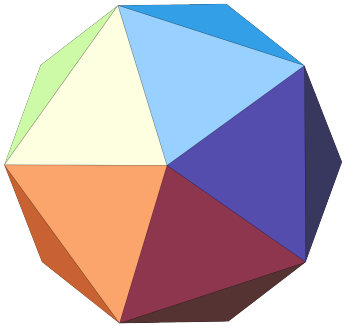
Exploiting Symmetry in Linear Programming*



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*Katrin Herr, R. Bödi, *Symmetries in linear and integer linear programming*, Oberwolfach Report 38/2010

Outline-Part I

- Background:
  - Linear Programs (LPs)
  - Symmetries of mathematical objects and group actions
- Symmetries of LPs and IPs
  - Symmetries preserving a linear cost function over a polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$
- Orbits, Fixed points and subspace (yes!) spanned by fixed points of $\mathcal{P}$ under action of $G \leq S_n$ preserving a linear cost function over $\mathcal{P}$

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Linear Programs

Canonical formulation:

$$max \;\; \mathbf{c}^t\mathbf{x}$$
$$s.t. \;\; \mathbf{Ax} \leq \mathbf{b}, \;\; x \in \mathbb{R}^n$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$

- $\mathbf{x} \in \mathbb{R}^n$ is *feasible* if it satisfies all constraints of the LP
- LP is *feasible* if it has atleast one feasible point
- Set of feasible solutions of an LP is $X := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} \leq \mathbf{b}\}$

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- Set of feasible solutions of an LP is $X := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} \leq \mathbf{b}\}$
- X is a *polyhedron*
- A *polyhedron* $\mathcal{P} \subseteq \mathbb{R}^n$ is intersection of finitely many affine halfspaces : $\mathcal{P} := \{x \in \mathbb{R}^n \mid \mathbf{Ax} \leq \mathbf{b}\}$

Linear Programs

- The feasible region of an LP is *convex*
- A *solution* of an LP is an element $\mathbf{x}^* \in \mathbb{R}^n$ that is feasible and maximizes the utility function

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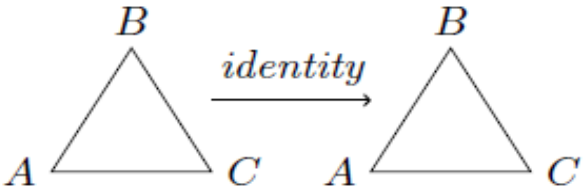
Permutations of a set

- Let $S = \{1, 2, 3\}$
- $\pi : S \to S$ given as: $\pi(1) = 2, \pi(2) = 1, \pi(3) = 3$.
- This is a bijective map.
- For any set S bijective map like $\pi : S \to S$ is called *permutation* of S.
- Such maps are also called *automorphisms*

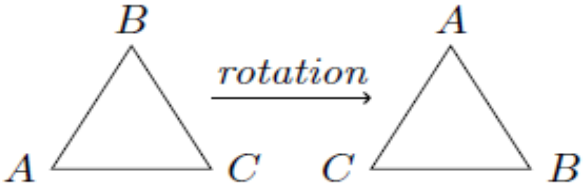
Permutations of $\mathbb{R}^n$

- Can be viewed as bijective maps $\pi : \mathbb{R}^n \to \mathbb{R}^n$
- We further restrict automorphisms of $\mathbb{R}^n$ to those that fix the origin i.e. $\pi(\mathbf{0}) = \mathbf{0}$ (less messy math!)
- We would always restrict the set of all permutations of an object to those that keep some property of the object invariant
- We could eg. further restrict automorphisms of $\mathbb{R}^n$ to isometries (permutations that preserve distance)

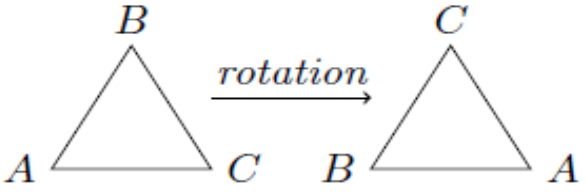
Permutations of vertices of equilateral triangle that preserve adjacency and distance



