Rateless Codes

Jie Ren

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ASPITRG, Drexel University

Outline

- Introduction of Rateless Codes
- Rateless Codes for Erasure Channel
- Rateless Codes for AWGN Channel

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  - Basic Idea of Rateless Codes
  - Applications of Rateless Codes
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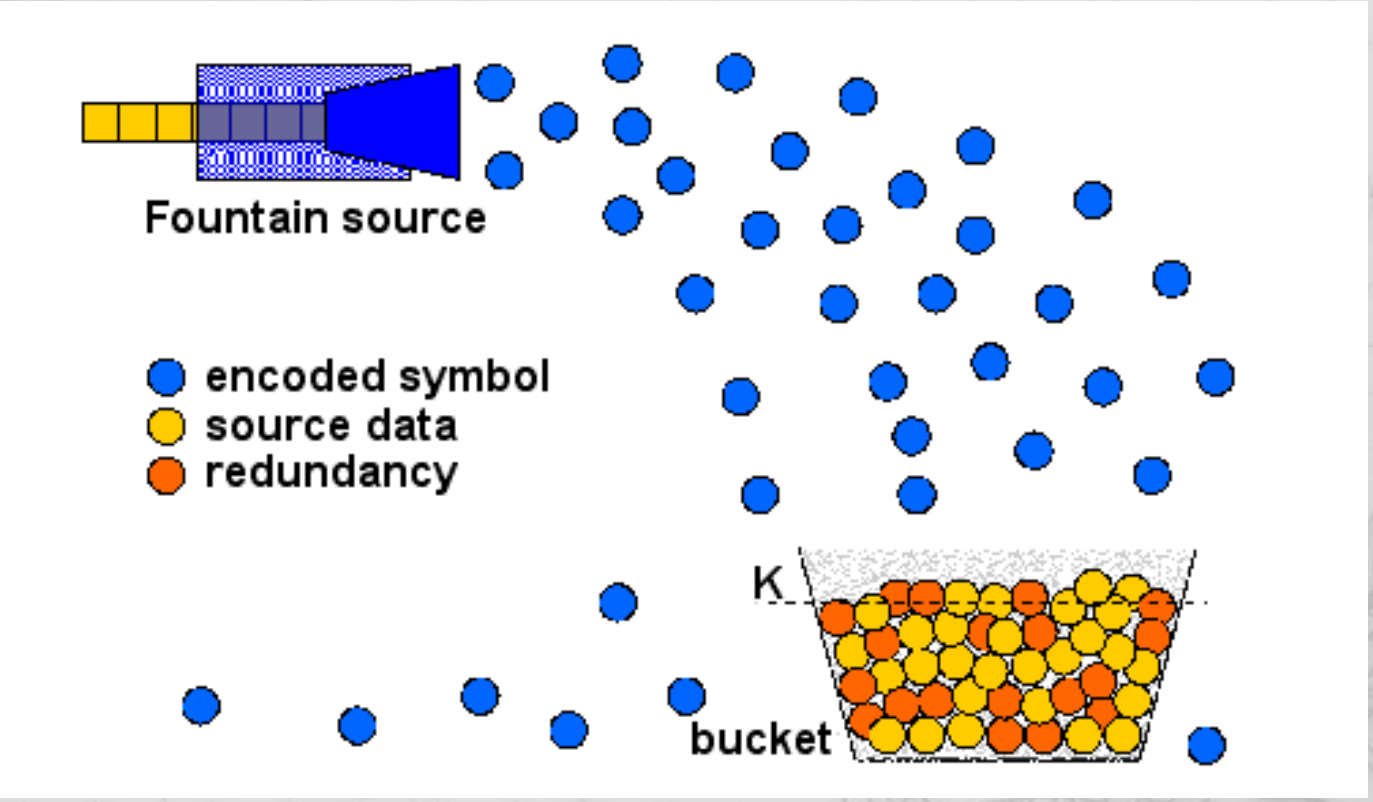
Basic Idea of Rateless Codes



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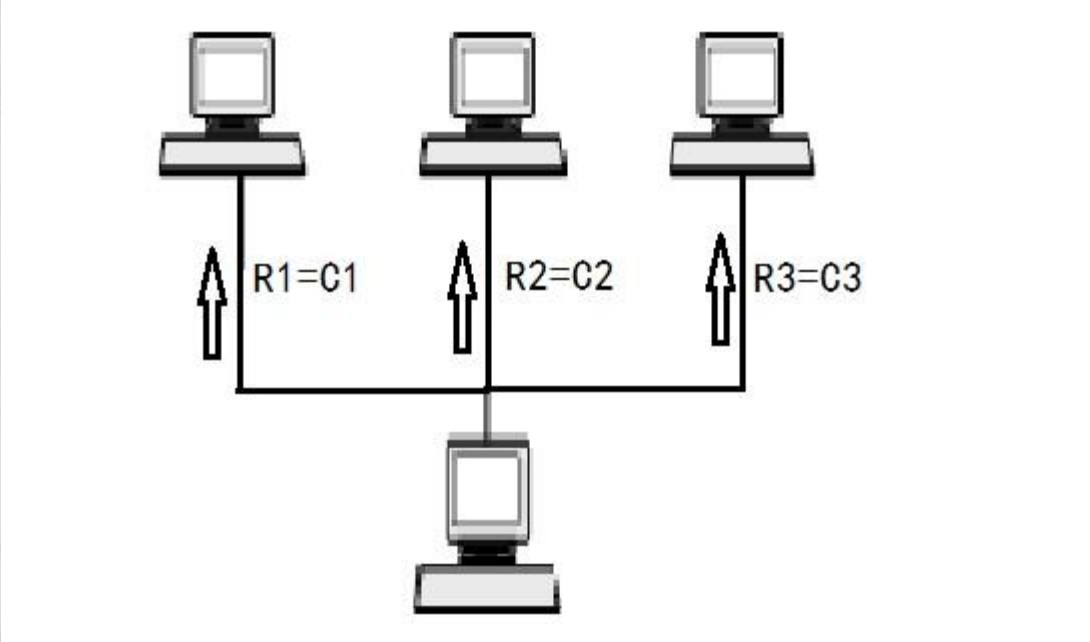
Applications of Rateless Codes

- One-to-many problem

  Each channel achieve its capacity

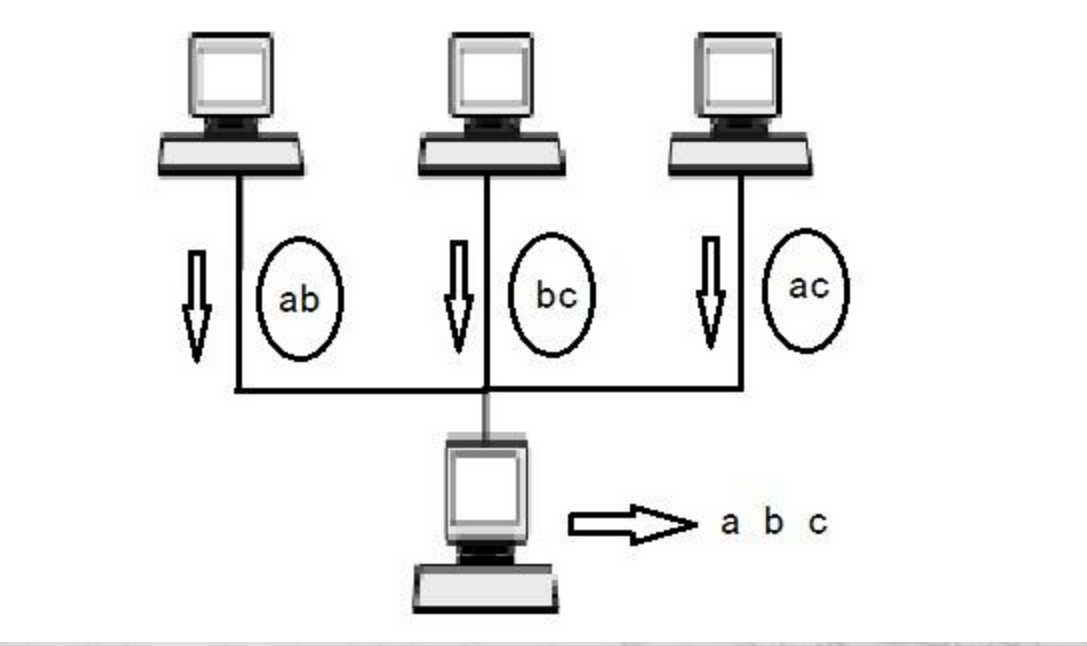
  No need of feed back



Applications of Rateless Codes

- Many-to-one problem

  Minimize delivery of redundant packets



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  - Enc/Dec
  - Performance
  - Proof
- Rateless Codes for AWGN Channel

