Coding for Computing

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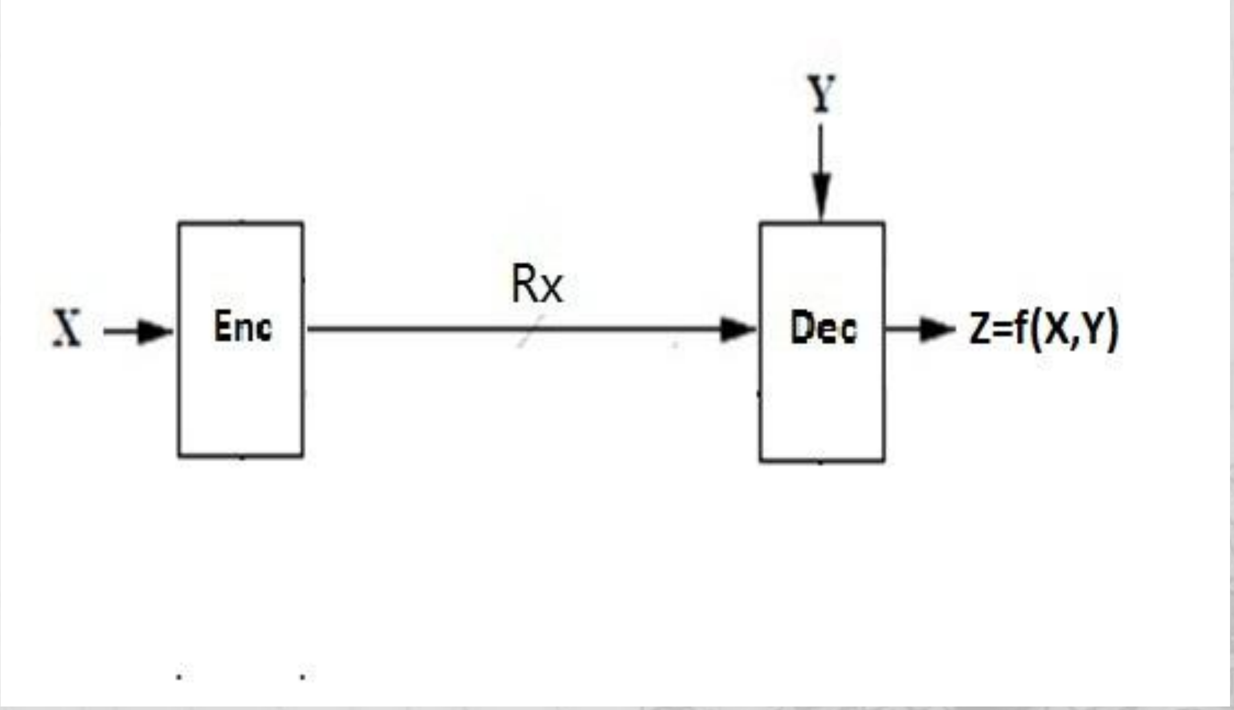
Outline

- Background and definitions
- Main Results
- Examples and Analysis
- Proofs

Background and definitions

- Problem Setup
- Graph Entropy
- Conditional Graph Entropy
- Characteristic Graph
- Wyner-Ziv

Problem Setup



Problem Setup

- Encode by block

  Transmits after observing the whole block of X

- Decode by block

  f(X,Y) must be determined for a block of (X,Y)

- Vanishing block-error probability

$$p(Z^N \neq f(X^N, Y^N)) \leq \varepsilon$$

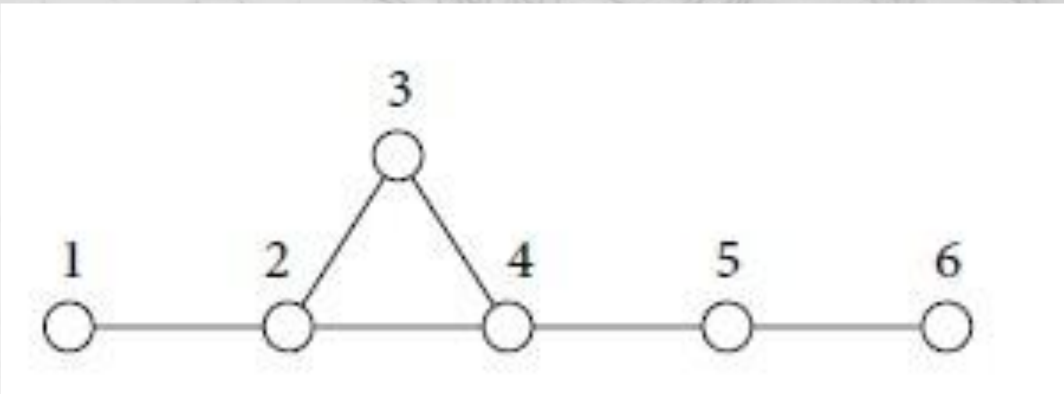
- Problem's rate

$$L_f(X|Y) = R/N$$

Graph Entropy

Maximum independent sets of G(V,E)

