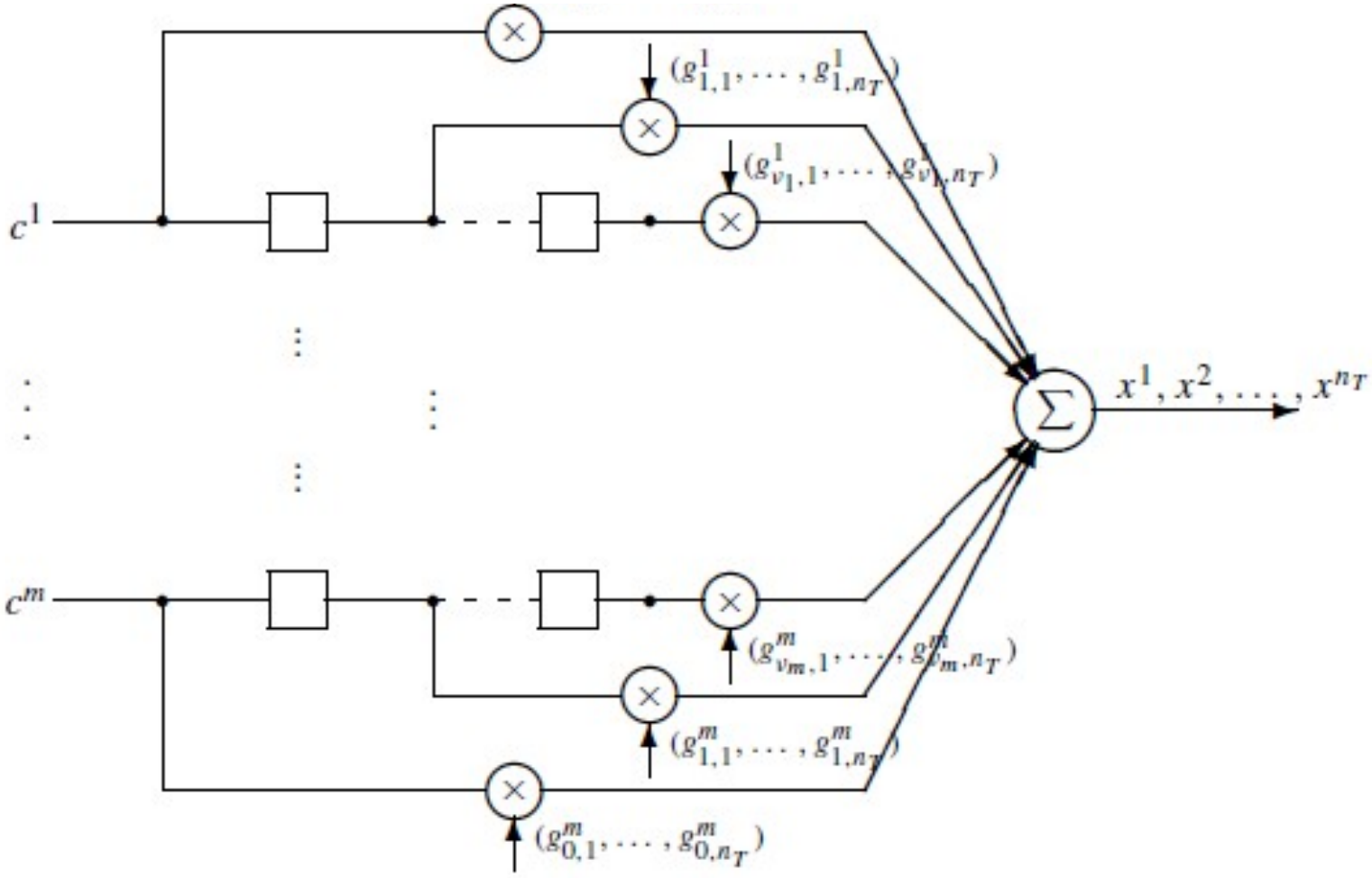
$$\boldsymbol{h}_e(z^{-1}) = \boldsymbol{h}_1(z^{-1}) + z^{-D}\boldsymbol{h}_2(z^{-1})$$

$$\boldsymbol{h}_e = \begin{bmatrix} \boldsymbol{h}_1 \\ 0_{n_r D,1} \end{bmatrix} + \begin{bmatrix} 0_{n_r D,1} \\ \boldsymbol{h}_2 \end{bmatrix}$$

Outline

- MIMO Wireless Communication Systems
- Space-Time Coding Performance Analysis
- Space-Time Block Codes
- Space-Time Trellis Codes
  - Delay Diversity Code
  - General STTC

Encoder Structure of STTC



Encoder Structure of STTC

- Generator Description
- Generator Polynomial Description

Example

- 4-state space-time trellis coded QPSK scheme with 2 transmit antennas
- Generator sequences:

$$\boldsymbol{g}^1 = [(02), (20)]$$
$$\boldsymbol{g}^2 = [(01), (10)]$$

Example

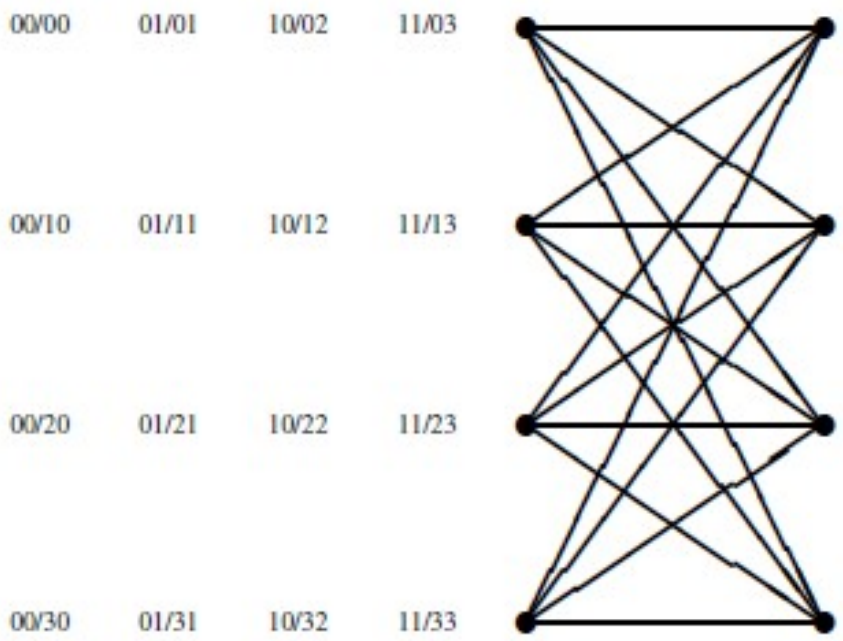


Figure 4.3 Trellis structure for a 4-state space-time coded QPSK with 2 antennas

Decoder Structure of STTC

- Maximum Likelihood Decoding
  - Employ Viterbi Algorithm
  - Minimize the path metric

$$\sum_{j=1}^{n_R} \left| r_t^j - \sum_{i=1}^{n_T} h_{j,i}^t x_t^i \right|^2$$