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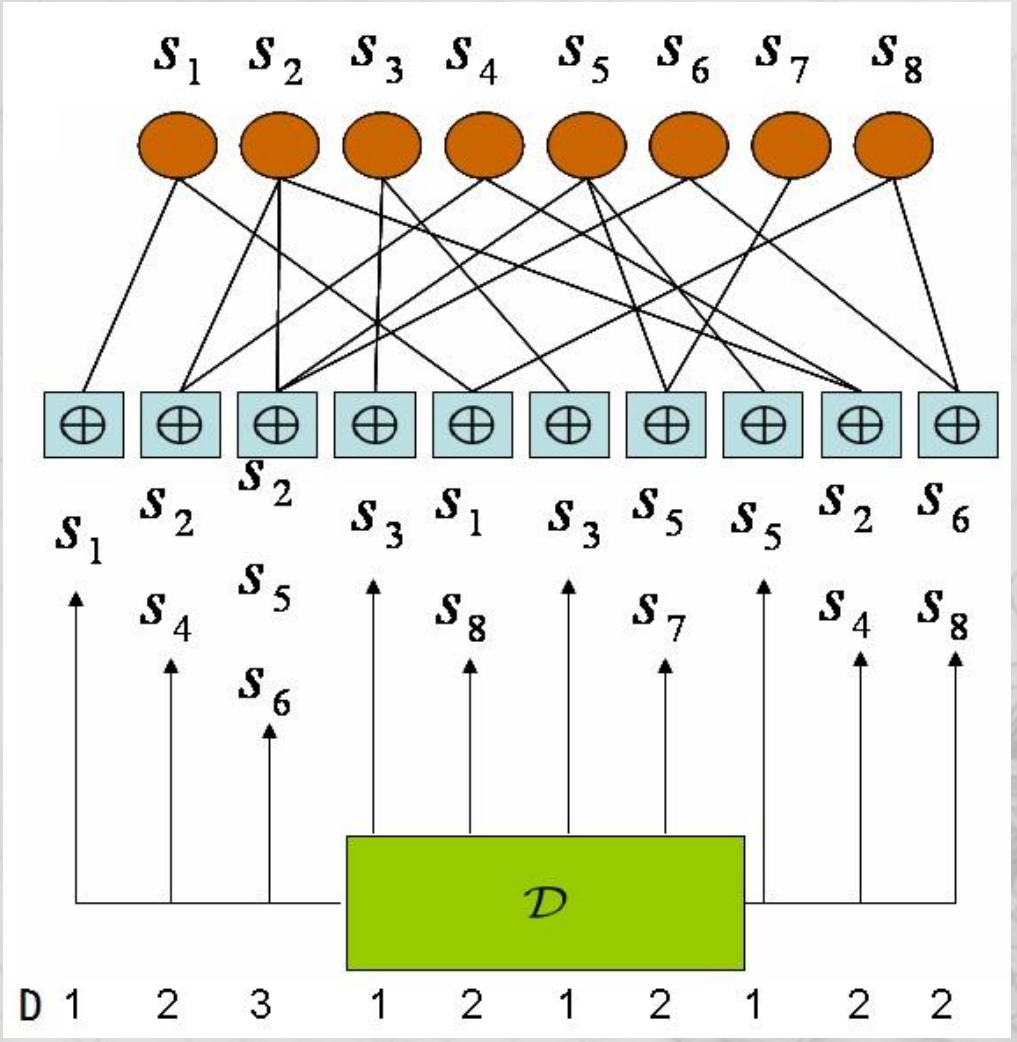
Rateless Codes for Erasure Channel



Rateless Codes for Erasure Channel

- Encoder

  k: input symbols of length l

  K: encoding symbols

  d: degree of encoding symbol

  Randomly choose the value d

  Choose d input symbols as neighbors(~uni)

  Calculate the XOR of the d neighbors

Rateless Codes for Erasure Channel

- Decoder

  1. Recover the encoding symbols with d=1
  2. Remove the input symbol as a neighbor

  Repeat

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Performance

- Encoding Complexity

$$O(\ln(k/\delta))$$

- Decoding Complexity

$$O(k * \ln(k/\delta))$$

- k input symbols can be recovered from any

$$k + O(\sqrt{k}\ln^2(k/\delta))$$

of the encoding symbols with probability 1-δ

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Proof

- The LT process
- The LT Distributions
- The Ideal Soliton Distribution
- The Robust Soliton Distribution

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- The LT process
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The LT process

- Definitons
  - Encoding Symbol
  - Input Symbol
  - Neighbor
  - Release Encoding Symbol
  - Ripple Set
  - Cover Set

The LT process

Procedures

Step 1 : release encoding symbols with d=1, put the input symbols in ripple

Step 2 : pick one from ripple, process it, put it in cover set

Step 3 : remove this symbol

Repeat

The LT process

The LT process ends when :

the ripple is empty at the end of some step

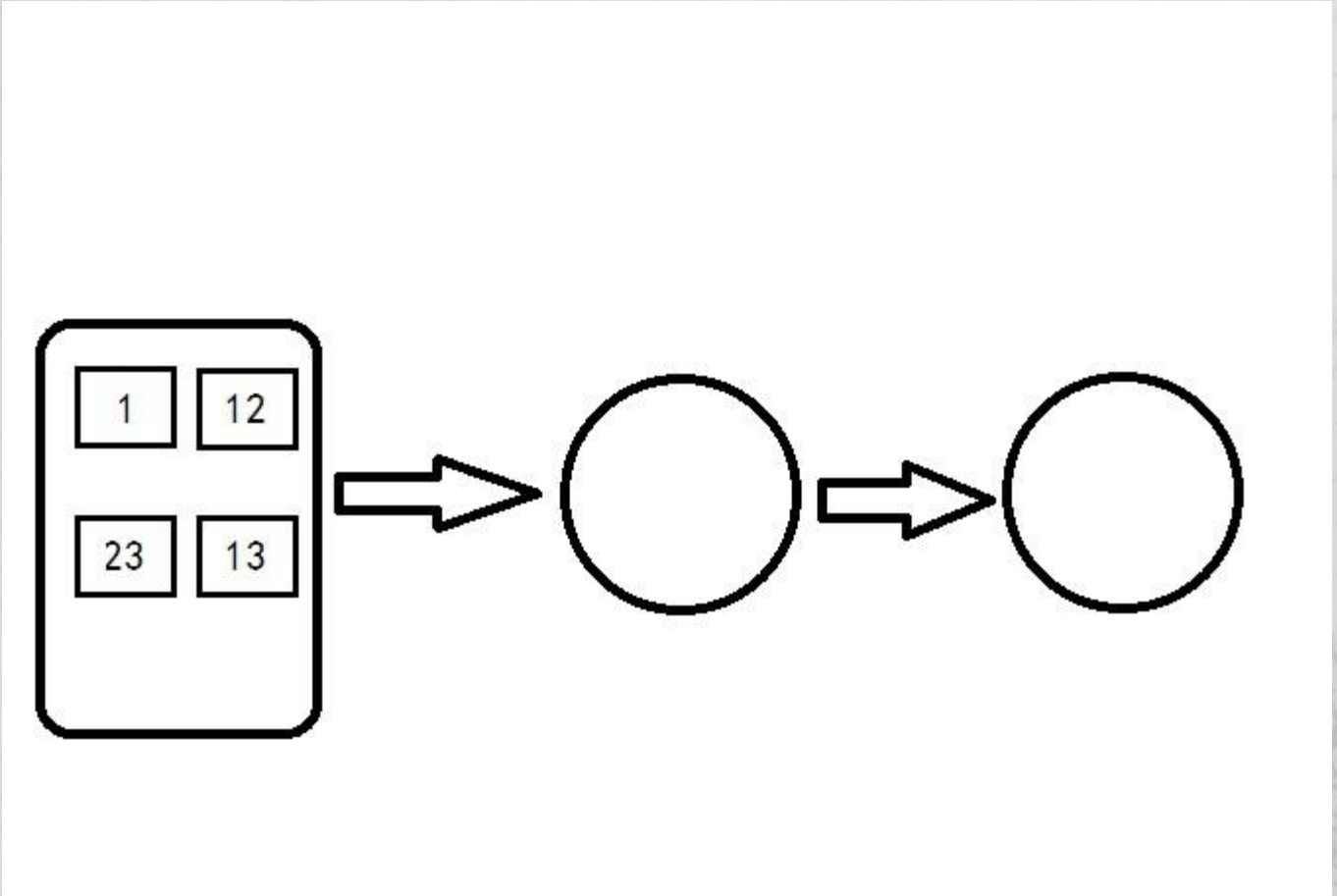
The LT process fails if :

there is at least one uncovered input symbol at the end

The LT process succeeds if :

all input symbols are covered at the end

The LT process



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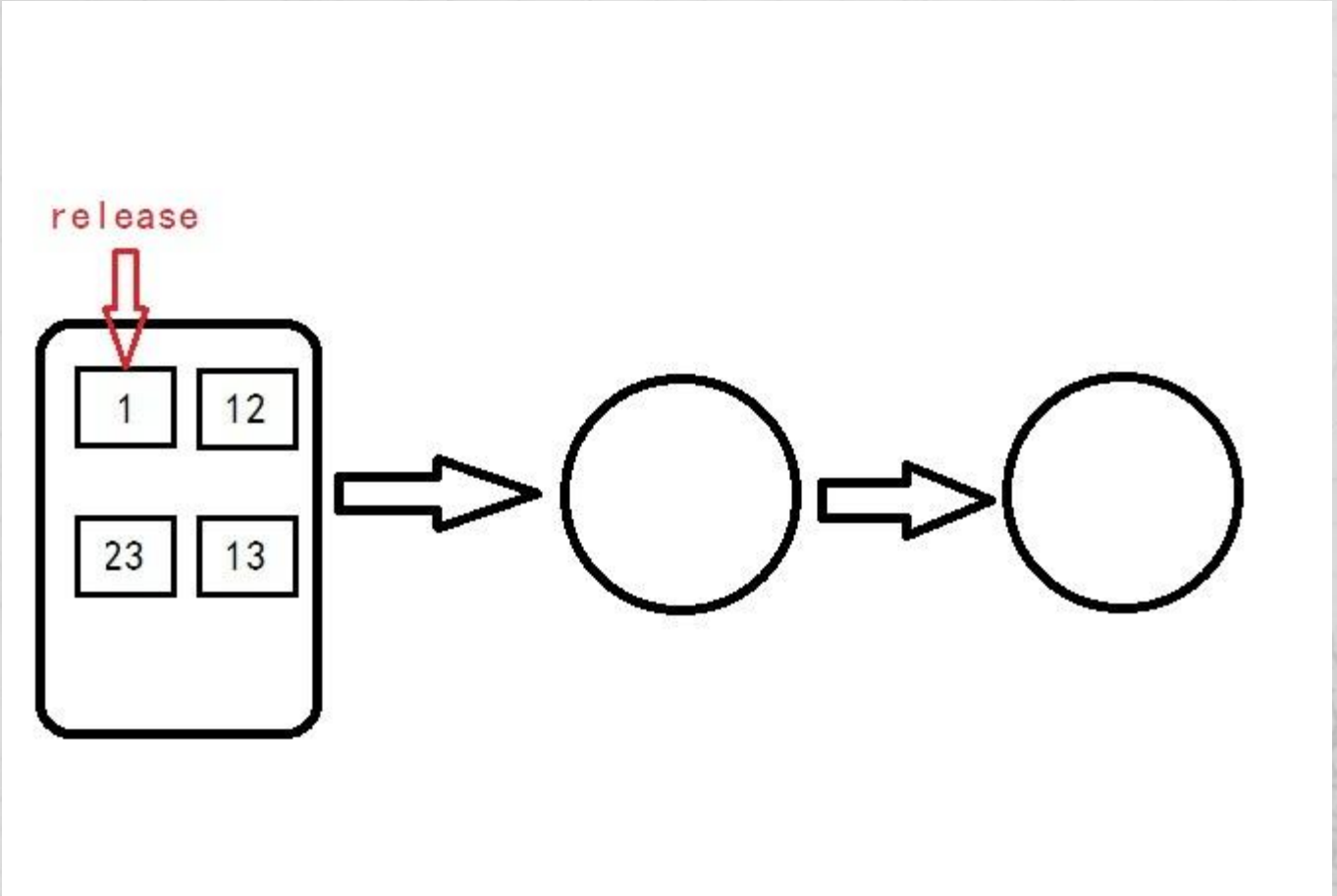

