Variable Rate Channel Capacity

Jie Ren

2013/4/26

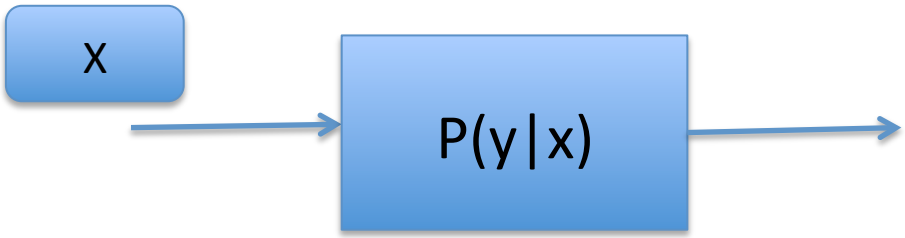
Reference

- This is a introduction of Sergio Verdu and Shlomo Shamai's paper.
- Sergio Verdu and Shlomo Shamai, "Variable-Rate Channel Capacity", IEEE Transactions on Information Theory, vol. 56, No. 6, June 2010.

Outline (1st talk)

- Conventional Channel Coding
- Setup of Variable Rate Coding

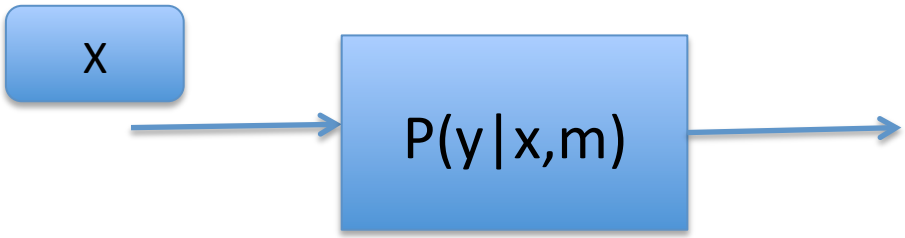
Conventional Channel Capacity



$$C = \sup_{p_X} I(X;Y)$$

Conventional Channel Capacity

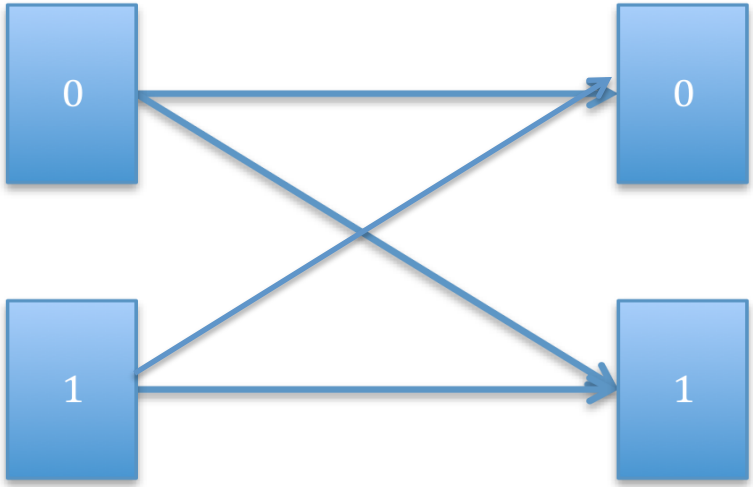
- What if Channel state varies?