$$\Gamma(G) = \{\{1,3,5\},\{1,3,6\},\{1,4,6\},\{2,5\},\{2,6\}\}$$

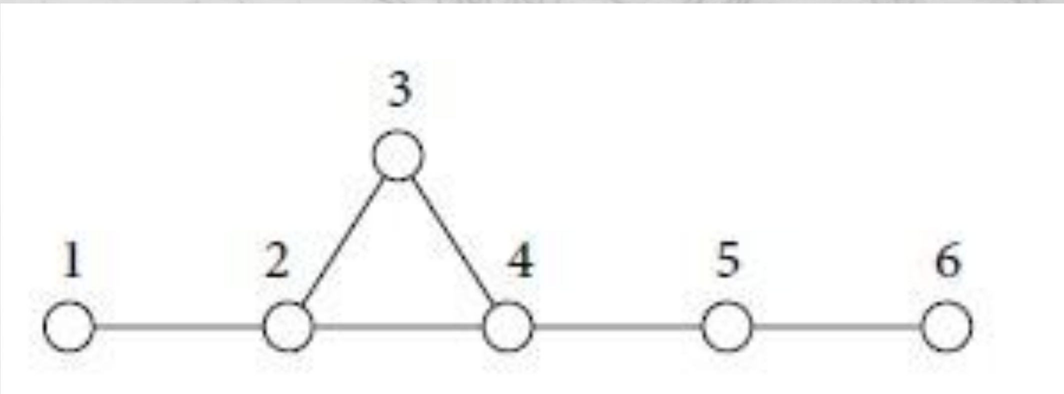


Graph Entropy

Define random viable W, $W \in \Gamma(G)$

$$p(w|x) = 0 \text{ if } x \text{ is not in } w$$

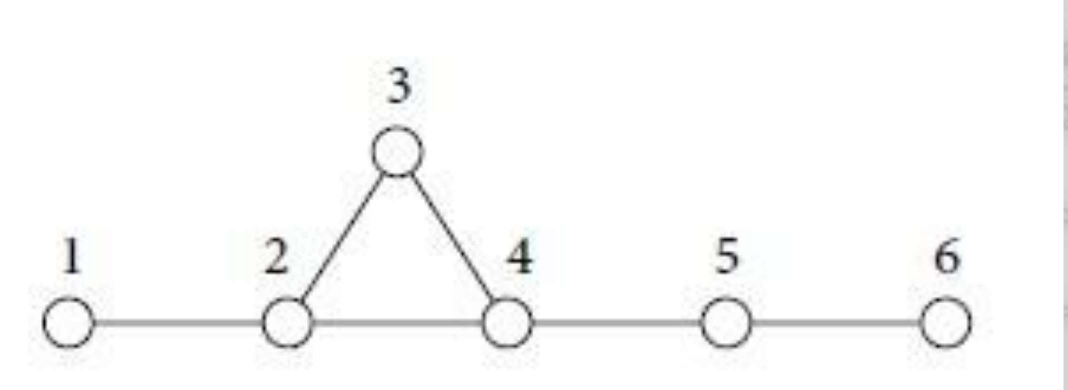
$$\sum_{w : x \in w} p(w|x) = 1$$



Graph Entropy

Graph Entropy:

$$H_G(X) = \min_{p(w|x)} I(X; W)$$



Conditional Graph Entropy

Extend the definition of graph entropy:

Let (X,Y) be a random pair and let a graph G be defined over the support set of X

$$H_G(X|Y) = \min_{\substack{W-X-Y \\ X \in W \in \Gamma(G)}} I(W;X|Y)$$

Characteristic Graph

- G(V,E) and f
- The vertex set is the support set of X
- The edge (x,x') exists if there is a y such that

$$p(x,y) > 0 \quad p(x',y) > 0$$

$$f(x,y) \neq f(x',y)$$

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Main Results

- Naive bound
- Theorem 1
- Theorem 2
- Theorem 3

Main Results(Naive bounds)

Inner bound

$$L_f(X|Y) \geq H(f(X,Y)|Y)$$

Outer bound

$$L_f(X|Y) \leq \min \{H(g(X)|Y) : g(X) \text{ and } Y \text{ determine } f\}$$

$$H(f(X,Y)|g(X),Y) = 0$$

Main Results(Naive bounds)

- lower bound: number of bits required when Px knows Y in advance
- upper bound : using slepian-wolf theorem
- Both bounds are tight in some special cases but not in general

Main Results(Theorem1)

Theorem 1 for every X, Y and f

$$L_f(X|Y) = H_G(X|Y)$$

Zero Error Worst Length Case

One-Way Complexity

$$\hat{C}_1(X|Y) = \lceil \log \omega(G(X|Y)) \rceil$$

The one-way complexity is the chromatic number of the characteristic hypergraph of (X,Y)

Main Results(Theorem2)

- An extention of Wyner-Ziv
- Allowing the distortion to depend on Y as well as on X and Z

$$R_D = \min_{\substack{p(u|x),g|E[d(x,y,g(u,y))] \leq D \\ U \leftrightarrow X \leftrightarrow Y}} I(X;U|Y)$$

Main Results(Theorem2)

- Applicant in our problem

$$d(x,y,z) = \begin{cases} 0, & \text{if } z = f(x,y) \\ 1, & \text{otherwise} \end{cases}$$

$$d(X,Y,Z) = p\left(Z \neq f(X,Y)\right)$$

$$d(X^N,Y^N,Z^N) = \frac{1}{N}\sum_{i=1}^{N} p(Z_i \neq f(X_i,Y_i))$$

Main Results(Theorem 2)

Lemma: For every X, Y and f

$$L_f(X|Y) \geq R(0)$$

$$p(Z^N \neq f(X^N, Y^N)) \geq \frac{1}{N}\sum_{i=1}^{N} p(Z_i \neq f(X_i, Y_i))$$

hence every rate achievable with vanishing block error is also achievable with zero distortion

Main Results(Theorem 2)