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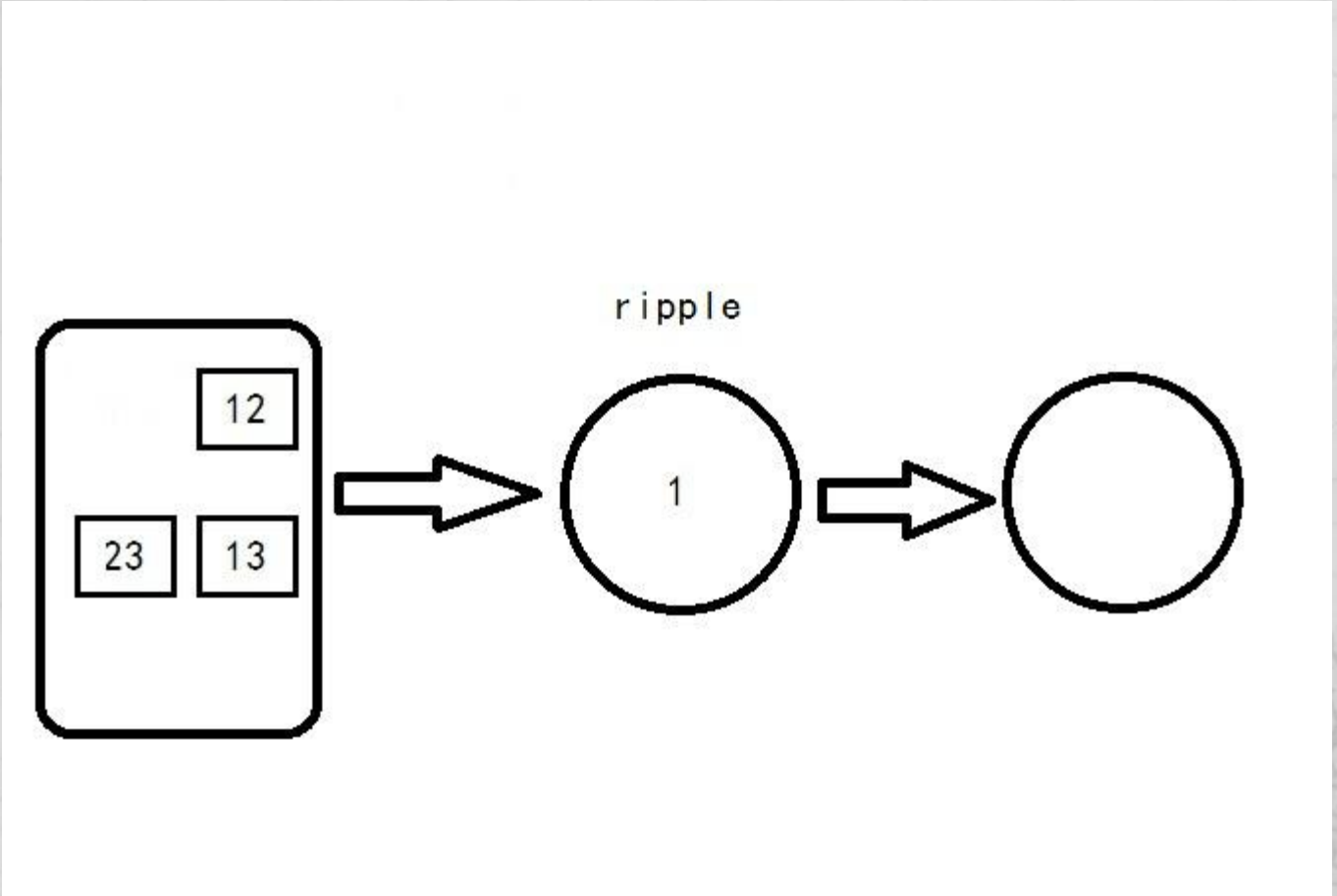

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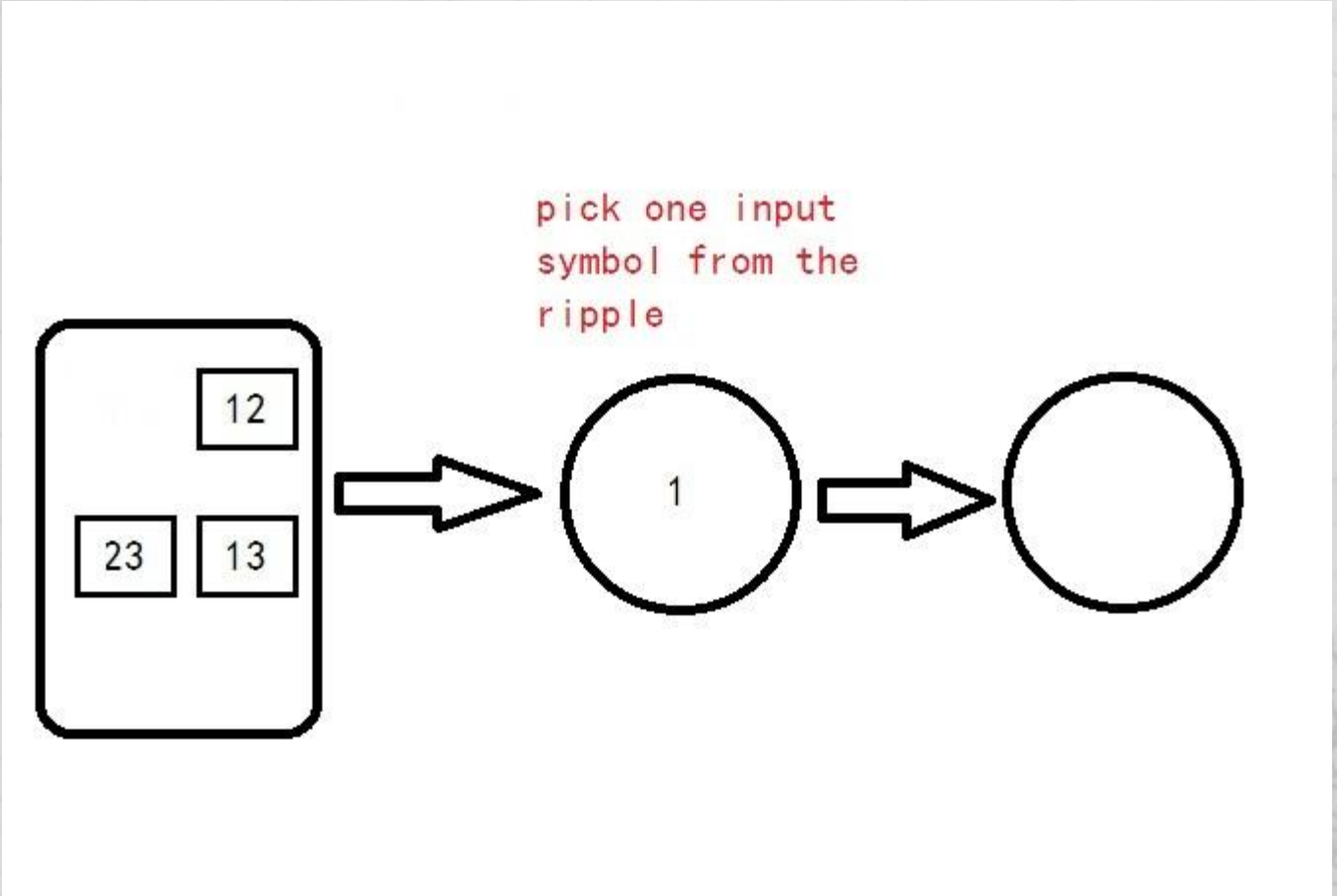