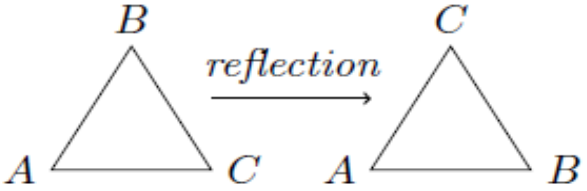
$$id = \begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix}$$



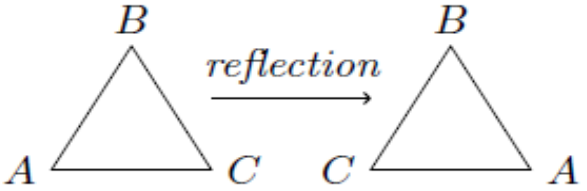
$$\rho_1 = \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix}$$



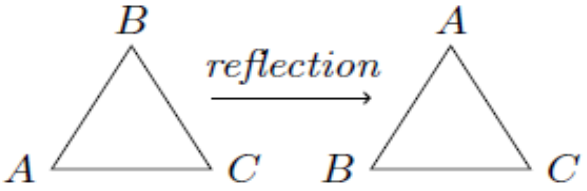
$$\rho_2 = \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}$$



$$\mu_1 = \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix}$$



$$\mu_2 = \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$$



$$\mu_3 = \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}$$

Figure Credits: Judson, Thomas W. Abstract Algebra: Theory and Applications. Boston, MA: PWS Pub., 1994. Print.

The Caylay Table for symmetries of equilateral triangle

$\circ$	id	ρ_1	ρ_2	μ_1	μ_2	μ_3
id	id	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	id	μ_3	μ_1	μ_2
ρ_2	ρ_2	id	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	id	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_2	id	ρ_1
μ_3	μ_3	μ_1	μ_2	ρ_1	ρ_2	id

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The Cayley Table for symmetries of equilateral triangle

We can *compose* two symmetries!

$\circ$	id	ρ_1	ρ_2	μ_1	μ_2	μ_3
id	id	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	id	μ_3	μ_1	μ_2
ρ_2	ρ_2	id	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	id	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_2	id	ρ_1
μ_3	μ_3	μ_1	μ_2	ρ_1	ρ_2	id

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Time to be more rigorous:

Groups and Group Actions

Group

- A **binary operation** or **law of composition** on set G is a function $G \times G \to G$ that assigns every pair $(a, b) \in G$ a unique element $a \circ b$, or ab in G is called composition of a and b.

- A **group** $(G, \circ)$ is a set G together with a law of composition $(a, b) \mapsto a \circ b$ satisfying axioms:
  1. operation is **associative** i.e.
  $\forall a, b, c \in G,\ (a \circ b) \circ c = a \circ (b \circ c)$
  2. $\exists$ **identity element** s.t. $\forall a \in G, a \circ e = e \circ a = a$
  3. $\exists$ **inverse element** a^{-1}, $\forall a \in G$ s.t. $a \circ a^{-1} = e$

Group

- **Order** of group G is $|G|$. Group is **finite** if its order is finite, else it is **infinite**
- If the composition operation is **commutative** the groups is said to be *abelian*

Examples of groups

- Set of integers under addition as operation is called $\mathbb{Z}$, Z forms a group under ordinary addition
- Set of integers *modulo* n is called $\mathbb{Z}_n$ and is also a group under *modulo* n addition
- Denote $\mathbb{M}_2$ to be set of all 2×2 matrices under matrix multiplication
- **General linear group:** $GL_2(\mathbb{R}) \subset \mathbb{M}_2$ and has all invertible 2×2 matrices under matrix multiplication.

Properties of Groups

- **Left and right cancellation laws:** $ba = ca \Rightarrow b = c$ and $ab = ac \Rightarrow b = c$
- Exponent notation(esp. for multiplication operator) for repetitive operations, inverse operation etc.
    1. $g^m g^n = g^{m+n}$
    2. $(g^m)^n = g^{mn}$
    3. $(gh)^n = (h^{-1}g^{-1})^{-n} \forall n \in \mathbb{Z}$
- For abelian G, $(gh)^n = g^n h^n$

Subgroups

- A **subgroup** H of G is is a subset closed under operation of group G
- **Trivial** if $H = \{e\}$
- **Proper** if $H \subsetneq G$
- Denoted as $H \leq G$ (general) and $H < G$(proper)