Theorem 2 For every X, Y and f

$$R(0) = H_G(X|Y)$$

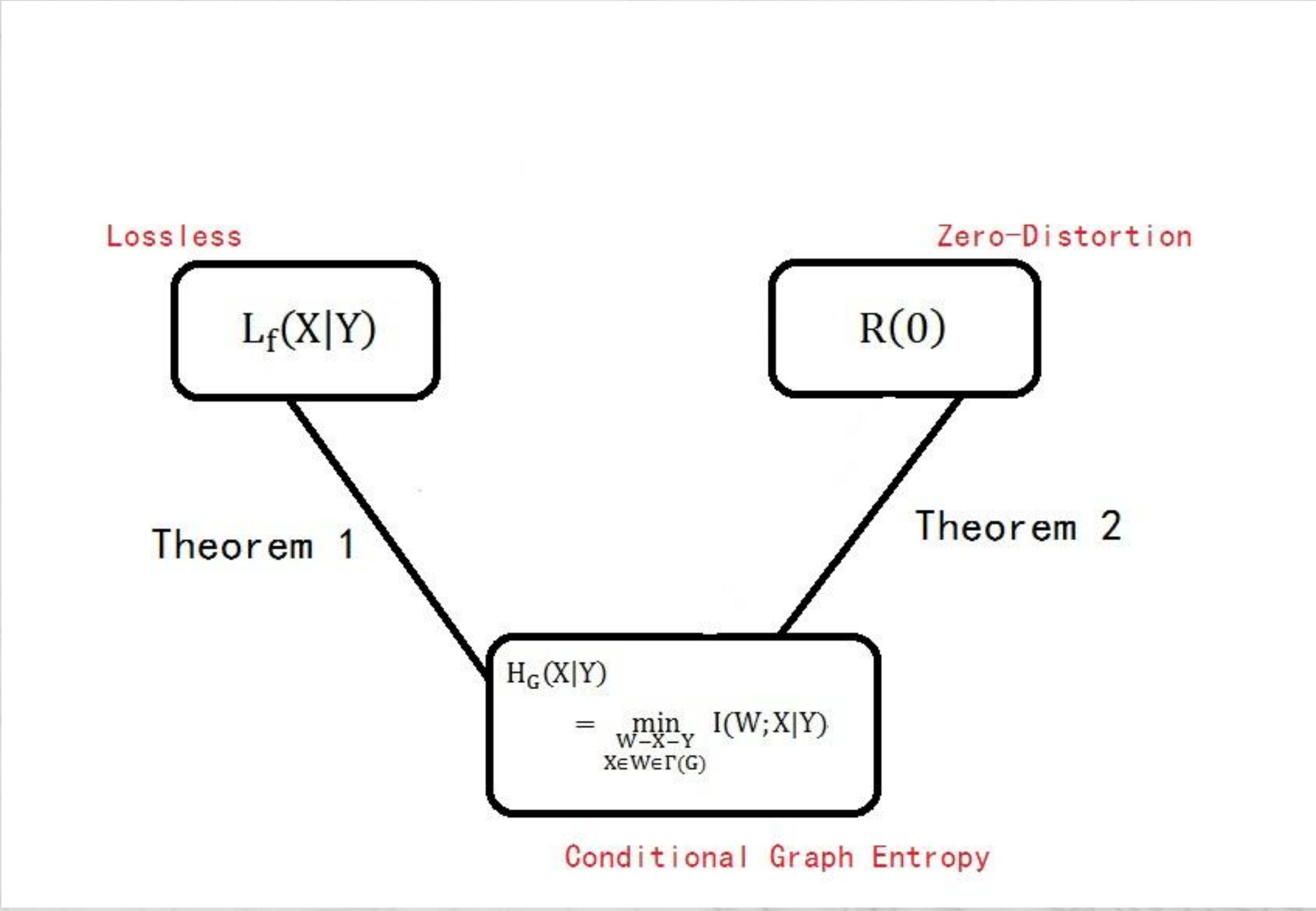
Main Results

Theorem 3 For every X, Y and f

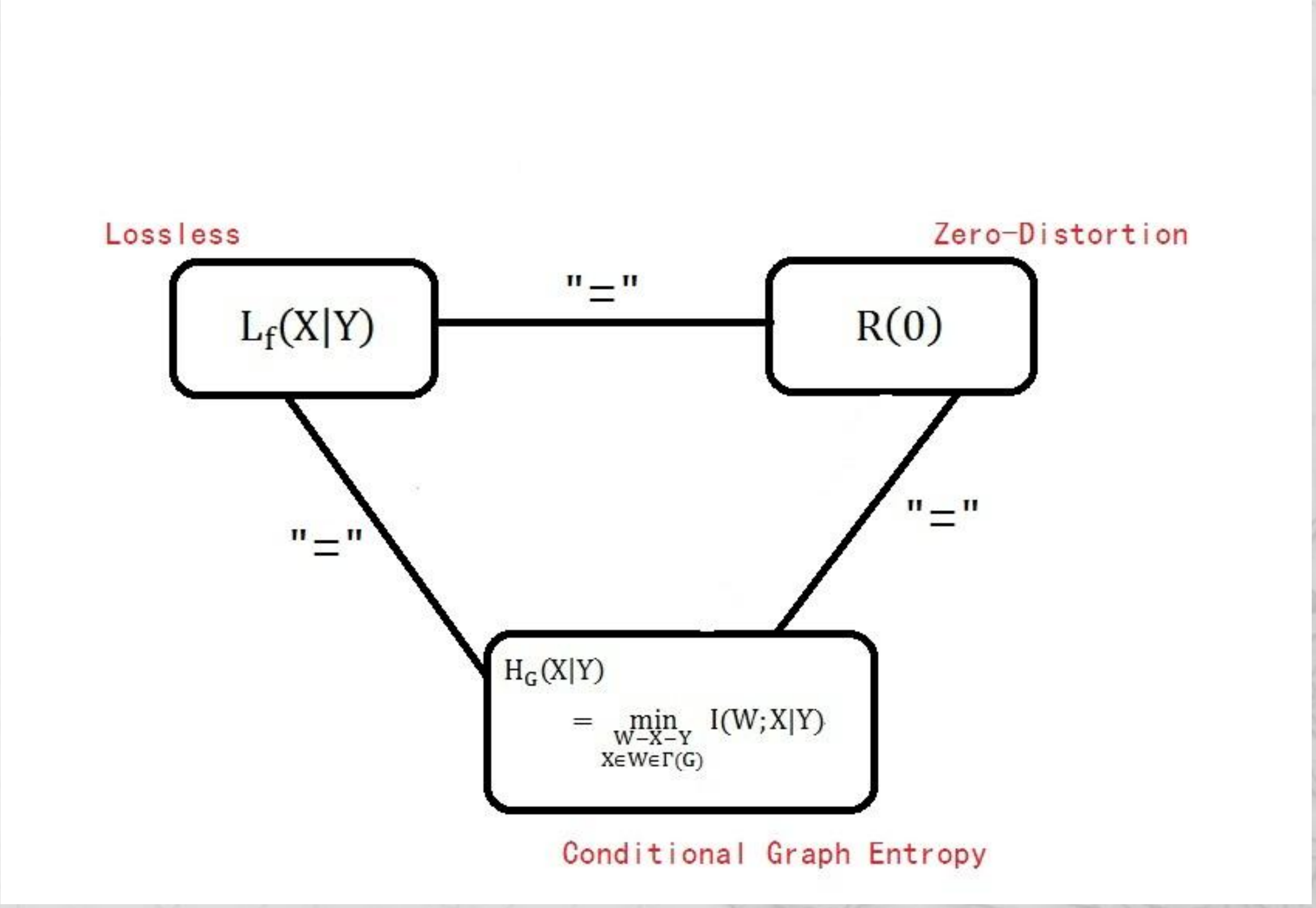
$$R_f^2(X|Y) = \{(r_x, r_y): r_x \geq I(V; X|UY) \text{and } r_y \geq I(U; Y|X)$$