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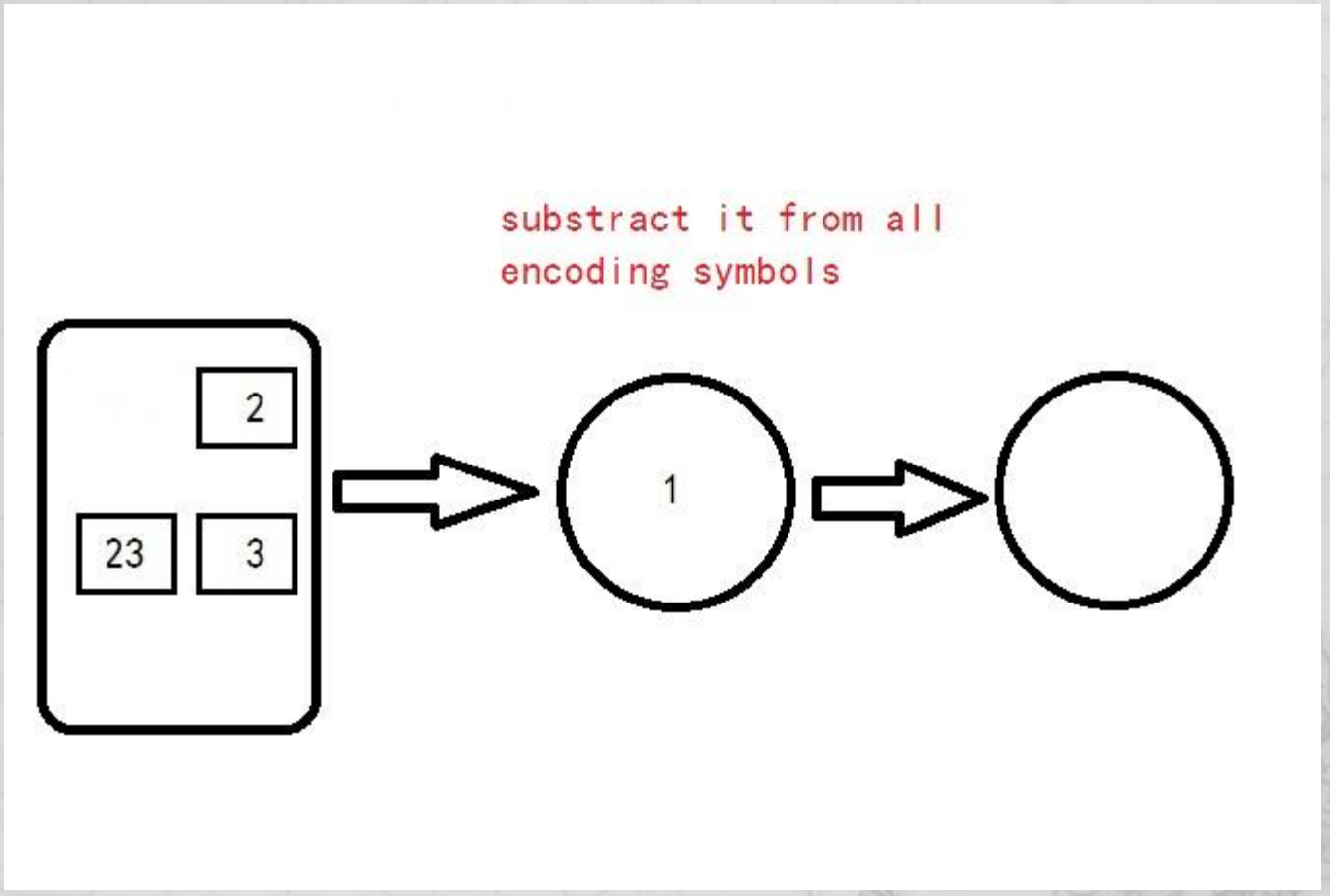

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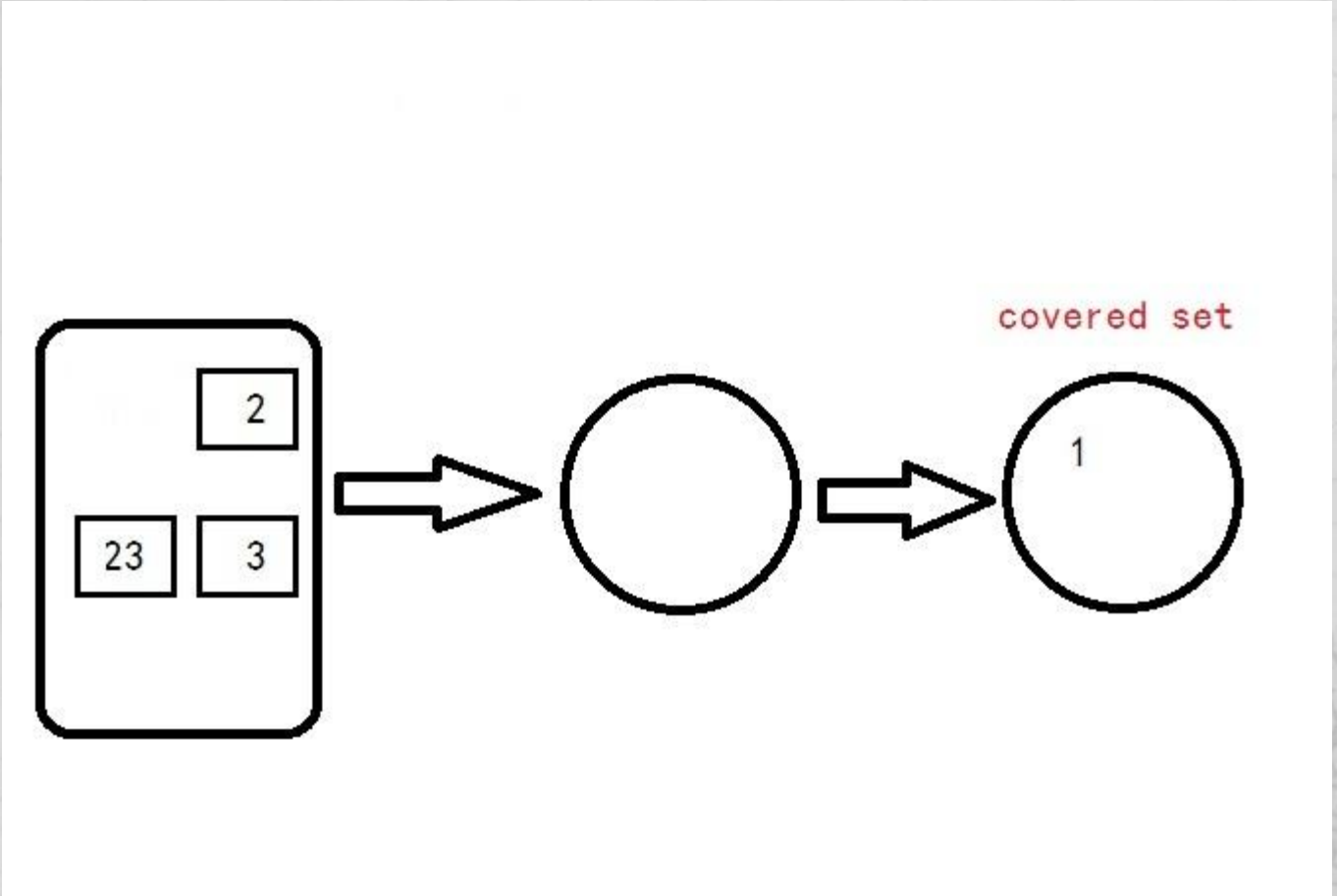

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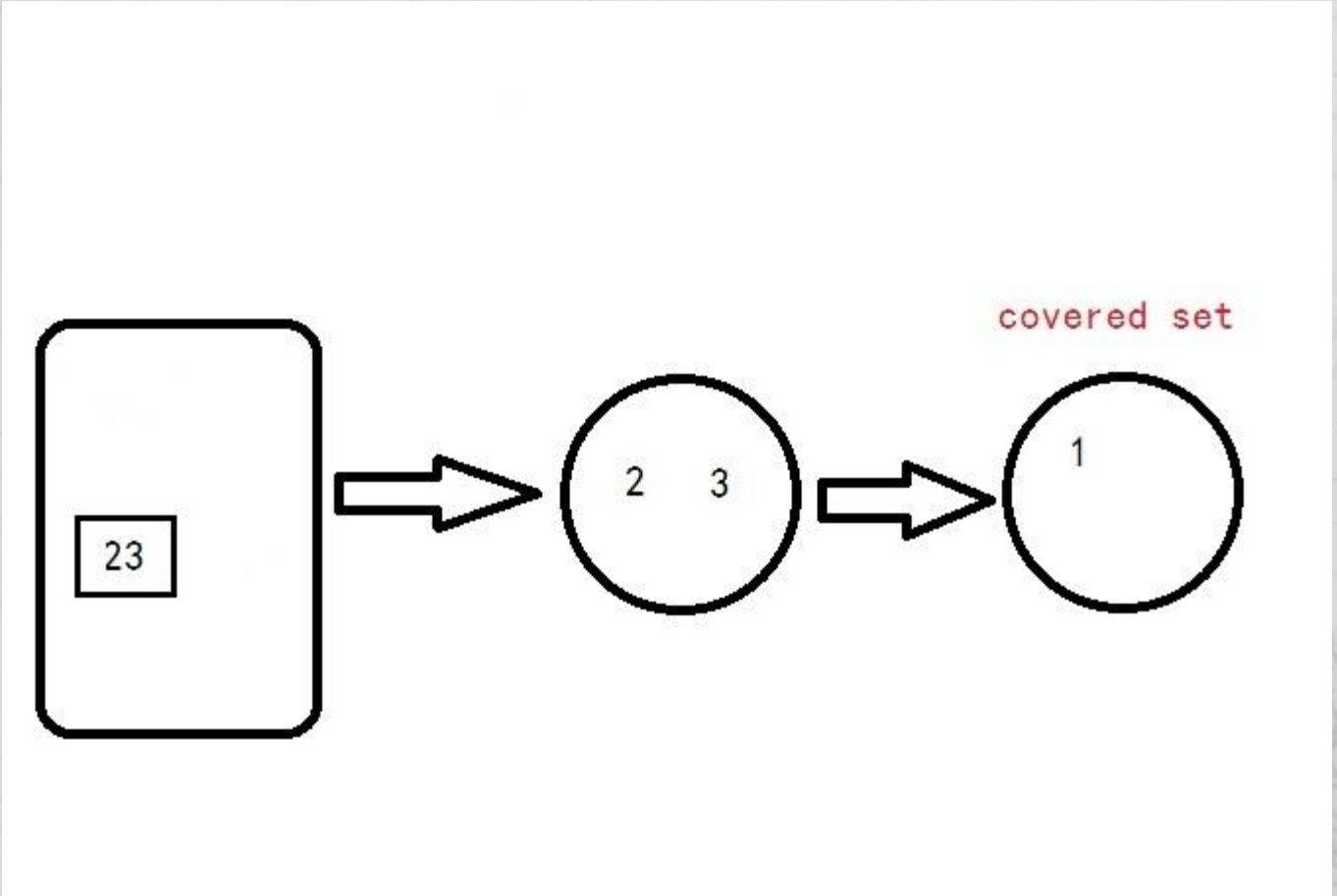