Permutation Group/Symmetry Group

- Group of permutations of elements of set X is called S_X
- When $X = \{1, 2, 3..., n\}$ it is denoted by S_n: **permutation group on n letters**
- The underlying set for S_n is the **set of ways of permuting** $\{1, 2, 3, ..., n\}$

Disjoint Cycle Notation

- We can refer to a permutation σ as
  $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$
- Or we can use the **disjoint cycle notation** as$(\ 4321)$
- (4321) $4 \mapsto 3, 3 \mapsto 2, 2 \mapsto 1, 1 \mapsto 4$
- σ above is described with with just one cycle.
- As a minimal representation, we need cycles to be disjoint
- $\therefore (4321)$ is trivially valid
- Some permutations may consist of more than one cycle e.g. $(12)(34)$

Transposition

- Example: $(135)(27)$ is minimal because cycles are disjoint but $(135)(347)$ reduces to (13475)
- **Transposition** is (not necessarily disjoint) cycle representation with each cycle having only 2 elements
- Conversion of a size n cycle:
  $(a_1, a_2, \ldots, a_n) = (a_1 a_n)(a_1 a_{n-1}) \ldots (a_1 a_3)(a_1 a_2)$
  e.g. $(16)(253) = (16)(23)(35)$
- **Lemma:** Number of cycles in transposition representation of id is even
- A permutation is **even** if it can be represented as even number of transpositions otherwise it is **odd**
- Set of all even permuatations A_n is a subgroup of S_n and its order is $n!/2$ = number of odd permutations

Group Actions

- **A left action** of G on X is a map $G \times X \mapsto X$ given by $(g, x) \mapsto gx$ where,

  1. $ex = x \ \ \forall x \in X$
  2. $(g_1 g_2)x = g_1(g_2 x), \forall x \in X$ and $\forall g_1, g_2 \in G$

- If conditions 1 and 2 are satisfied we call X as the ***G-set*** (That is same as saying G 'can act' on X)

- Every group trivially acts on every set through the map $(g, x) \mapsto x$

- **Example:** $GL_2(\mathbb{R})$ acting on $X = \mathbb{R}^2$ via matrix-vector multiplication. We can verify that condition 1 is satisfied with $e = I_2$ and condition 2 is satisfied because matrix multiplication is associative.

G-equivalence

- **G-equivalence:** If G acts on a set X and $x, y \in X$, then x is said to be G-***equivalent*** to y if $\exists g \in G$ s.t. $gx = y$. We denote G-***equivalence*** of x and y as $x \sim_G y$
- One can check that G-***equivalence*** is an equivalence relation by verifying its reflexive, symmetry and transitive properties

Orbits

- Each G-equivalent partition of X so formed is called ***orbit*** of X under G.
- We denote each orbit by a representative. e.g. orbit containing X is called $\mathcal{O}_x$
- **Example:** $G = \{(1), (123), (132), (45), (123)(45), (132)(45)\}$ and $X = \{1, 2, 3, 4, 5\}$
- X is a G-set
- The orbits are:

  1. $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}_3 = \{1, 2, 3\}$ and
  2. $\mathcal{O}_4 = \mathcal{O}_5 = \{4, 5\}$

Fixed point sets

- ***Fixed point set*** X_g of $g \in G$ is the set $\{x | x \in X_g$ if $x \in X$ and $gx = x\}$
- **Example:** $X = \{1, 2, 3, 4, 5, 6\}$ and suppose G is permutation group given by permutations $\{(1), (12)(3456), (35)(46), (12)(3654)\}$
- Fixed point sets in X are:

  1. $X_{(1)} = X$
  2. $X_{(35)(46)} = \{1, 2\}$
  3. $X_{(12)(3456)} = X_{(12)(3654)} = \phi$

Stabilizer Subgroup

- Subgroup of G, denoted as G_x that fixes given x called ***stabilizer subgroup***
- **Example:** $X = \{1, 2, 3, 4, 5, 6\}$ and suppose G is permutation group given by permutations $\{(1), (12)(3456), (35)(46), (12)(3654)\}$
- The stabilizer subgroups are:

  1. $G_1 = G_2 = \{(1), (35)(46)\}$
  2. $G_3 = G_4 = G_5 = G_6 = \{(1)\}$

Kernel of the action

- Consider action of group G on A
- K=$\{g \in G \mid ga = a \ \ \forall a \in A\}$
- Why?