$$C = \inf_{m} \sup_{P_x} I(X; Y_m)$$

Example: BSC

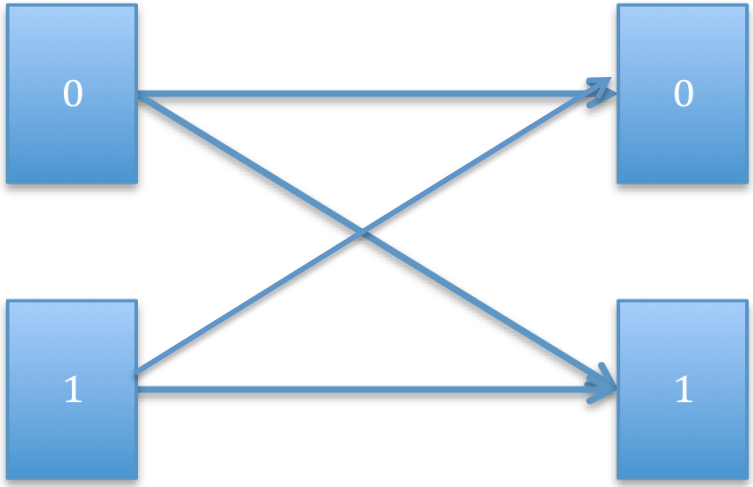
- Crossover Probability p



$$C = 1 - h(p)$$

Example: BSC

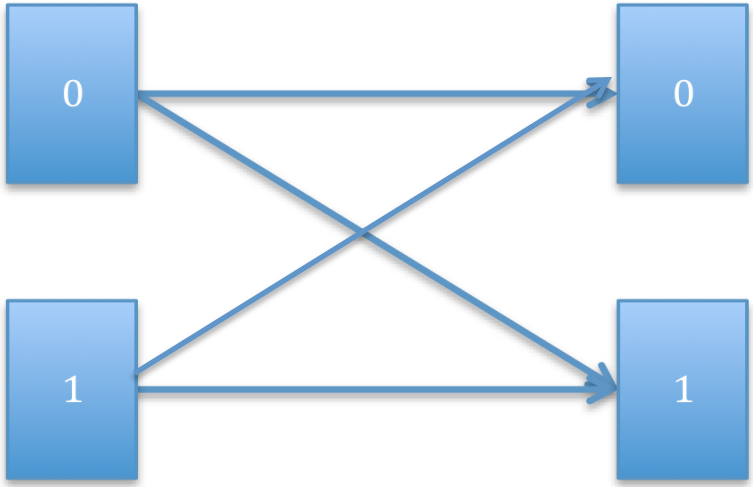
- Crossover Probability p=0.5



$$C = 1 - h(0.5) = 0$$

Example: BSC

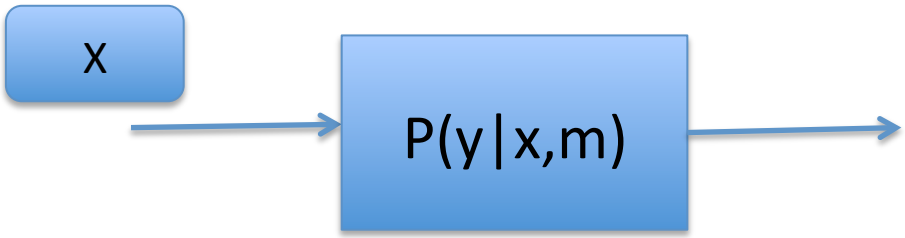
- Crossover Probability p~Uni(0,0.5)



$$C = \inf_{p} 1 - h(p) = 0$$

Conventional Channel Capacity

- What if Channel state varies?



$$C = \inf_{m} \sup_{P_x} I(X; Y_m)$$

Outline

- Conventional Channel Coding
- Setup of Variable Rate Coding
- Variable-to-Fixed Capacity
- Fixed-to-Variable Capacity
- Variable-Blocklength Capacity
- Examples

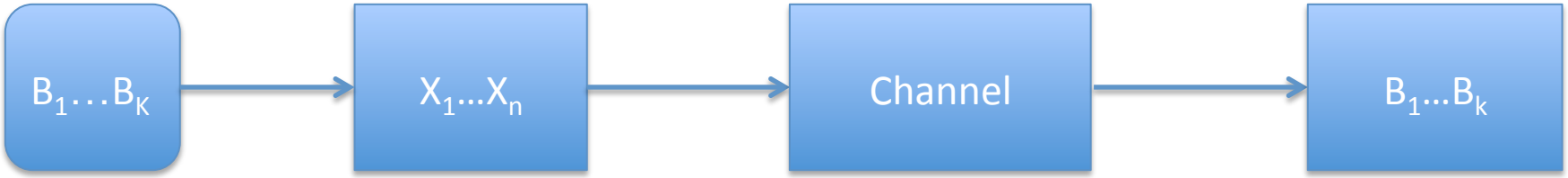
Setup of Variable Rate Coding

- Fixed-to-fixed coding
- Variable-to-fixed coding
- Fixed-to-variable coding
- Variable-blocklength coding