$$\text{for some admissible U and V}\}$$

Proof of Theorem 1&2



Proof of Theorem 1&2



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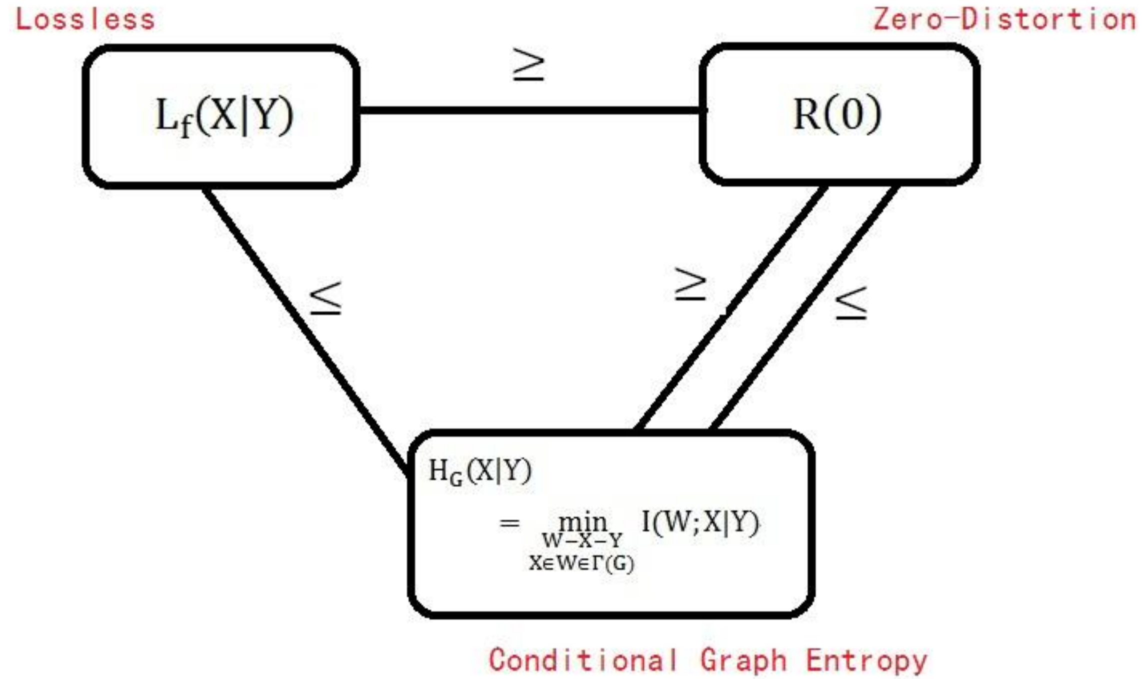

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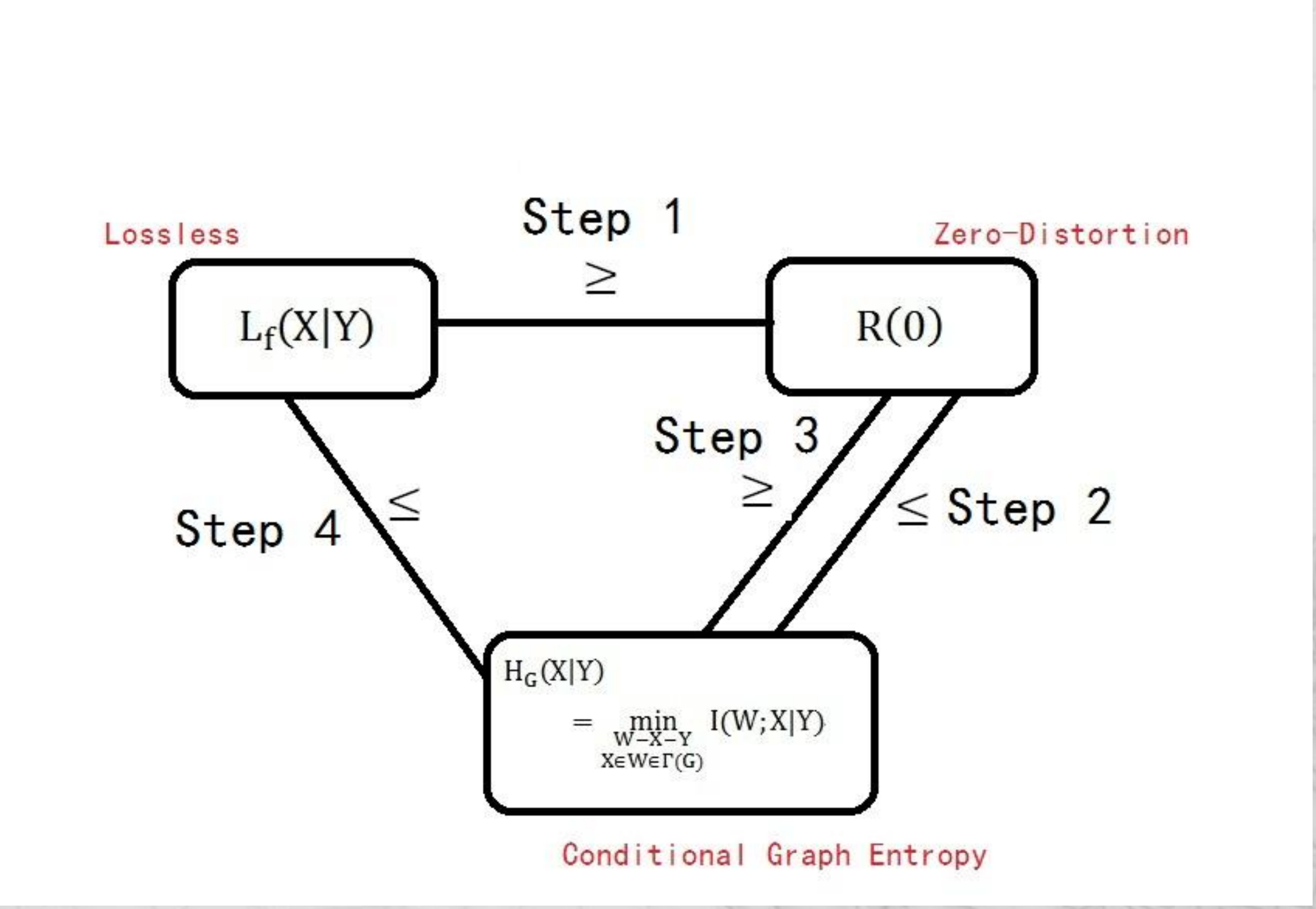