Kernel of the action

- Consider action of group G on A
- K=$\{g \in G \mid ga = a \ \ \forall a \in A\}$
- Why?
  - Trivially, $id \in K$
  - Suppose $k_1, k_2 \in K$ and $a \in A$
  - Then, $(k_1 k_2)a = k_1(k_2 a) = k_1 a = a \Rightarrow k_1 k_2 \in K$
  - Now let $k \in K$ and $a \in A$
  - $k^{-1}a = k^{-1}(ka) = (k^{-1}k)a = 1 \Rightarrow k^{-1} \in K$ ■

Cosets

- For any $g \in G$ and H $\leq G$ the set $gH = \{gh : h \in H\}$ is called **left coset** of H with **representative** g
- Similarly, $Hg = \{hg : h\}$ is the right coset
- **Example:** Let $H \leq \mathbb{Z}_6$ and $H = \{0, 3\}$ then 6 elements of G give following 6 cosets:

1. $0 + H = \{0, 3\}$
2. $1 + H = \{1, 4\}$
3. $2 + H = \{2, 5\}$
4. $3 + H = \{0, 3\}$
5. $4 + H = \{1, 4\}$
6. $5 + H = \{2, 5\}$

Cosets

- Only 3 of these cosets are unique. They are:
  $\{0,3\}, \{1,4\}, \{2,5\}$
- Further,$\{0,3\} \cup \{1,4\} \cup \{2,5\} = \mathbb{Z}_6$
- This leads to following theorem:
  For any $H \leq G$, left cosets of H in G partition G
  OR
  The group G is disjoint union of left cosets of H in G
- Number of left cosets=number of right cosets

Cosets

- For $H \leq G$ ***Index*** of H in G is number of left(right) cosets of H in G denoted by $[G : H]$
- ***Theorem(Lagrange):*** Let G be a finite group and let $H \leq G$. Then $|G|/|H| = [G : H]$ is the number of distinct cosets of H in G. Hence, if H is a subgroup of G then, number of elements in H must divide number of elements in G.
- **Converse is not true**

Normal Subgroups

- A subgroup H of a group G is *normal* in G if $gH = hg, \ \ \forall g \in G$
- Every Subgroup of an abelian group is trivially a normal subgroup
- Let $H \leq S_3$ s.t. $H = \{(1), (12)\}$
- $(123)H = \{(123), (13)\}$ and $H(123) = \{(123), (23)\} \Rightarrow H$ is not normal
- Let $N \leq S_3$ $N = \{(1), (123), (132)\}$
- Cosets of N are $(12)N = N(12) = \{(12)(13)(23)\} \Rightarrow N$ is normal

Semidirect product

- Let G be a group with a subgroups H (not necessarily normal) and a normal subgroup N
- Then, G is a *semidirect product* of N by H if
  - $H \cap N = \{id\}$
  - $HN = G$
- S_3 is semidirect product of A_3 by $H = \{id, (12)\}$
- Why?

Semidirect product

- Let G be a group with a subgroups H (not necessarily normal) and a normal subgroup N
- Then, G is a *semidirect product* of N by H if
  - $H \cap N = \{id\}$
  - $HN = G$
- S_3 is semidirect product of A_3 by $H = \{id, (12)\}$
- Why?
  S_3 has even permutations $\{(32)(31), (21)(23), id\}$ and odd permutations $\{(13), (12), (23)\}$
  We can compose (12) with even permutations (as those in A_3) and obtain odd permutations