Fixed-to-fixed

$$f^{k\to n}: \{0,1\}^k \to x^n$$

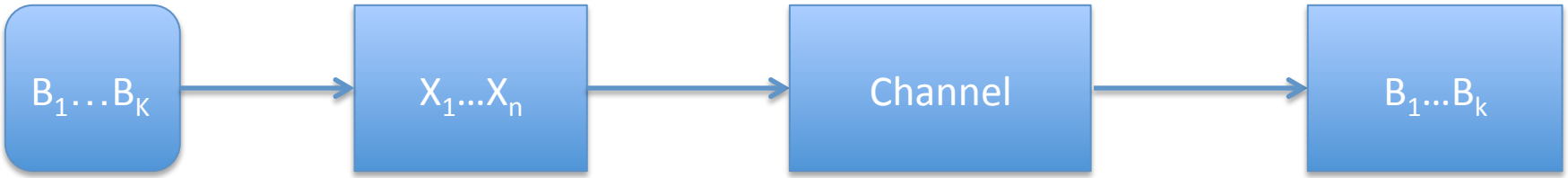
$$g^{n\to k}: y^n \to \{0,1\}^k$$



Fixed-to-fixed

$$\lim_{n\to\infty} \epsilon_n = 0$$

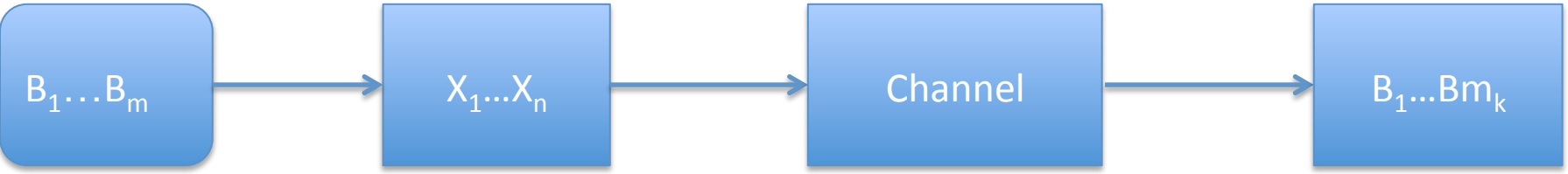
$$\lim_{n\to\infty} \inf \frac{k}{n} = R$$



Variable-to-Fixed

$$f^{:n}: \{0,1\}^{m_n} \rightarrow x^n$$

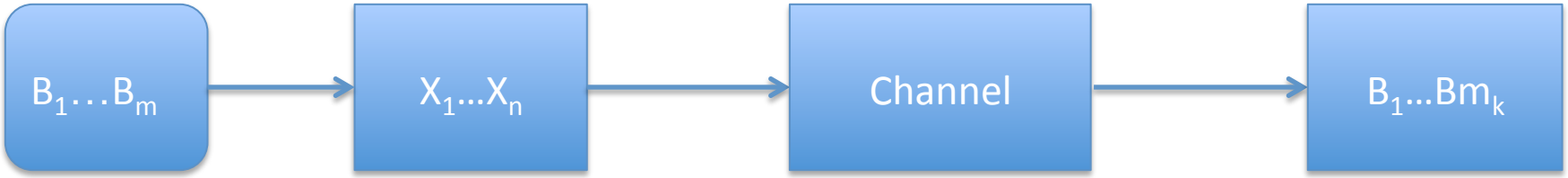
$$g^{n:}: y^n \rightarrow \{0,1\}^{m_n}$$



Variable-to-Fixed

$$L_n = \max_{(B_1 \ldots B_k) = g^{n:k}(Y^n)} k$$

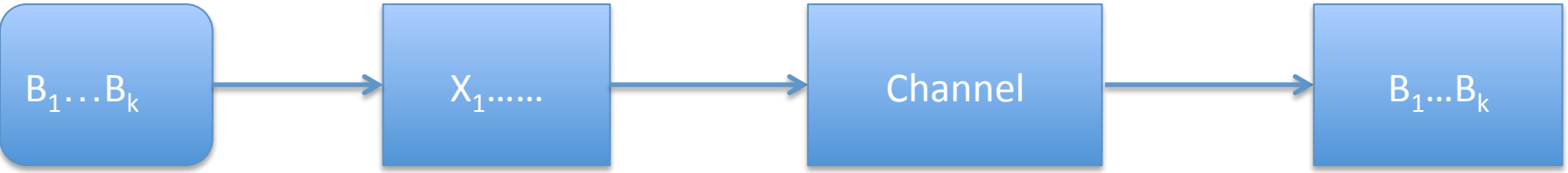
$$R = \lim_{n \to \infty} \inf \frac{E[L_n]}{n}$$