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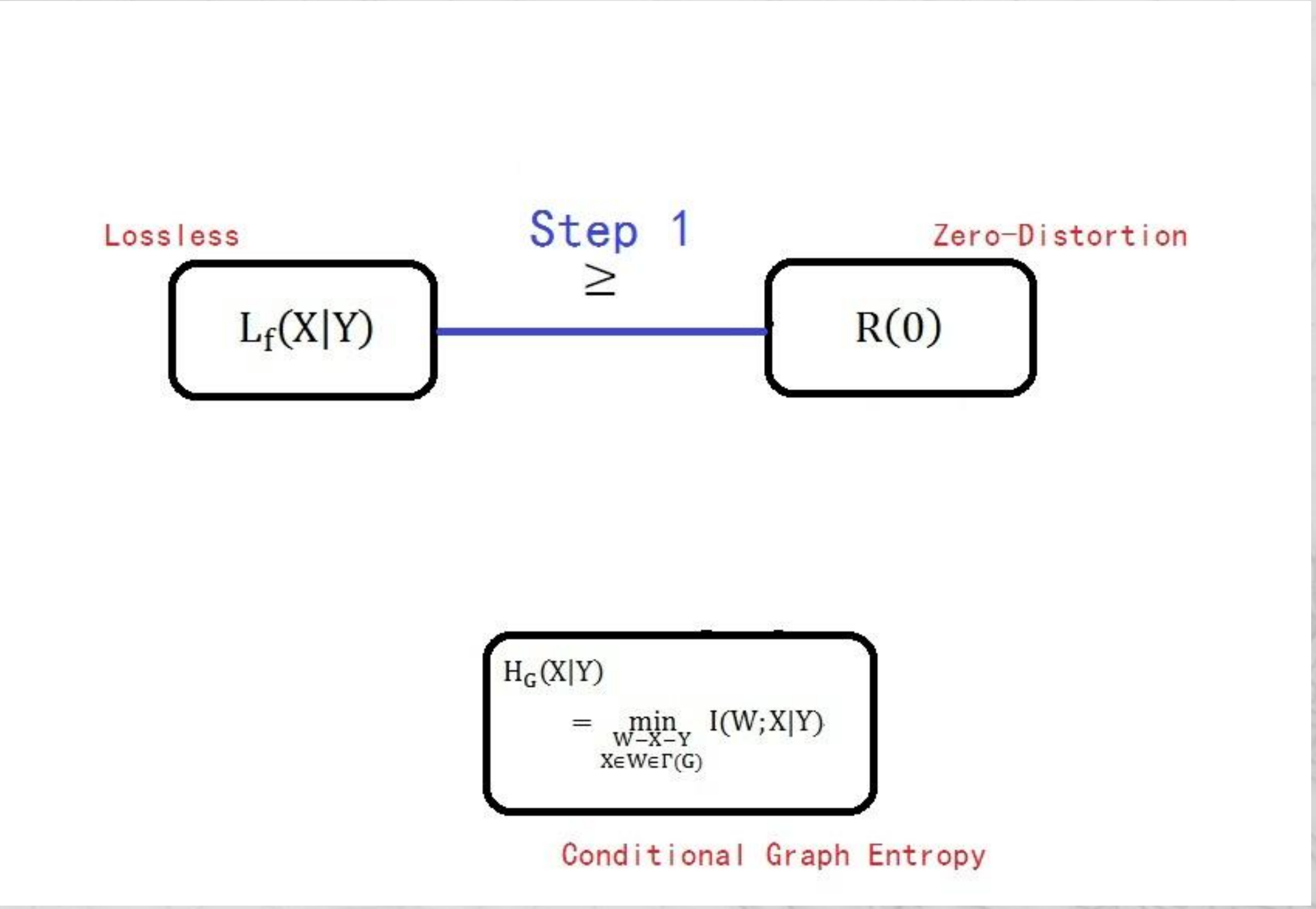