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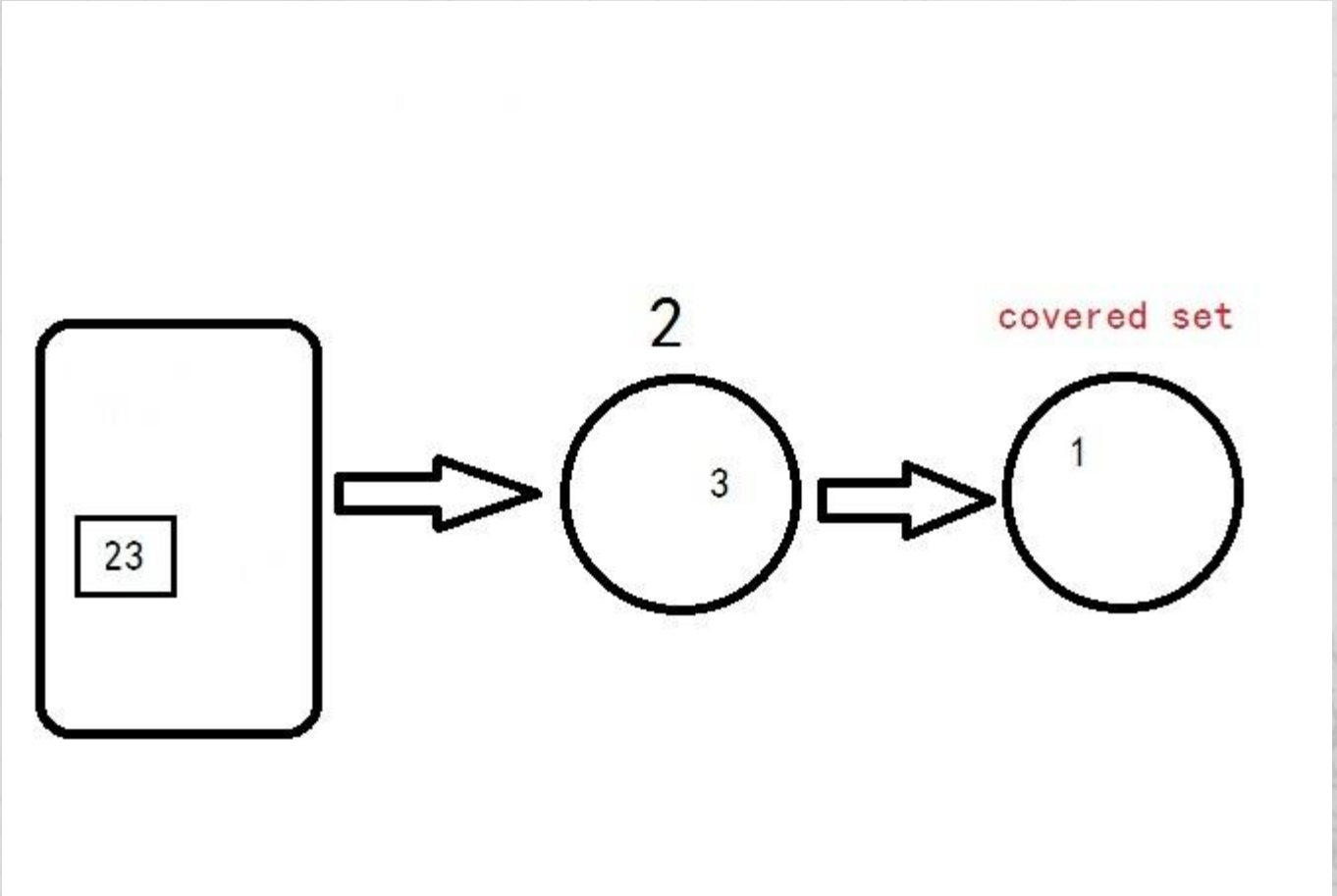

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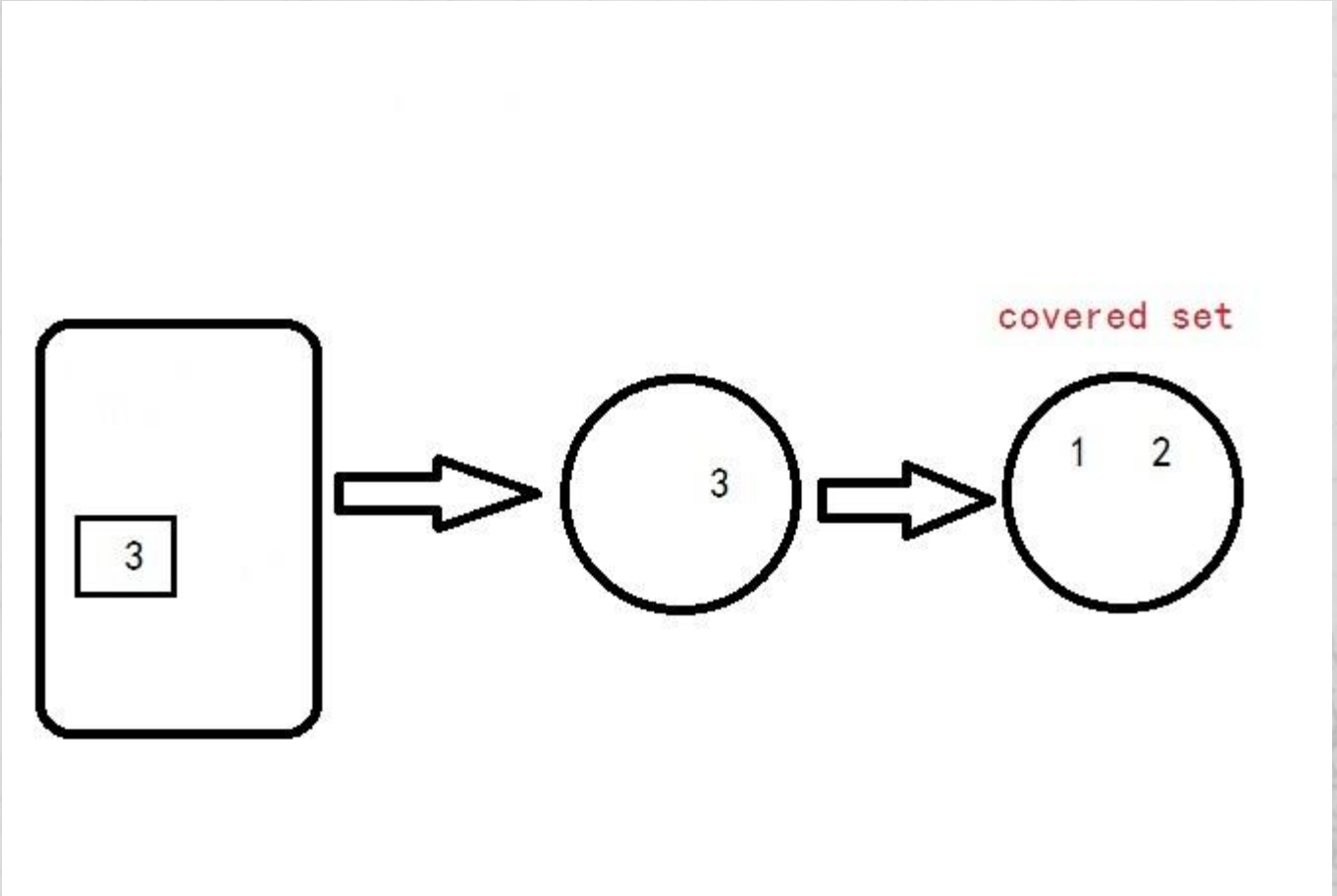