Fixed-to-variable

$$f^{k:}: \{0,1\}^k \rightarrow x^{\infty}$$

$$\{g^{n:k}: y^n \rightarrow \{0,1\}^k\}_{n=1}^{\infty}$$

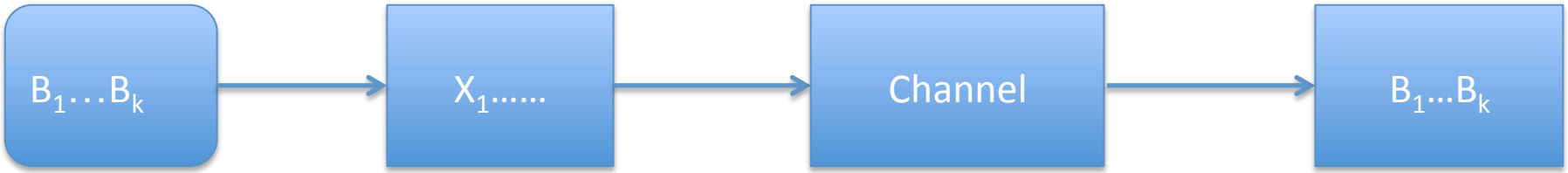


Fixed-to-variable

$$N_k = \min_{g^{n:k}(Y^n)=B^k} n$$

$$[C] = \max_{f,g} \lim_{k\to\infty} \inf \frac{k}{E[N_k]}$$

$$[\bar{C}] = \max_{f,g} \lim_{k\to\infty} \inf E\left[\frac{k}{N_k}\right]$$



Variable-blocklength

$$f^{k\rightarrow}: \{0,1\}^k \rightarrow x^+$$

$$g^{\rightarrow k}: y^+ \rightarrow \{0,1\}^k$$

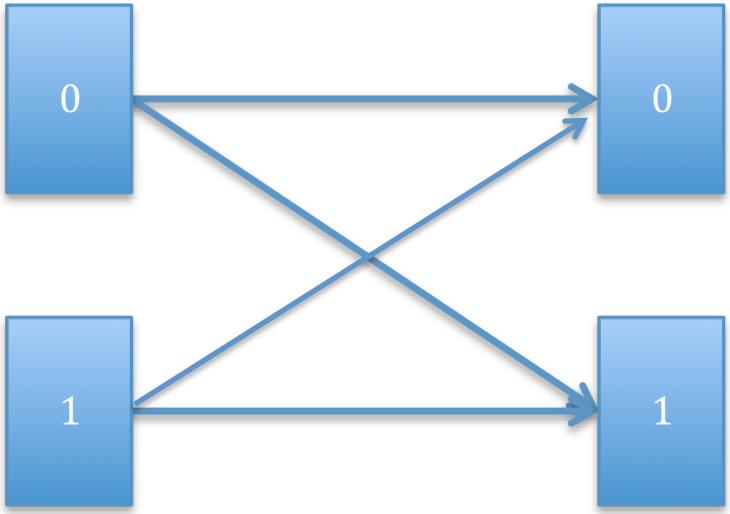
Notations

$$C: fixed - to - fixed\ capacity$$
$$\langle C \rangle: variable - to - fixed\ capacity$$
$$[C]: fixed - to - variable\ capacity$$
$$[\bar{C}]: upper - fixed - to - fixed\ capacity$$
$$[[C]]: variable - blocklength\ capacity$$

Basic Relationship

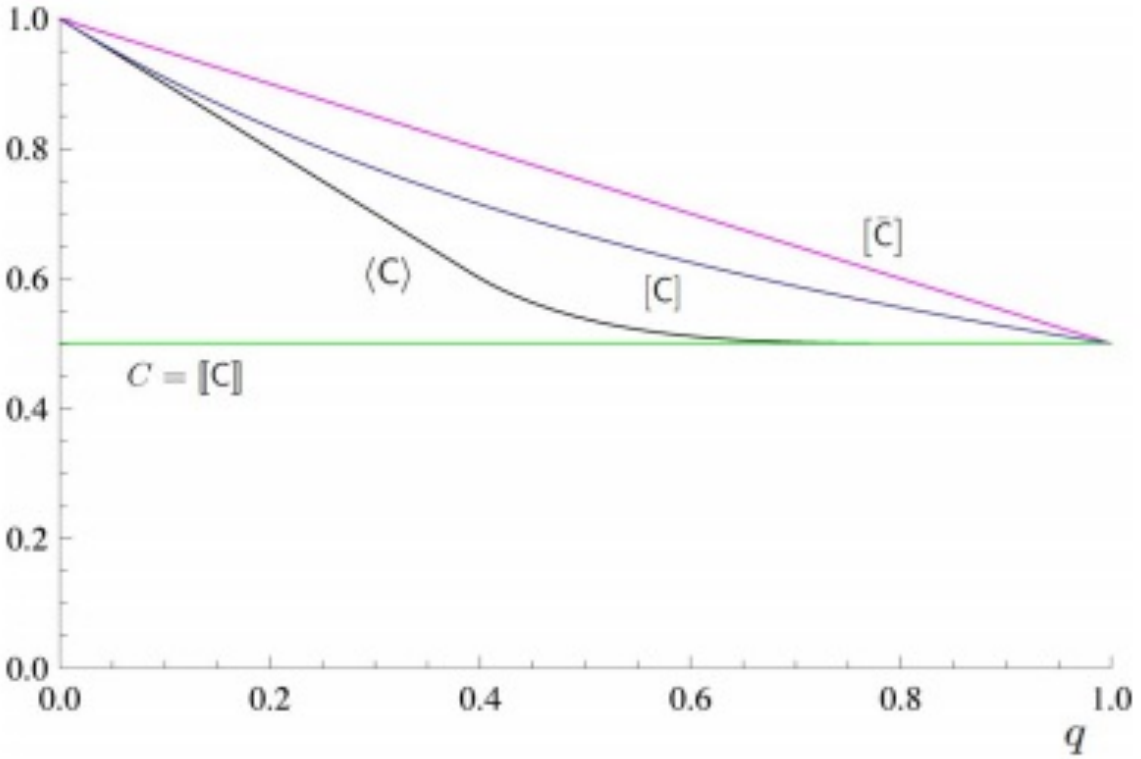
$$C \leq \min\left([C], \langle C \rangle\right) \leq \max\left([C], \langle C \rangle\right) \leq [\bar{C}] \leq \lim_{\epsilon \uparrow 1} C_{\epsilon}$$