Step 1

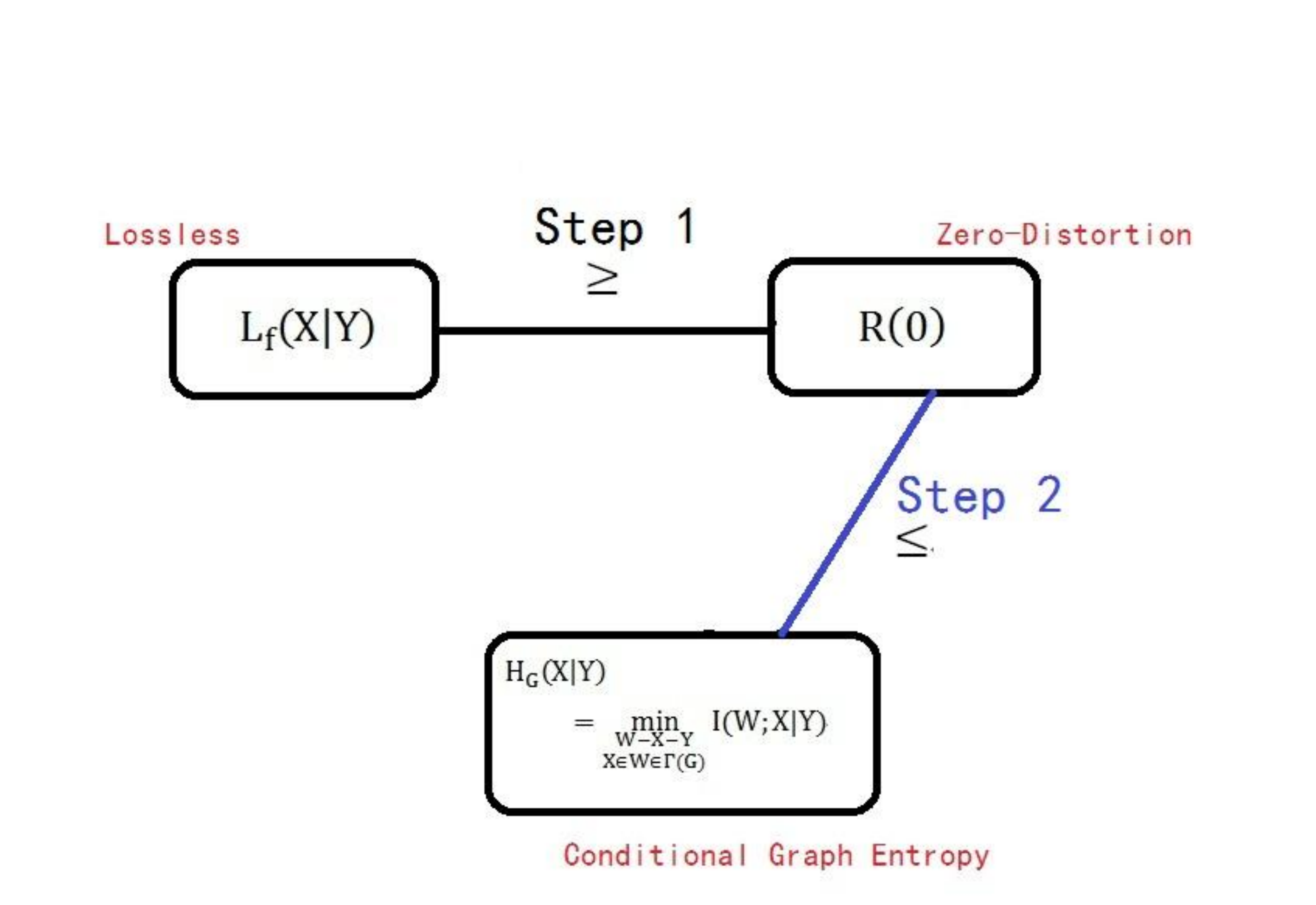
Lemma 1: For every X, Y and f

$$L_f(X|Y) \geq R(0)$$

$$p(Z^N \neq f(X^N, Y^N)) \geq \frac{1}{N}\sum_{i=1}^{N} p(Z_i \neq f(X_i, Y_i))$$

hence every rate achievable with vanishing block error is also achievable with zero distortion

Proof of Theorem 1&2



Step 2

To show

$$\min_{\substack{V-X-Y \\ g|Ed(X,Y,g(V,Y))\leq 0}} I(V;X|Y) \leq \min_{\substack{W-X-Y \\ X\in W\in\Gamma(G)}} I(W;X|Y)$$

Need to show that if $X \in W \in \Gamma(G)$

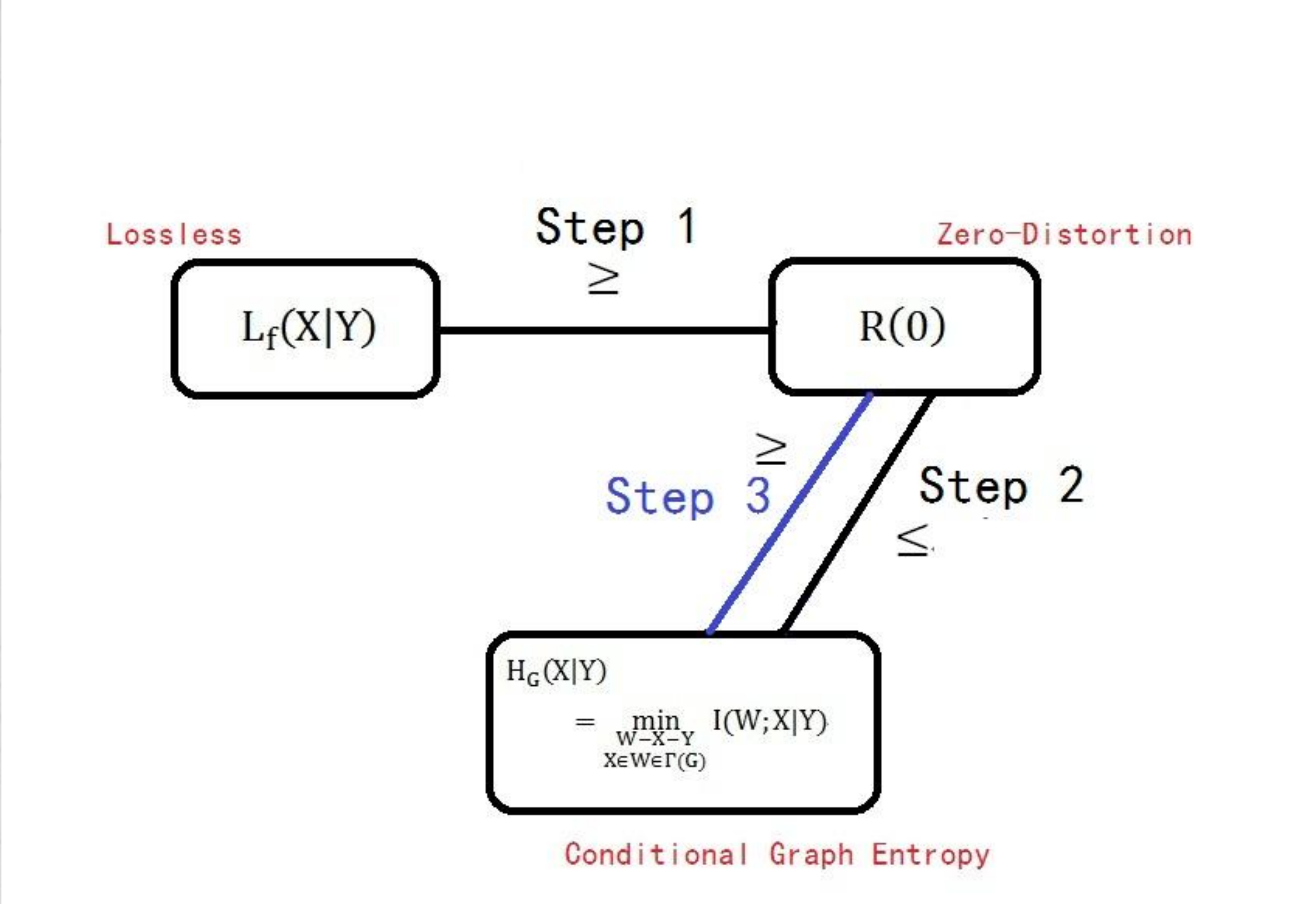
Then, there is a function g over $\Gamma(G) \times Y$ s.t.