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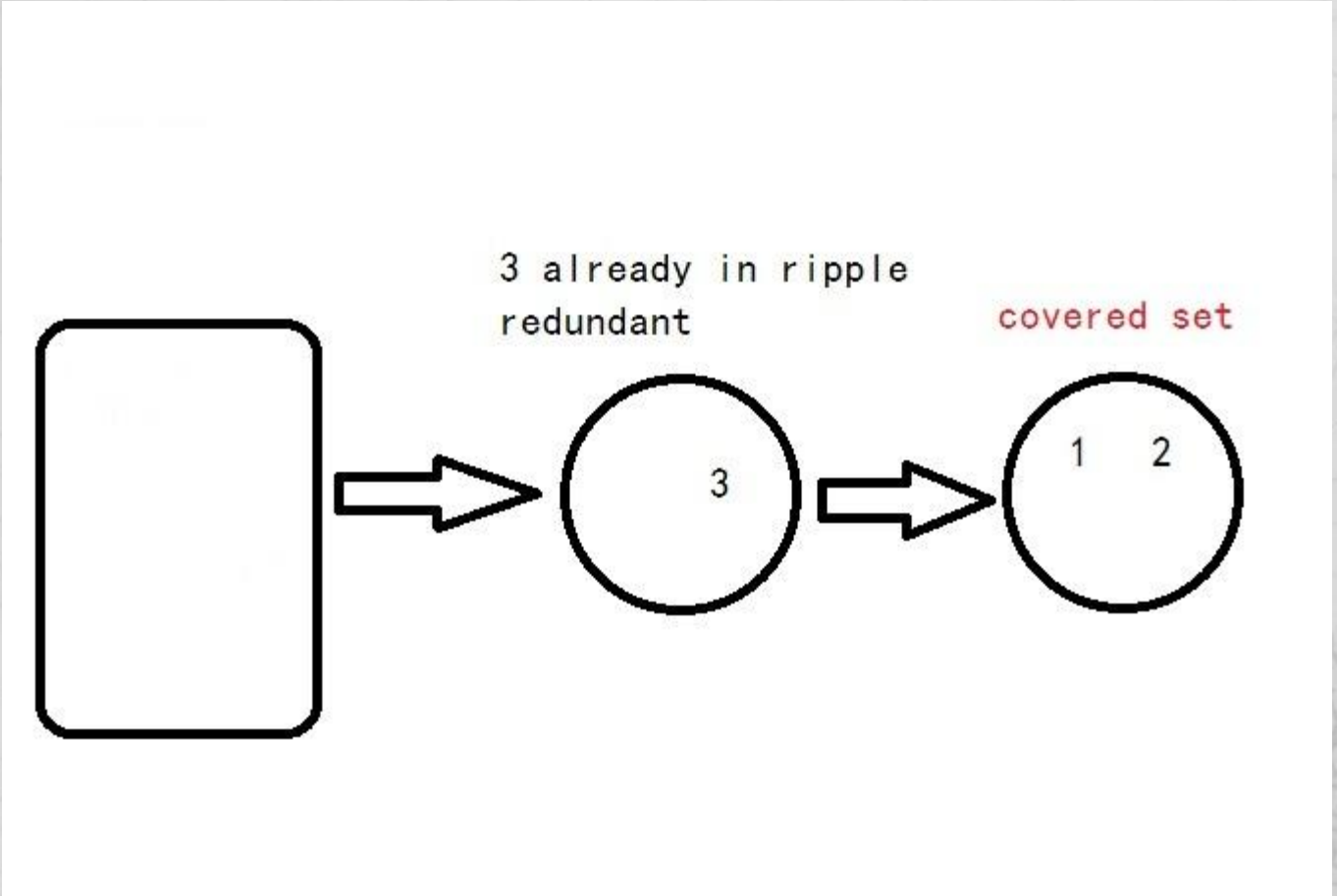

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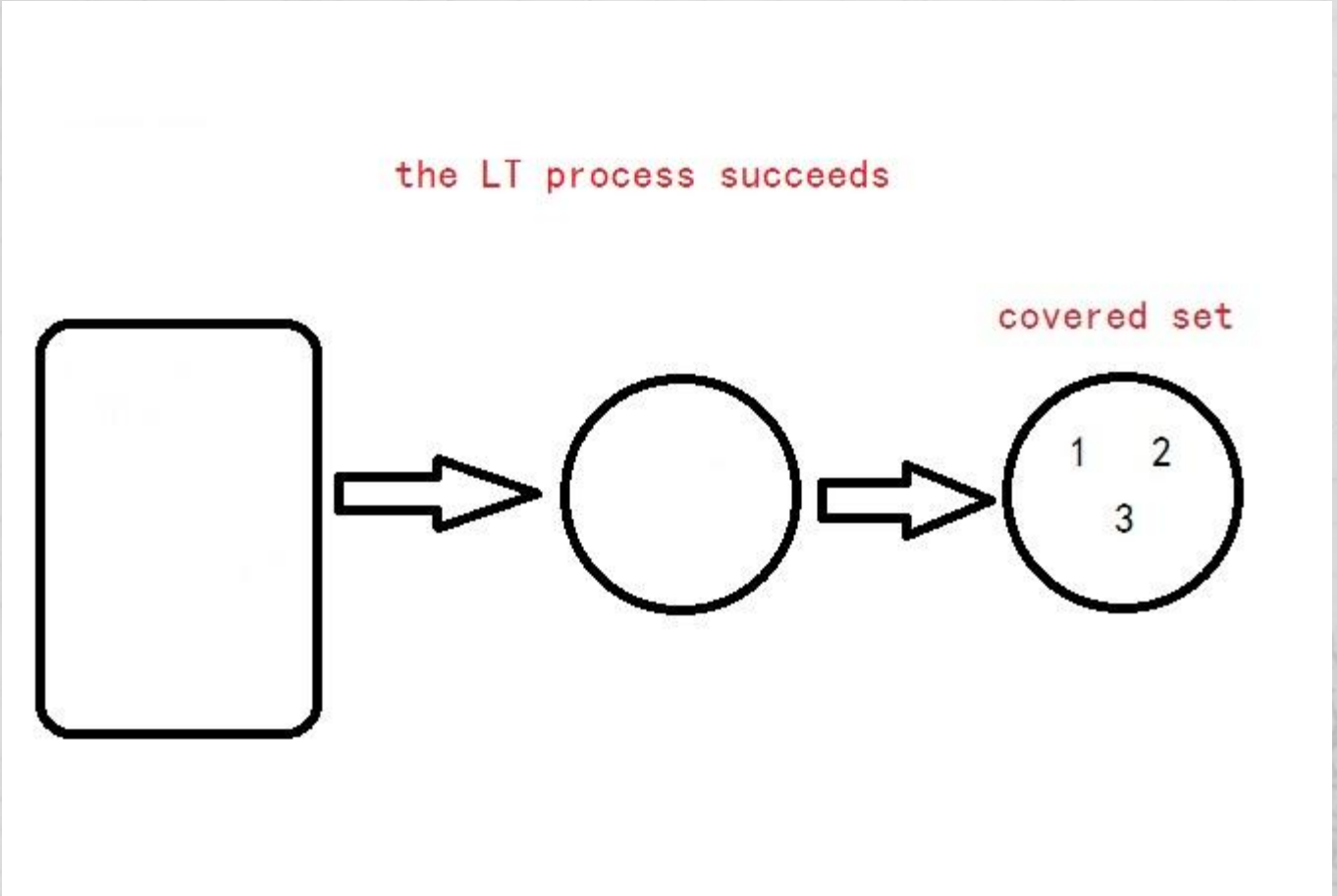


The LT process

- Small fraction of the input symbols in the ripple at the beginning
- The goal of the degree distribution design:
  - "Slowly" release encoding symbols
  - Keep the ripple small
  - Prevent redundant coverage
- Why is that?

The LT process

Consider the three situation with 4 input symbols

Situation 1:

{1} {1,2} {2,3} {3,4} good distribution

Situation 2:

{1} {1,2} {1,3} {1,4} bad distribution

Situation 3:

{1} {1,2} {1,2,3} {1,3,4}

seems good but bad actually

Proof

- The LT process
- The LT Distributions
- The Ideal Soliton Distribution
- The Robust Soliton Distribution

The LT distribution

Two design goals of the degree distribution

- As few encoding symbols as possible on average
- The average degree of the encoding symbols is as low as possible

The LT distribution

Degree Release Probability : q(i,L)

Overall Release Probability : r(L)

$$r(L) = \sum_i r(i,L) = \sum_i p(i) \cdot q(i,L)$$

Degree Release Probability

$$q(1,k) = 1$$

For i = 2,…,k, for all L=k-i+1,…,1

$$q(i,L) = \frac{i(i-1)L\prod_{j=0}^{i-3} k-(L+1)-j}{\prod_{j=0}^{i-1} k-j}$$

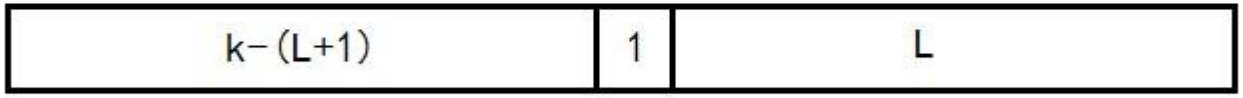
For all other i and L

$$q(i,L) = 0$$

Degree Release Probability

$$q(i,L) = \frac{i(i-1)L\prod_{j=0}^{i-3} k - (L+1) - j}{\prod_{j=0}^{i-1} k - j}$$

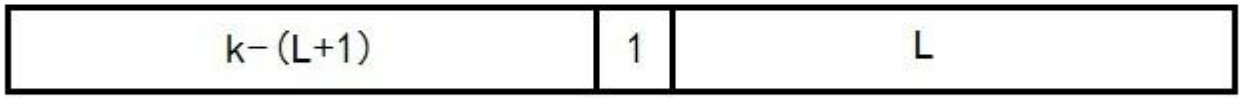
Degree Release Probability



k-(L+1) symbols are covered
k-Lth symbol is under processing
L symbols remain uncovered