Example: BSC



$$P = \begin{cases} 0.11 & q \\ 0 & 1-q \end{cases}$$

Example: BSC



Comparison of capacities for a binary symmetric channel whose crossover probability is 0.11 with probability q and 0 with probability 1

Variable Rate Channel Capacity

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2013/4/30

Outline (2^{nd} talk)

- Reviews
- Theorems
- Examples

Review: Definitions

- Fixed-to-fixed capacity

$$\lim_{n\to\infty} \epsilon_n = 0$$

$$\lim_{n\to\infty} \inf \frac{k}{n} = R$$

Review: Definitions

- Variable-to-fixed capacity

$$L_n = \max_{(B_1 \ldots B_k) = g^{n:k}(Y^n)} k$$

$$R = \lim_{n \to \infty} \inf \frac{E[L_n]}{n}$$

Review: Definitions

- Fixed-to-variable & upper fixed-to-variable

$$N_k = \min_{g^{n:k}(Y^n)=B^k} n$$

$$[C] = \max_{f,g} \lim_{k\to\infty} \inf \frac{k}{E[N_k]}$$

$$[\bar{C}] = \max_{f,g} \lim_{k\to\infty} \inf E\left[\frac{k}{N_k}\right]$$

Review: Definitions

- ε capacity

$$\lim_{n\to\infty} \epsilon_n \leq \varepsilon$$

$$\lim_{n\to\infty} \inf \frac{k}{n} = R$$

Basic Relationship

$$C \leq \min\left([C], \langle C \rangle\right) \leq \max\left([C], \langle C \rangle\right) \leq [\bar{C}] \leq \lim_{\epsilon \uparrow 1} C_{\epsilon}$$

State-Dependent Channels

$$P_{Y^n|X^n}(b^n|a^n) = \sum_{l=1}^{K} \pi_l \, P_{Y^n|X^n,S}(b^n|a^n, l)$$

State-Dependent Channels

- Finite alphabets

$$P_{Y^n|X^n,S}(b^n|a^n,l) = \prod_{i=1}^{n} W_l(b_i|a_i)$$

Outline (2nd talk)

- Reviews
- Theorems
- Examples

Theorem 7

- Two possible states

$$P_{Y^n|X^n}(b^n|a^n) = \pi_1 \prod_{i=1}^{n} W_1(b_i|a_i) + \pi_2 \prod_{i=1}^{n} W_2(b_i|a_i)$$

- Variable-to-fixed capacity in SDC

$$\langle C \rangle = \max_{P_{XU}} \{\min\{I(U, W_1), I(U, W_2)\} + \max\{\pi_1 I(X, W_1|U), \pi_2 I(X, W_2|U)\}\}$$

Theorem 12

- Fixed-to-fixed capacity in SDC

$$C = \max_{P} \min_{l} I(P, W_l)$$

Theorem 13

- Fixed-to-variable capacity in SDC

$$[C] \geq \left(\min_{P_X} \sum_{l=1}^{K} \frac{\pi_l}{I(P_X, W_l)} \right)^{-1}$$