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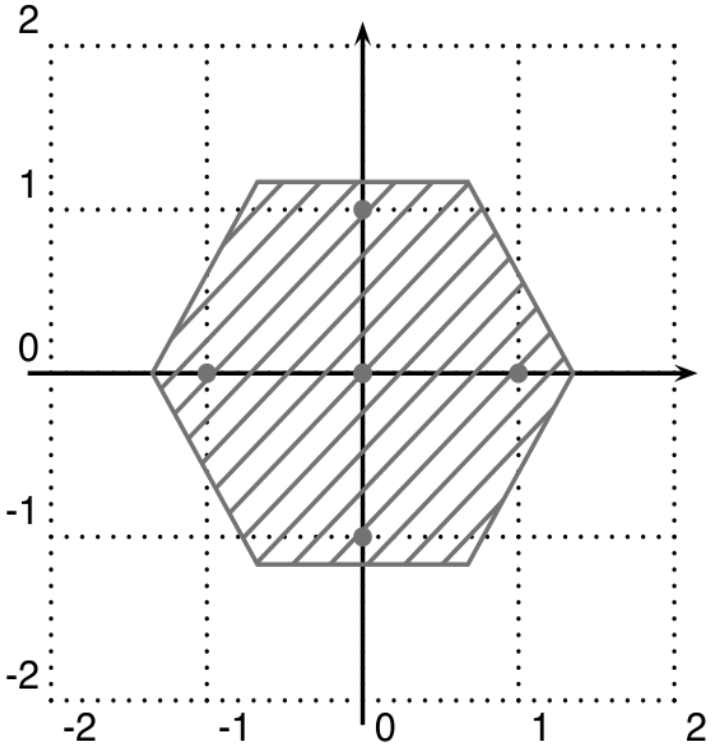
Symmetries of an LP

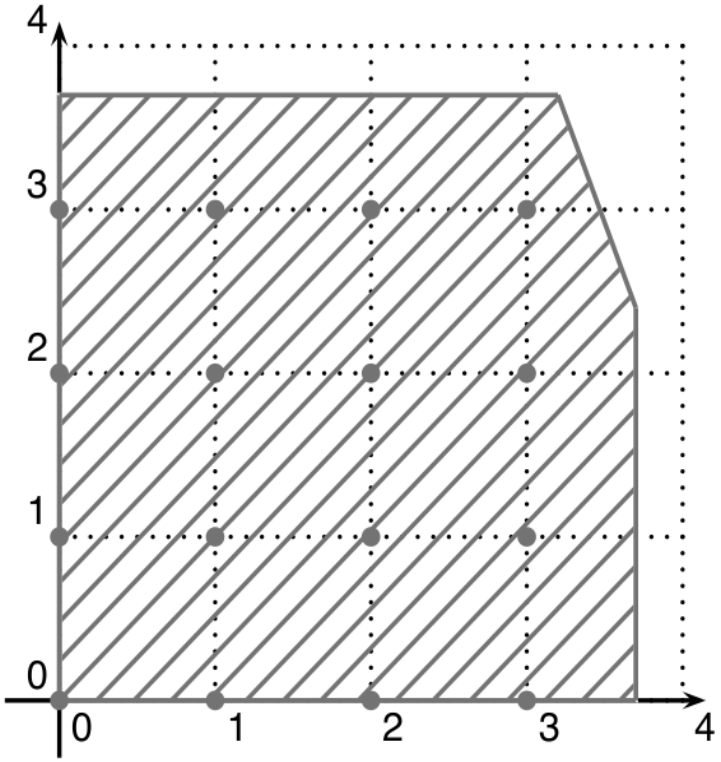
- Roughly speaking, symmetry of LP is a group action that preserves both the cost vector and the inequality system or underlying polyhedron
- $Aut(\mathbb{R}^n)$ is the group of all permutations $\phi : \mathbb{R}^n \to \mathbb{R}^n$ that preserve its algebraic structure i.e. $\phi(\mathbf{x} + \mathbf{x}) = \phi(\mathbf{x}) + (y)$ and $\phi(a\mathbf{x}) = a\phi(\mathbf{x}), \forall x, y \in \mathbb{R}^n, \forall a \in \mathbb{R}$
- $Aut(\mathbb{R}^n) \cong GL_n(\mathbb{R})$
- $Aut^I(\mathbb{R}^n) \leq Aut(\mathbb{R}^n)$ is the group of permutations that also preserve distance i.e. $d(\mathbf{x}, \mathbf{y}) = d(g\mathbf{x}, g\mathbf{y}),\ \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \forall g \in Aut^I(\mathbb{R})$
- $Aut^I(\mathbb{R}^n) \cong O_n(\mathbb{R})$

Symmetries of an LP

- $O_n(\mathbb{R})$ is orthogonal group
- It is the group of $n \times n$ orthogonal matrices, where the group operation is given by matrix multiplication
- Orthogonal matrix is a real matrix whose inverse equals its transpose i.e. $A^t A = I, \ \ \forall A \in O_n(\mathbb{R})$
- Left multiplication by orthogonal matrix preserves distance
- (Hand wavy proof:) It preserves dot product which is equivalent to preserving distance
- Hence symmetries of LP should be some $G \leq O_n(\mathbb{R})$ that preserve the cost function

What about integer programs?