Degree Release Probability

consider an encoding
symbol with degree i



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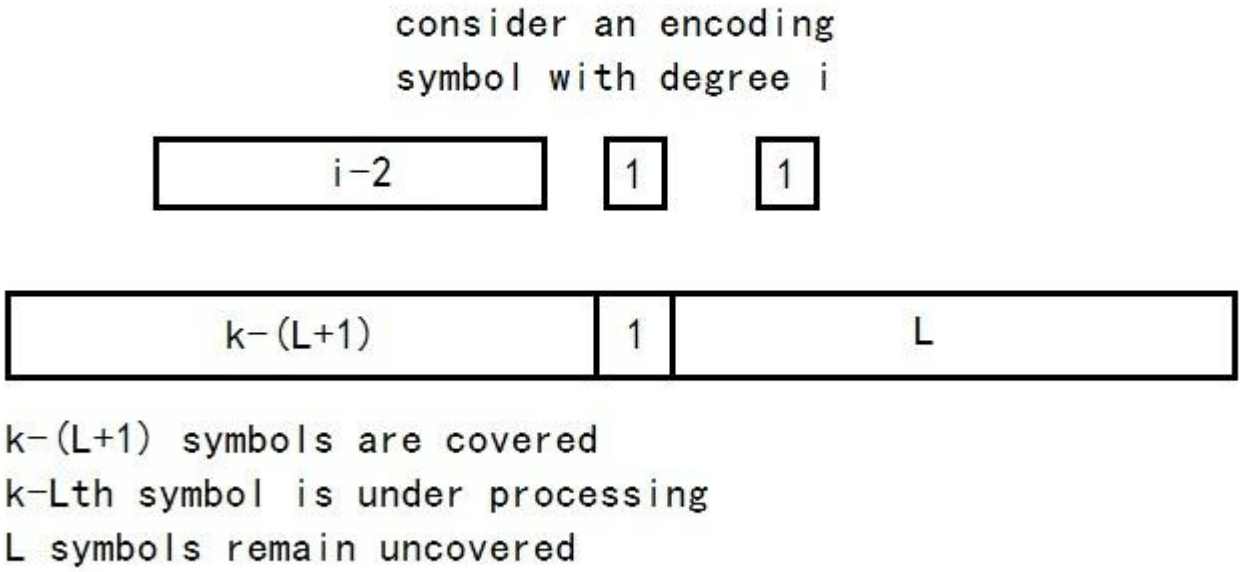
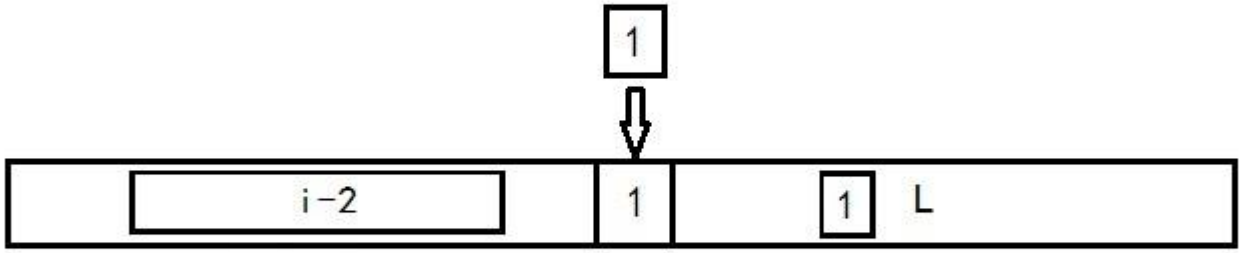
Degree Release Probability



Degree Release Probability

an encoding symbol with degree i is released if
i-2 symbols are covered
1 symbol is under processing
Another one is uncovered



k-(L+1) symbols are covered
k-Lth symbol is under processing
L symbols remain uncovered

Degree Release Probability

i-2 symbols are covered:

$$\binom{k-(L+1)}{i-2}\binom{L+1}{2}/\binom{k}{1}$$

1 symbol is covering, 1 remains uncovered:

$$\binom{L}{1}/\binom{L+1}{2}$$

Degree Release Probability

$$q(i,L) = \frac{\binom{k-(L+1)}{i-2}\binom{L+1}{2}}{\binom{k}{1}} * \frac{\binom{L}{1}}{\binom{L+1}{2}}$$

$$= \frac{i(i-1)L\prod_{j=0}^{i-3} k-(L+1)-j}{\prod_{j=0}^{i-1} k-j}$$

Degree Release Probability

An encoding symbol is chosen to be d=i and is released when L input symbols remain uncovered

$$r(i, L) = p(i) \cdot q(i, L)$$

Overall Release Probability

$$r(L) = \sum_i r(i, L) = \sum_i p(i) \cdot q(i, L)$$

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The Ideal Soliton Distribution

For i=1

$$\rho(1) = 1/k$$

For i=2,…,k,

$$\rho(i) = 1/i(i-1)$$

The Ideal Soliton Distribution

Theorem 1 (uniform release probability)

For the Ideal Soliton distribution,

$$r(L) = \frac{1}{k} \quad \text{for all } L = k, \ldots, 1$$

Proof of Theorem 1

For L = k

All encoding symbols with d=1 are released

$$r(L) = \sum_i r(i, L) = \sum_i p(i) \cdot q(i, L)$$

$$r(L) = p(1) \cdot q(1, L) = 1 \cdot \frac{1}{k}$$

Proof of Theorem 1

For L = k-1,…,1

$$r(i,L) = p(i) \cdot q(i,L)$$

$$r(i,L) = \frac{1}{i(i-1)} \cdot \frac{i(i-1)L\prod_{j=0}^{i-3} k-(L+1)-j}{\prod_{j=0}^{i-1} k-j}$$

$$r(i,L) = \frac{L\prod_{j=0}^{i-3} k-(L+1)-j}{\prod_{j=0}^{i-1} k-j}$$

Proof of Theorem 1

For L = k-1,…,1

$$r(i,L) = \frac{L\prod_{j=0}^{i-3} k - (L+1) - j}{\prod_{j=0}^{i-1} k - j}$$

$$k \cdot r(i,L) = \frac{L\prod_{j=0}^{i-3} k - (L+1) - j}{\prod_{j=1}^{i-1} k - j}$$

Proof of Theorem 1

Rewrite the formula

$$k \cdot r(i, L) = \frac{L \prod_{j=0}^{i-3}(k-1) - L - j}{\prod_{j=0}^{i-2}(k-1) - j}$$

$$= \frac{(k-1) - L}{k-1} \cdots \frac{(k-1) - (i-3) - L}{(k-1) - (i-3)} \cdot \frac{L}{(k-1) - (i-2)}$$

Proof of Theorem 1

Consider:

Throw balls uniformly among k-1 bins

Eliminating each bin once occupied

i-1th ball first comes into one of L designated bins

$$= \frac{(k-1)-L}{k-1} \cdots \frac{(k-1)-(i-3)-L}{(k-1)-(i-3)} \cdot \frac{L}{(k-1)-(i-2)}$$

Proof of Theorem 1

$$k \cdot r(L) = k \cdot \sum_{i=1}^{k-L+1} r(i, L) = 1$$

Hence,

$$r(L) = \frac{1}{k}$$

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The Robust Soliton Distribution

Let $R = c \cdot \ln(k/\delta)\sqrt{k}$ for suitable c>0

Define:

$$\tau(i) = \begin{cases} \frac{R}{ik} & \text{for } i = 1, \ldots, \frac{k}{R} - 1 \\ \frac{R \ln\left(\frac{R}{\delta}\right)}{k} & \text{for } i = k/R \\ 0 & \text{for } i = \frac{k}{R}, \ldots, k \end{cases}$$

The Robust Soliton Distribution

Add the ideal Soliton distribution and normalize to get the robust Solition distribution:

$$\beta = \sum_{i=1}^{k} \rho(i) + \tau(i)$$

For all i = 1,…,K

$$\mu(i) = (\rho(i) + \tau(i))/\beta$$

The Robust Soliton Distribution

Idea:

Keep the ripple size = R instead of 1

Using $K=k\beta$ encoding symbols instead of k

The Robust Soliton Distribution

How we design this Robust Soliton Distribution?

A released encoding symbol adds to the ripple $\frac{L}{(L-R)}$

Requires $\frac{(L-R)}{L}$ released symbols to add one to the ripple

The Robust Soliton Distribution

Density of encoding symbols with degree i=k/L should be proportional to

$$\frac{L}{i(i-1)(L-R)} = \frac{k}{i(i-1)(k-iR)}$$

$$= \frac{1}{i(i-1)} + \frac{R}{(i-1)(k-iR)} \approx \rho(i) + \tau(i)$$

The Robust Soliton Distribution

Theorem 2 (number of encoding symbols)

$$K = k + O(\sqrt{k} \cdot \ln^2(k/\delta))$$

The Robust Soliton Distribution

Theorem 3 (average degree of an encoding symbol)

$$D = O(\ln (k/\delta))$$

The Robust Soliton Distribution

Euler-Mascherorri Constant

$$\gamma = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} - \ln(n)$$

$$\gamma = 0.577\ldots$$

The Robust Soliton Distribution

Theorem 4 (high probability recovery)

The decoder fails to recover the data with probability at most δ from a set of K encoding symbols

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  - Code Structure
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Rateless Codes for AWGN Channel