Theorem 14

- Suppose

$$C_l = \max_{P_X} I(P_X, W_l) = I(P_X^*, W_l)$$

- Then the fixed-to-variable capacity in SDC

$$[C] = \left(\sum_{l=1}^{K} \frac{\pi_l}{C_l} \right)^{-1}$$

Theorem 15

- Upper-fixed-to-variable capacity in SDC

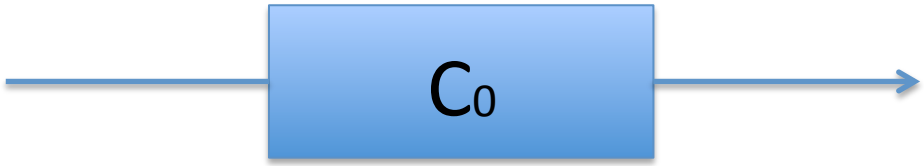
$$[\bar{C}] = \max_{P_X} \sum_{l=1}^{K} \pi_l I(P_X, W_l)$$

Outline (2nd talk)

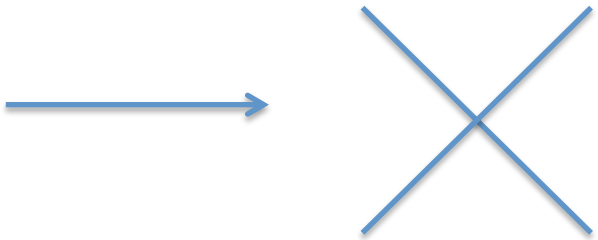
- Reviews
- Theorems
- Examples

Example 1

- With probability q



- With probability 1-q



Example 1

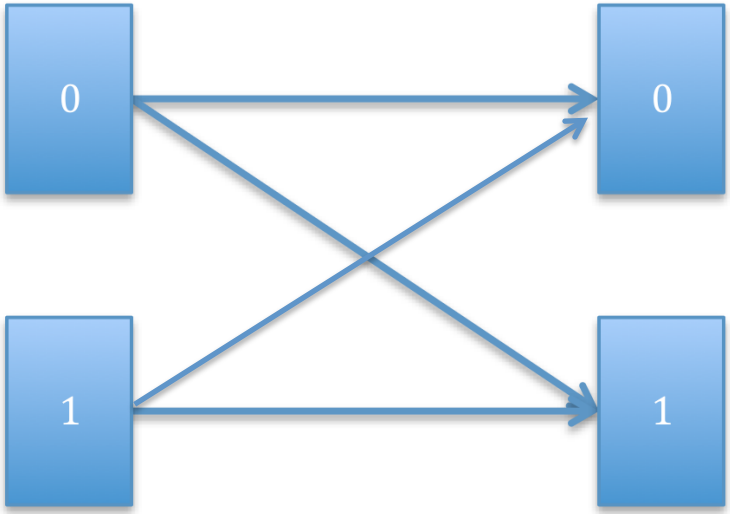
- By definition

$$C = [C] = 0$$
$$\lim_{\epsilon \uparrow 1} C_\epsilon = C_0$$
$$\langle C \rangle = [\bar{C}] = qC_0$$

Example 2

- With probability π_0 : BSC(δ_0)
- With probability π_1 : BSC(δ_1), $0.5>\delta_1>\delta_0$

Example 2



$$P = \begin{cases} \delta_0 & \pi_0 \\ \delta_1 & \pi_1 \end{cases}$$

Example 2

- By theorem 12

$$C = 1 - h(\delta_1)$$

Example 2

- By theorem 7

$$\langle C \rangle = 1 - \pi_0 h(\delta_0) - \min_{0 \le \gamma \le 1} h(\gamma * \delta_1) - \pi_0 h(\gamma * \delta_0)$$

$$\gamma * \delta = (1 - \delta)\gamma + (1 - \gamma)\delta$$

Example 2

- By theorem 14 & 15

$$[C] = \left(\frac{\pi_0}{1 - h(\delta_0)} + \frac{\pi_1}{1 - h(\delta_1)}\right)^{-1}$$

$$[\bar{C}] = 1 - \pi_0 h(\delta_0) - \pi_1 h(\delta_1)$$

Example 3

- Channel alphabet {0…1023}
- With probability 0.9: no error
- With probability 0.1: $y_i=(x_i>0)$

Example 3

- By theorem 13

$$[C] \geq \left(\min_{P_X} \sum_{l=1}^{K} \frac{\pi_l}{I(P_X, W_l)} \right)^{-1} = 4.4$$