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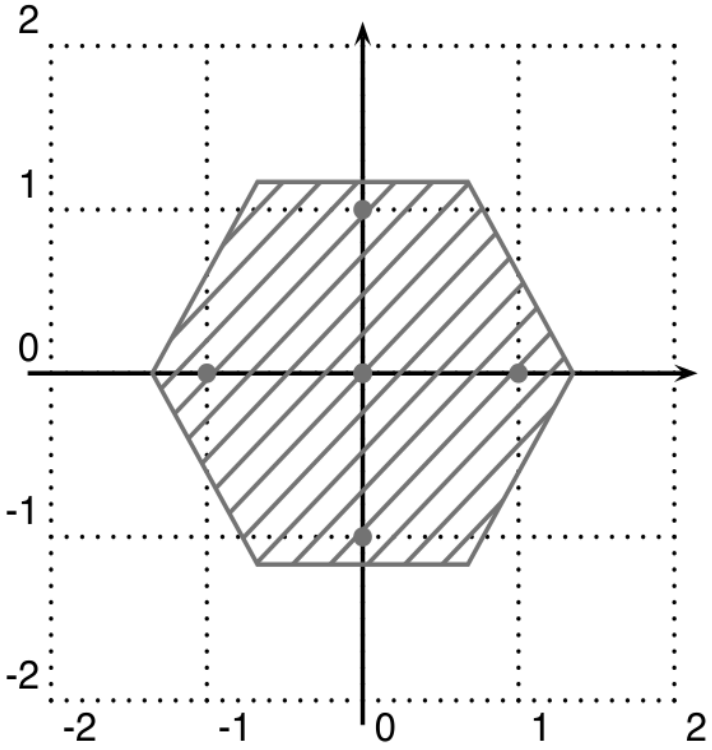