- Channel Model

$$Y_m^N = \beta X_m^N + Z_m^N, \quad m = 1,2, \ldots,$$

β channel gain

σ^2 noise variance

β and σ^2 are known a priori at the Dec

P power limit

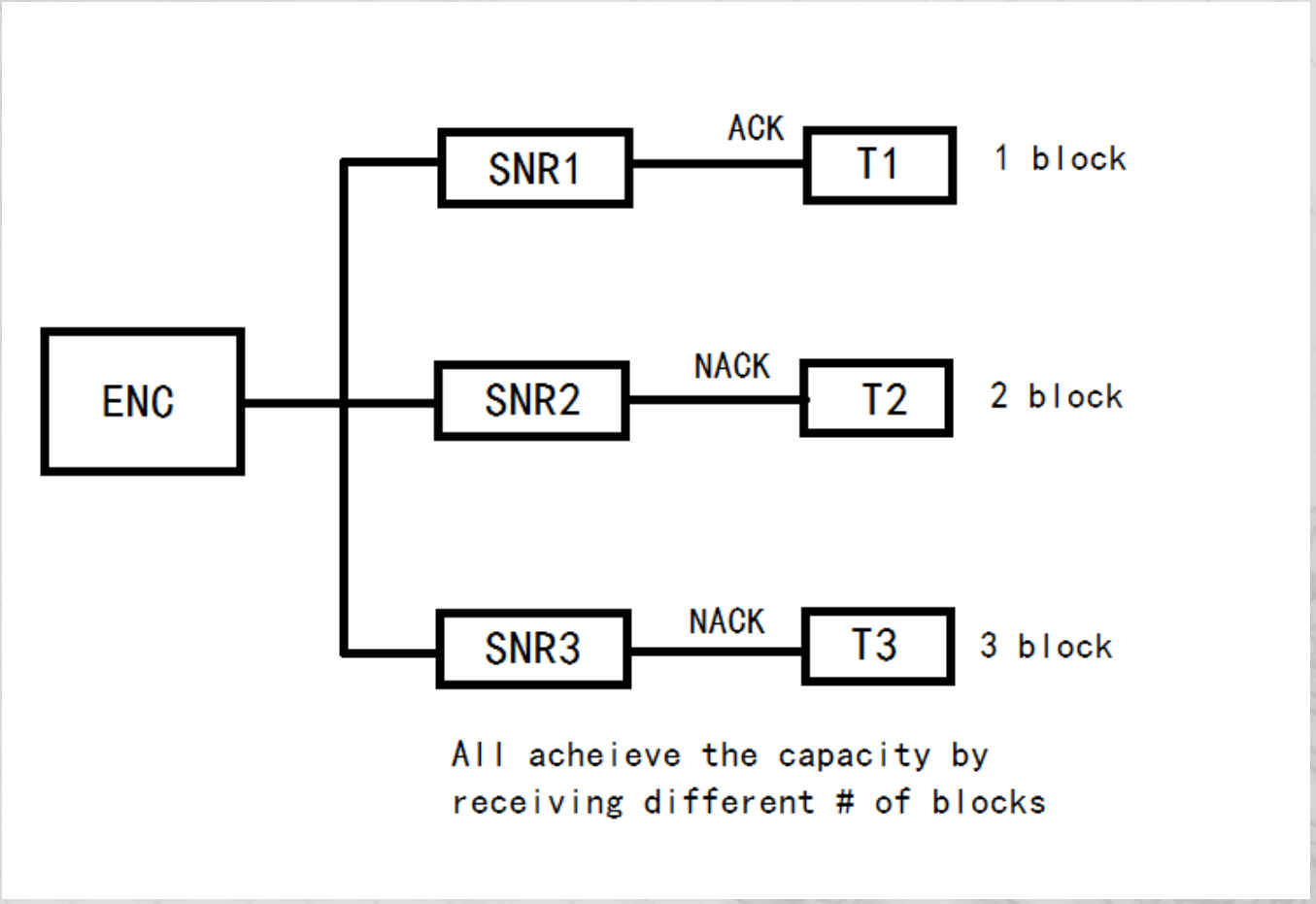
a sequence of code blocks $x_1(w), x_2(w), \ldots,$

receiver get sufficient blocks of y to recover w

receiver always starts receiving from m=1

Rateless Codes for AWGN Channel

- System Model



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Rateless Codes for AWGN Channel

- Layering
- Repetition
- Time-varying Weighting

Rateless Codes for AWGN Channel

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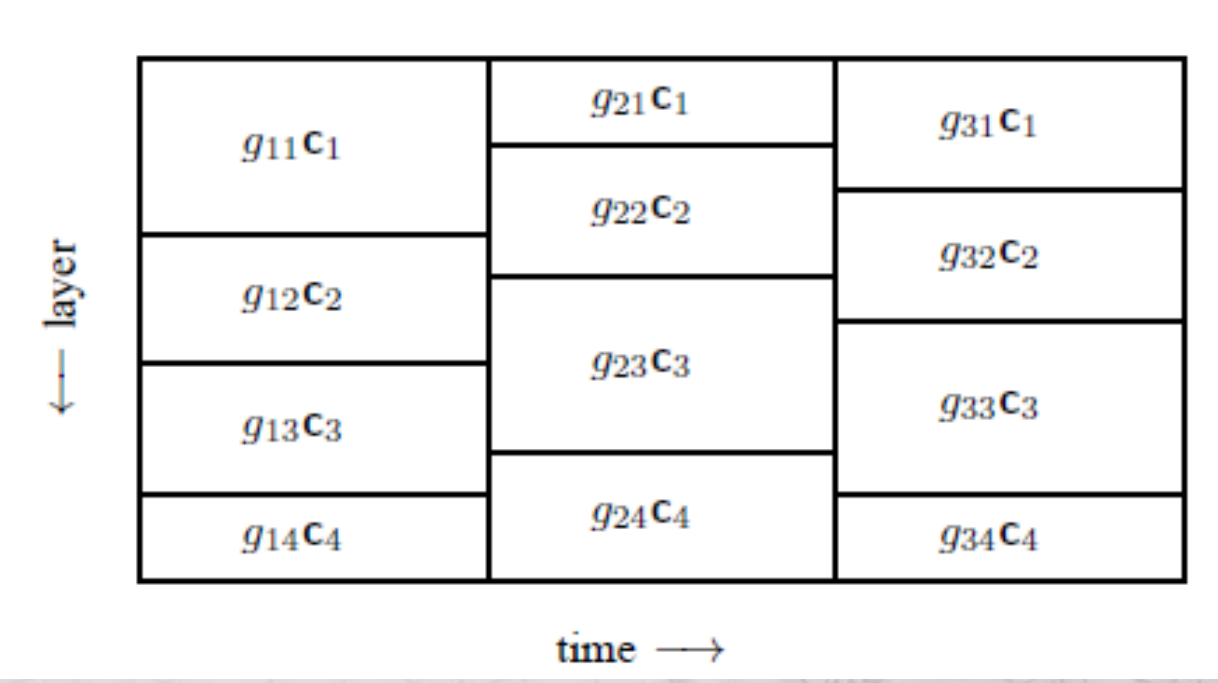
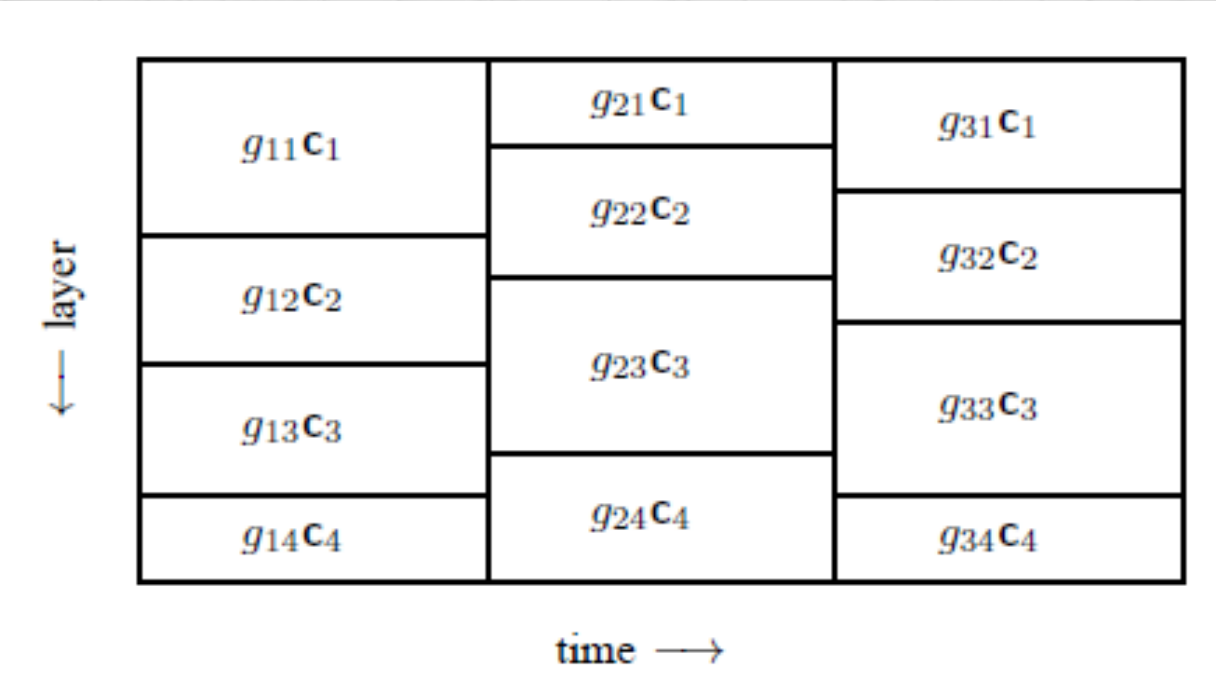


Practical Rateless Codes for AWGN

$$x = g_{11}c_1 + g_{12}c_2 + g_{13}c_3 + g_{14}c_4$$

$$|g_{11}|^2 + |g_{12}|^2 + |g_{13}|^2 + |g_{14}|^2 = P$$



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Perfect Rateless Codes

- Use a 2-layers design as an example

  two channel gain thresholds

  $\beta \geq \alpha_1$

  decoding requires one block

  $\alpha_1 \geq \beta \geq \alpha_2$

  decoding requires two blocks

  $\alpha_2 > \beta$

  decoding is not possible

Perfect Rateless Codes

One-block decoding

$$Y_1^N = \alpha_1 X_1^N + Z_1^N$$

Two-block decoding

$$Y_1^N = \alpha_2 X_1^N + Z_1^N$$

$$Y_2^N = \alpha_2 X_2^N + Z_2^N$$

Perfect Rateless Codes

SNR thresholds

$$\mathrm{SNR}_1 = \mathrm{P}\alpha_1^2/\delta^2$$

$$\mathrm{SNR}_2 = \mathrm{P}\alpha_2^2/\delta^2$$

Channel capacity

$$\mathrm{I}_1 = \log\left(1 + \mathrm{SNR}_1\right)$$

$$\mathrm{I}_2 = 2\log\left(1 + \mathrm{SNR}_2\right)$$

Perfect Rateless Codes

The smallest of channel gain we can achieve occur when

$$I_1 = I_2 = R$$

Reference

[1] Uriz Erez, Mitchell D. Trott, Gregory W. Wornell “Rateless Coding for Gaussian Channels” IEEE Trans. Inform. Theory, vol. 58, pp. 530-547, Feb. 2012.

[2] M.G. Luby, “LT Codes,” The 43rd Annual IEEE Symposium on Foundations of Computer Science, 2002.