Example 3

- Suboptimal scheme

$$N_k = \begin{cases} \frac{k}{10} & 0.9 \\ \frac{11k}{10} & 0.1 \end{cases}$$

$$\frac{k}{E[N_k]} = 5$$

Example 4

- Binary erasure channel BEC(E)
  - E random variable
  - Stay constant during the duration of the codeword

$$F(x) = P[1 - E \leq x]$$

Example 4

$$[C] = (E[\frac{1}{1-E}])^{-1} = (\int_1^{\infty} F(\frac{1}{\alpha})d\alpha)^{-1}$$

$$[\bar{C}] = 1 - \int_0^1 F(x)dx$$

$$\langle C \rangle = \max_{0 \le s \le 1} s - sF(s)$$

Example 4

- E~Uni(0,1)

$$C = [C] = 0$$

$$\langle C \rangle = \frac{1}{4}$$

$$[\bar{C}] = \frac{1}{2}$$

Example 5

- Gaussian channel with nonergodic fading

$$y_i = Hx_i + n_i$$
$$H \in [H_{min}, H_{max}]$$
$$\inf\{x: F_H(x) > 0\} = H_{min}$$
$$\inf\{x: F_H(x) = 1\} = H_{max}$$
$$\frac{1}{n}\sum_{i=1}^{n} |x_i|^2 \leq P$$

Example 5

- Gaussian channel with nonergodic fading

$$C = \log(1 + H_{min}P)$$
$$\lim_{\epsilon \uparrow 1} C_\epsilon = \log\,(1 + H_{max}P)$$

Example 5

- Gaussian channel with nonergodic fading

$$[C] = \left(E\left[\frac{1}{\log(1+HP)}\right]\right)^{-1}$$

$$[\bar{C}] = E[\log(1+HP)]$$

Example 5

- Gaussian channel with nonergodic fading

$$\langle C \rangle = \int_{x_0}^{x_1} \left(1 - F_H(x)\right) \left(\frac{\dot{f}_H(x)}{f_H(x)} + \frac{2}{x} \right) dx \log e$$

Variable Rate Channel Capacity

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2013/5/7

Outline (3rd talk)

- Theorems and proofs

Theorems and Proofs

$$C \leq \min\left([C], \langle C \rangle\right) \leq \max\left([C], \langle C \rangle\right) \leq [\bar{C}] \leq \lim_{\epsilon \uparrow 1} C_{\epsilon}$$

Theorem 1

- For all channels

$$C \leq \langle C \rangle$$

Proof

- Consider a fixed-to-fixed code

$$\lim_{n\to\infty} \epsilon_n = 0$$

$$\liminf_{n\to\infty} \frac{k}{n} = R$$

- Let

$$f^{:n} = f^{k\to n}(b_{1:k})$$

$$g^{n:}(y^n) = (g^{n\to k}(y^n), 0, \dots 0)$$

Theorem 2

- If the channel input alphabet is finite

$$\langle C \rangle \leq \lim_{\epsilon \uparrow 1} C_\epsilon$$

Proof

- Prove by contradiction

$$C_2 = \frac{1}{2}\langle C \rangle + \frac{1}{2}\lim_{\epsilon \uparrow 1} C_\epsilon$$

$$P\left[\frac{L_n}{n} \leq C_2\right] \leq \epsilon_0$$

Theorem 3

- State-dependent channel, suppose each component satisfies strong converse

$$\langle C \rangle \leq \sum_{i=1}^{K} \pi_i C_i$$

Proof

- The bound would be tight if the encoder knows the channel state
- Hence it's an upper bound in general

Theorem 4

- Capacity region or 2-user DMBC with degraded message sets

$$C = \bigcup_{P_{X,U}} \left\{ (R_1, R_2): \begin{array}{c} R_1 \leq \min\{I(U, Y_1), I(U, Y_2)\} \\ R_2 \leq I(X; Y_2|U) \end{array} \right\}$$

Proof

- Thomas M. Cover, “Elements of Information Theory”, page 568

Theorem 5

- In state-dependent channels

$$\langle C \rangle \geq \max_{P} \max_{(R_{1:K}) \in C^K} \sum_{j=1}^{K} R_j \sum_{i=j}^{K} \pi_{P^{-1}(i)}$$

Proof

- Assume identity permutation
- Show

$$\langle C \rangle \geq \sum_{j=1}^{K} R_j \sum_{i=j}^{K} \pi_{(i)}$$

Theorem 6

- Theorem 5 holds with equality if the K-user broadcast channels satisfy the strong converse