$f(x,y) = g(w,y)$ whenever $p(w,x,y) > 0$

Hence

$$E\left[d\left(X,Y,g(W,Y)\right)\right] = 0$$

Proof of Theorem 1&2



Step 3

To show

$$\min_{\substack{V-X-Y \\ g|Ed(X,Y,g(V,Y))\leq 0}} I(V;X|Y) \geq \min_{\substack{W-X-Y \\ X\in W\in \Gamma(G)}} I(W;X|Y)$$

Suppose that V-X-Y and that there exists g such that

$Ed(X,Y,g(V,Y)) \leq 0$ We define W and prove:

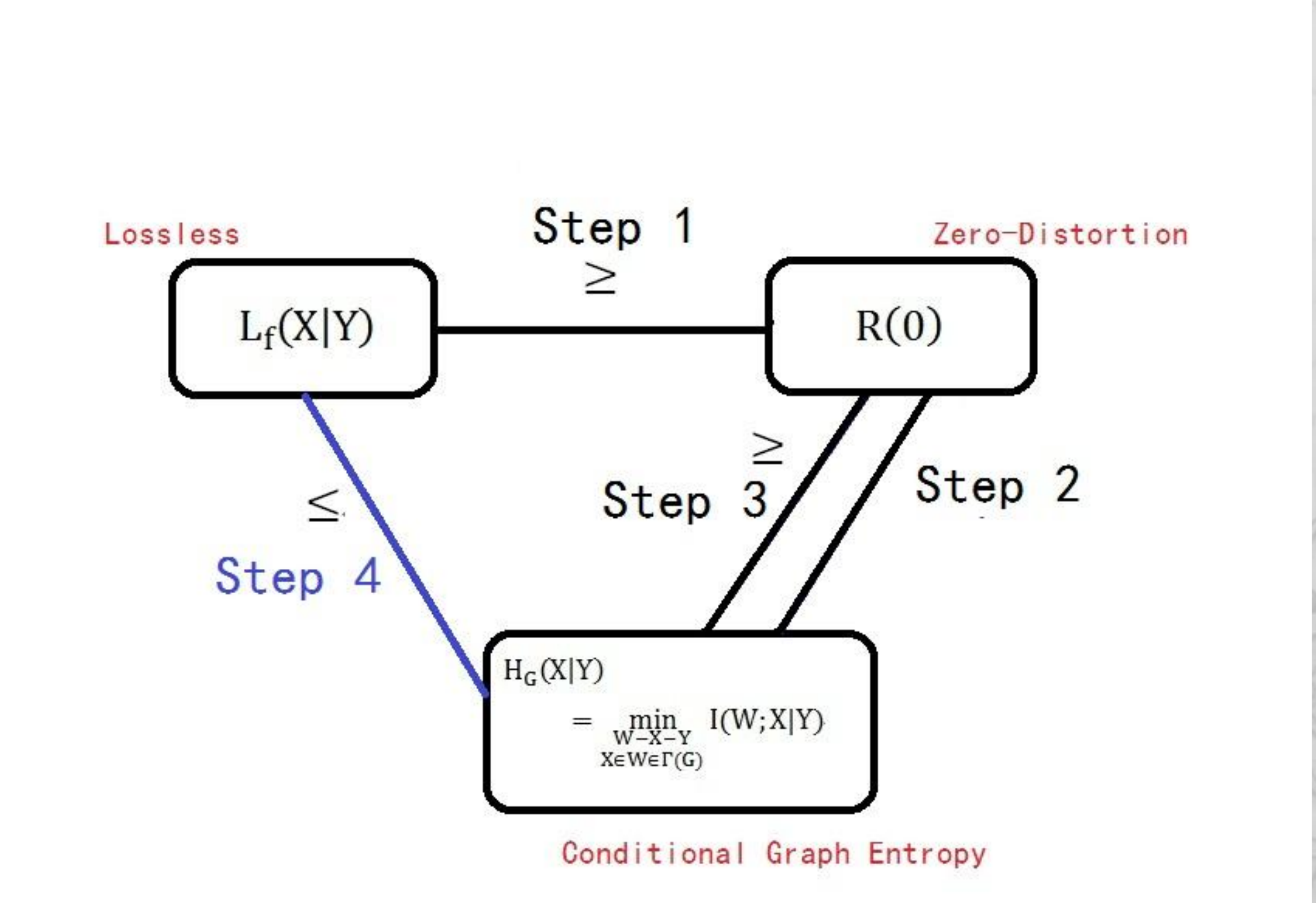
1. $X \in W \in \Gamma(G)$
2. $W - X - Y$
3. $I(W;X|Y) \leq I(V;X|Y)$

Outline

- Finish the proof of theorem 1 and 2

- Analysis 2-Way Communication Model
  - Definitions
  - Results(Theorem 3)
  - Proof

Proof of Theorem 1&2



Step 4

Introduce Robust Typical Set

$$T_\epsilon^N(X) = \{\vec{x} \in X^N | \left|\frac{1}{N} N(x|\vec{x}) - p_X(x)\right| < \frac{\varepsilon}{|X|} p_X(x)\}$$

Compare with Strong Typical Set

$$T_\epsilon^N(X) = \{\vec{x} \in X^N | \left|\frac{1}{N} N(x|\vec{x}) - p_X(x)\right| < \frac{\varepsilon}{|X|}\}$$

Step 4

High-level idea

Design a protocol based on conditional graph entropy

Define

$$J^* = \{j: (W^j, X) \in T_2\}$$

$$K^* = \{k: \left(W^k, Y\right) \in T_3 \text{ and } \Phi(k) = \Phi(J)\}$$

If $|K^*|=1$ then we can decode the message

Step 4

Encoder:

If J* is empty, transmits an error message. Otherwise, transmits $\Phi(J)$

Decoder:

If |K*|≠1, declares an error. Otherwise, proceeds to determine

$$g\left(W_i^K, Y_i\right) \text{ for all } i$$

Step 4

Error Analysis:

E1: $J^* = \emptyset$

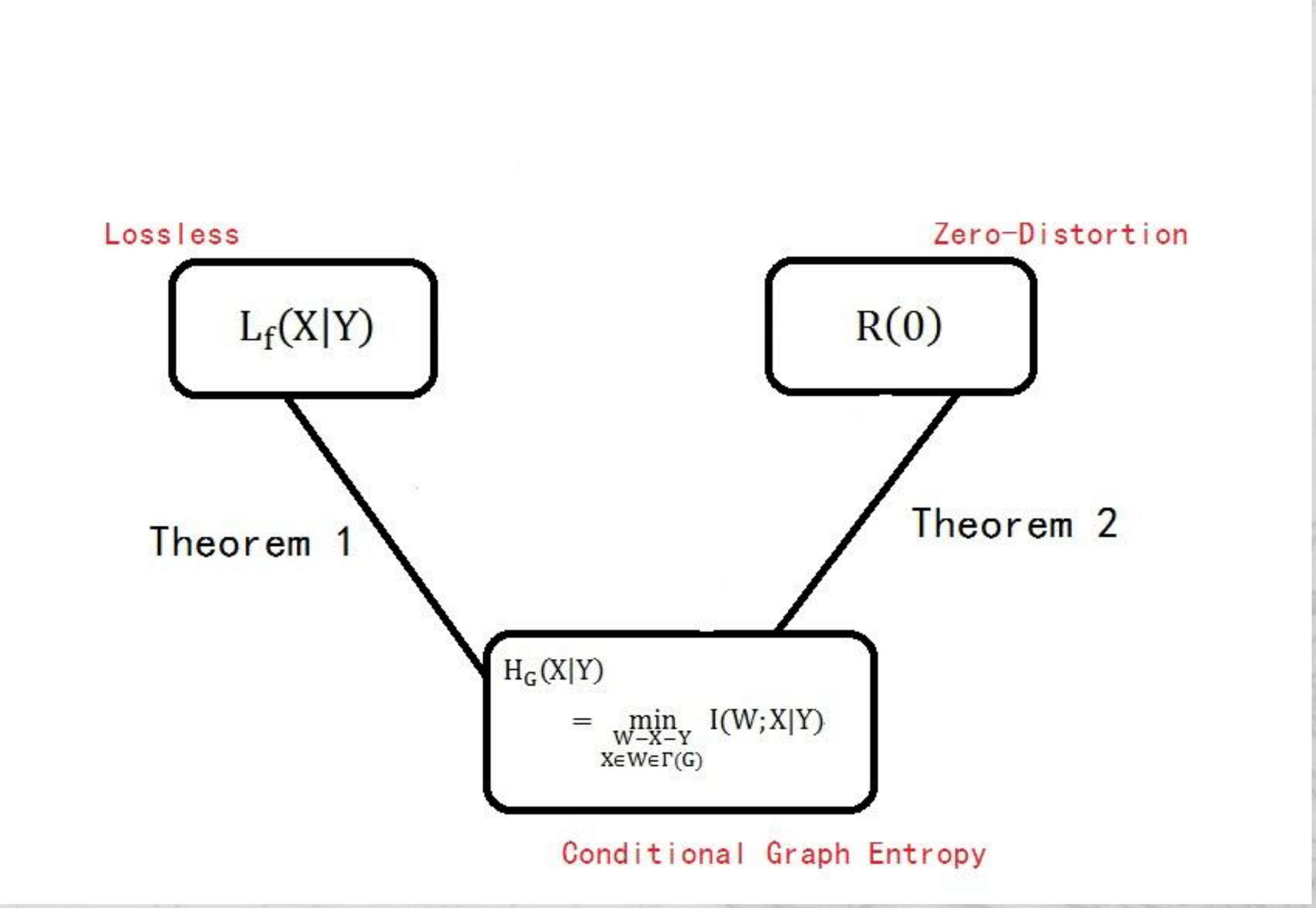
E2: $J^* \neq \emptyset$ and $(W^J, Y)$is not in T_3

E3: $J^* \neq \emptyset$ and $\exists$ k $\neq$ J suth that $(W^K, Y) \in T_3$

and, $\Phi(k) = \Phi(J)$

Need to show these errors are exponentially small

Proof of Theorem 1&2



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Definitions

Independent instances of a random pair (X,Y)

$$\{(X_i, Y_i)\}_{i=1}^{\infty}$$

A two-message protocol consists of:

Y-encoding function $\varphi : Y^n \to \{0,1\}^l$

X-encoding function $\varepsilon : \{0,1\}^l \times X^n \to \{0,1\}^k$

Decoding function $\Psi : \{0,1\}^l \times \{0,1\}^k \times X^n \to Z^n$

Definitions

Block-error probability

$$p[\Psi(\varphi(Y^n), \varepsilon(\varphi(Y^n), X^n), Y^n) \neq f(X^n, Y^n)]$$

Where

$$f(X^n, Y^n) = f(X_1, Y_1), f(X_2, Y_2), \ldots, f(X_n, Y_n)$$

Definitions

Rate region

$$R_f^2(X|Y)$$

Two-message communication complexity

$$L_f^2(X|Y) = \min\{r_x + r_y : (r_x, r_y) \in R_f^2(X|Y)\}$$

Outline

- Finish the proof of theorem 1 and 2

- Analysis 2-Way Communication Model

  Definitions

  Results(Theorem 3)

  Proof

Theorem 3

Two random variables U and V defined over finite alphabets are admissible if

1) U – Y – X
2) V – UX – Y
3) U, V, and Y determine f(X,Y)

Theorem 3

Theorem 3 :

For every (X,Y) and f

$$R_f^2(X|Y) = \{r_x + r_y : r_x \geq I(V;X|UY)$$

$$\text{and } r_y \geq I(U;Y|X) \text{for some admissible U and V}\}$$

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Proof of Theorem 3

Converse:

1. Obtain an outer bound on S_3=(rx,ry,D). Show $S_3 \subseteq S_3^*$

Where S_3^* defined in terms of X, Y, T, U, V

2. Show S_3^* can be expressed in terms of only X, Y, U, V

3. Show S_2, the 2-D slice with D=0 is contained in the 2-D region S'_2