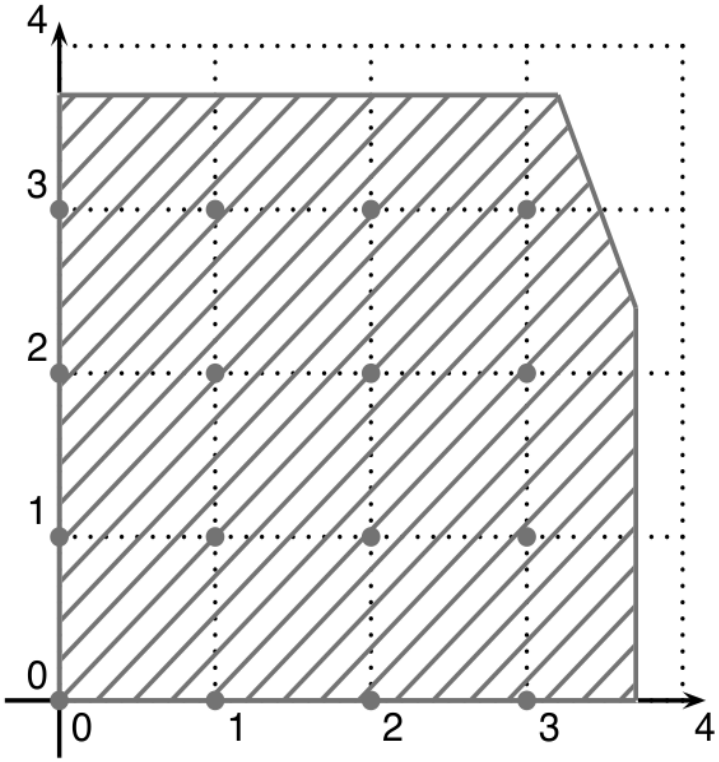

In general, symmetries of LPs and IPs don't coincide

Symmetries of an integer program

- We would want the standard lattice to map to itself under action of $G \leq O_n(\mathbb{R})$
- $M \in O_n(\mathbb{R})$ that satisfy this property must map standard basis vector $\mathbf{e}_j$ to an integer vector i.e. $\mathbf{Me}_j = (m_{1j}, ..., m_{nj})^t$
- This means all columns of M must be integral i.e. $M \in O_n(\mathbb{R}) \cap GL_n(\mathbb{Z})$ where $GL_n(\mathbb{Z})$ is the group of integrally invertible matrices
- Since every integer orthogonal matrix is integrally invertible (duh), $O_n(\mathbb{Z}) \cong O_n(\mathbb{R}) \cap GL_n(\mathbb{Z}) \leq O_n(\mathbb{R})$
- Any $M \in O_n(\mathbb{Z})$ preserves distance, $(m_{1j}, ..., m_{nj})^t$ is an integer vector of length one
- $\therefore (m_{1j}, ..., m_{nj})^t \in \{\pm e_i \mid 1 \leq i \leq n\}, \ \ \forall 1 \leq j \leq n$

Symmetries of an integer program

- $O_n(\mathbb{Z}) \subseteq S$ where S is the set of signed permutation matrices

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

- In fact $O_n(\mathbb{Z})$ is the same set as the set of all signed permutation matrices(call it S)
  Every $M \in S$ is orthogonal and integral $\Rightarrow S \subseteq O_n(\mathbb{Z})$

Symmetries of an integer program

- $O_n(\mathbb{Z})$ acts on the set $\{\mathbb{R}e_i \mid 1 \leq i \leq n\} \subseteq \mathrm{R}^n$
- Consider set $\mathcal{D}_{sign}$ of all sign-changing matrices in $O(\mathbb{Z})$
- These are diagonal matrices D with ± 1 as entries on diagonal
- $M\mathcal{D}_{sign} = \mathcal{D}_{sign}M \;\; \forall M \in O(\mathbb{Z})$
- $\mathcal{D}_{sign}$ is normal subgroup of $O_n(\mathbb{Z})$
- Hence $O_n(\mathbb{Z}) = \mathcal{P}_n \rtimes \mathcal{D}_{sign}$ where, $\mathcal{P}_n \leq O_n(\mathbb{Z})$ corresponds to subgroup of $n \times n$ permutation matrices
- If LP or IP has feasible region $X \subseteq \mathbb{R}^n_{\geq 0}$ then $O_n(\mathbb{Z})$ shrinks to $\mathcal{P}_n$

Relationship between symmetries of IP and LP

- $M \in O_n(\mathbb{Z})$ preserves cost of LP $\Rightarrow$ $M \in O_n(\mathbb{Z})$ preserves cost of IP
- $M \in O_n(\mathbb{Z})$ preserves cost of IP $\not\Rightarrow$ $M \in O_n(\mathbb{Z})$ preserves cost of LP

Symmetries of LP as a group

- $G \leq S_n$ permutes the standard basis vectors which induces an atomorphism on $\mathbb{R}^n$, hence $G \cong G' \leq S_n$
- Consider LP $max \;\; \mathbf{c}^t\mathbf{x}$ s.t. $\mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \in \mathbb{R}^n_{\geq 0}$
- Define *symmetry of matrix* $\mathbf{A} \in \mathbb{R}^{m \times n}$ to be an element g$\in S_n$ such that there exists a row permutation $\sigma \in S_m$ with $\mathbf{P}_\sigma A P_\sigma = A$ where $\mathbf{P}_\sigma$ and $\mathbf{P}_g$ are permutation matrices corresponding to σ and g
- The full symmetry group of $A \in \mathbb{R}^m$ is $\{g \in S_n \mid \exists \sigma \in S_m : P_\sigma A P_g = A\}$
- Symmetry of a linear inequality system $\mathbf{Ax} \leq b$ is $g \in S_n$ is a symmetry of matrix A via row permutation $\sigma \in S_m$ which satisfies $b^\sigma = b$

Symmetries of LP as a group

- The *full symmetry group* of an LP(or corresponding IP) is: $\{g \in S_n \mid \mathbf{c}^g = \mathbf{c}, \exists \sigma \in S_m : (\mathbf{b}^\sigma = \mathbf{b} \wedge P_\sigma A P_g = A)\}$

End of part I

- Questions?