Proof

- Number the channel states in order of increasing expected number of recovered bits

$$r_i = \lim_{n \to \infty} \inf \frac{k_i}{n}$$

Proof

- For an arbitrarily small δ

$$R_1 = r_1 - \delta$$
$$R_i = r_i - r_{i-1}$$

Proof

- Hence,

$$R_{1:K}\ is\ \epsilon - achievable \xrightarrow{Strong\ Converse} R_{1:K}\ is\ achievable$$

Theorem 7

- Suppose the channel has finite input/output alphabets and is memoryless.

$$\langle C \rangle = \max_{P_{XU}} \{ \min\{ I(U, W_1), I(U, W_2) \} + \max\{ \pi_1 I(X, W_1 | U), \pi_1 I(X, W_2 | U) \} \}$$

Proof

- When the constituent channels are discrete memoryless, the broadcast channel with degraded message sets satisfies the strong converse.
- Theorem 7 follows from 4 and 6

Theorem 8

- For all channels

$$C \leq [\bar{C}]$$

Proof

- Consider a fixed-to-fixed code

$$\lim_{n\to\infty} \epsilon_n = 0$$

$$\liminf_{n\to\infty} \frac{k}{n} = R$$

- Define fixed-to-variable code

$$f^{k:} = (f^{k\to n}(b_{1:k}), a, a, \dots)$$

Theorem 9

- Suppose the channel has finite memory

$$C \leq [C]$$

Proof

- Similar proof as theorem 8

Theorem 10

- Assume either the input or output alphabets are finite

$$[C] \leq [\bar{C}] \leq \lim_{\epsilon \uparrow 1} C_\epsilon$$

Proof

- Jensen's inequality
- Counterpart of Theorem 2

Theorem 11

- For all channels

$$\langle C \rangle \leq [\bar{C}]$$

Proof

- Assume

$$\lim_{k\to\infty} \inf E\left[\frac{k}{N_k}\right] = \lim_{k\to\infty} \sup E\left[\frac{k}{N_k}\right]$$

Proof

- Non-anticipatory setting
- Compatible variable-to-fixed encoder

$$f^{:(n+1)}\left(b_1, \ldots, b_{m_n}\right) = \left(f^{:n}\left(b_1, \ldots, b_{m_n}\right), A\right)$$

Proof

- Construct a sequence of rateless encoders
  - Let the ith component be equal to the ith component of variable-to-fixed code
  - For n sufficiently large

Proof

$$N_{L_i} \; ? \; i$$

$$N_{L_i} = \lceil f^{-1}(\lfloor f(i) \rfloor) \rceil$$

Proof

$$N_{L_i} \leq i$$

$$\frac{L_n}{n} \leq \frac{L_n}{N_{L_n}}$$

Theorem 12

- In state-dependent channel

$$C = \max_{P} \min_{l} I(P, W_l)$$

Proof

- Shannon capacity
- Has to satisfy the worst channel

Theorem 13

- In state-dependent channel

$$[C] \geq \left(\min_{P_X} \sum_{l=1}^{K} \frac{\pi_l}{I(P_X, W_l)} \right)^{-1}$$

Proof

- Achievability proof
- Scheme:
  - Fixed-to-variable
  - Repition after the first transmission

Theorem 14

- In state-dependent channel, suppose

$$C_l = \max_{P_X} I(P_X, W_l) = I(P_X^*, W_l)$$

- Then theorem 13 holds with equality

$$[C] = \left(\min_{P_X} \sum_{l=1}^{K} \frac{\pi_l}{I(P_X, W_l)} \right)^{-1}$$

Proof

- Converse proof
- Data-processing inequality

$$\frac{\mathrm{E}[N_K]}{K} \geq \sum_{s=1}^{K} \frac{\pi_s}{C_s}$$

$$\frac{\mathrm{K}}{\mathrm{E}[N_K]} \leq \left(\sum_{s=1}^{K} \frac{\pi_s}{C_s} \right)^{-1}$$

Theorem 15

- In state-dependent channel

$$[\bar{C}] = \max_{P_X} \sum_{l=1}^{K} \pi_l I(P_X, W_l)$$

Proof

- Achievability
  - Assume the decoder has knowledge of the ergodic mode
  - Apply theorem 8

$$[\bar{C}] \geq C = \max_{P_X} \sum_{l=1}^{K} \pi_l I(P_X, W_l)$$

Proof

- Converse

$$\frac{1}{n} I\left(B_{1:k}; \hat{B}_{1:k}\right) \leq \frac{1}{n} I(f^{k:}(B_{1:k}); Y^n) \leq I(Q_x, W_l)$$

$$E\left[\frac{k}{N_k} | S = l\right] \leq I(Q_x, W_l)$$