What happens to symmetries when we add extra inequalities to the system?

Consider the following system system of linear inequalities:

$$\begin{pmatrix} A \\ A' \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} b \\ b' \end{pmatrix}$$

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It is made up of 2 different systems of inequalities:

$\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{A'x} \leq \mathbf{b'}$

$\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{A'} \in \mathbb{R}^{m' \times n}, \mathbf{b} \in \mathbb{R}^{m}$

Symmetries of combined system

Let $G \leq S_n$ be a symmetry group of both inequality systems

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What does that mean?

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$$\forall \ \ g \in G, \exists \sigma \in S_m : P_\sigma A P_g = A$$

$$\forall \ \ g \in G, \exists \sigma \in S_{m'} : P_\sigma A' P_g = A'$$

Symmetries of combined system

Let $G \leq S_n$ be a symmetry group of both inequality systems

Theorem The group G is also a symmetry group of the system

$$\begin{pmatrix} A \\ A' \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} b \\ b' \end{pmatrix}$$

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$$\text{i.e } \mathbf{P}_\sigma \mathbf{A} \mathbf{P}_g = \mathbf{A}, \mathbf{P}'_\sigma \mathbf{A}' \mathbf{P}'_g = \mathbf{A}', \mathbf{P}_\sigma \mathbf{b} = \mathbf{b}, \mathbf{P}'_\sigma \mathbf{b}' = \mathbf{b}'$$

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Theorem The group G is also a symmetry group of the system

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{A}' \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} b \\ b' \end{pmatrix}$$

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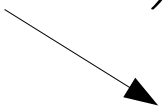

Row permutation we need to prove above theorem

Part-II

- Orbits
- The set of Fixed Points
- Solution in the subspace of fixed points via Orbit-Stabilizer theorem aka dimensionality reduction

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Orbits

Let $G \leq S_n$ and an element $\mathbf{x} \in \mathbb{R}^n$, the *orbit* $\mathbf{x}^G$ of $\mathbf{x}$ wrt G is defined as:

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Feasibility and Orbits

Consider symmetry group G of an LP Λ with feasible region X, which leaves X invariant

Claim: A point $\mathbf{x} \in \mathbb{R}^n$ is feasible for Λ iff every element of the orbit $\mathbf{x}^G$ is feasible for Λ

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Why?

Utility and orbits

Theorem: Let $G \in S_n$ be a symmetry group of an LP Λ. Given $\mathbf{x} \in \mathbb{R}^n$, the utility function of Λ is constant on the orbit $\mathbf{x}^G$

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$$\mathbf{c}^t\mathbf{x}^g = (\mathbf{c}^g)\mathbf{x}^g = \sum_{i \in I_n} (\mathbf{c}^g)_i (\mathbf{x}^g)_i = \sum_{i \in I_n} \mathbf{c}_{ig}\,\mathbf{x}_{ig} = \sum_{i^g \in I_n} \mathbf{c}_{ig}\,\mathbf{x}_{ig} = \mathbf{c}^t\mathbf{x}$$

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- G also acts on the set of standard basis vectors $B := \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}, B \subseteq \mathbb{R}^n$
- Under this action B splits into orbits, which we would denote by O and the set of all orbits as $\mathcal{O}$

Orbits of bases

WLOG, the orbits of G on B are given by:
$O_1 = \{\mathbf{e}_1, \ldots, \mathbf{e}_{n_1}\}$,
$O_i = \{\mathbf{e}_{s_i-1+1}, \ldots, \mathbf{e}_{s_i-1+n_i}\}$
for $i \in \{2, \ldots, k\}$, where
k is the number of orbits,
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s_i is defined by $s_i := \sum_{j=1}^{i} n_j$
Corollary: Let $\mathbf{e}_i, \mathbf{e}_j \in B$ be two elements of the same orbit O under a group G. Then the entries $\mathbf{c}_i$ and $\mathbf{c}_j$ of the utility vector $\mathbf{c}$ are equal

Structure of cost vector

As a result, the cost vector has following structure:

$$\mathbf{c} = (\underbrace{\gamma_1, \ldots, \gamma_1}_{n_1}, \underbrace{\gamma_2, \ldots, \gamma_2}_{n_2}, \ldots, \underbrace{\gamma_k, \ldots, \gamma_k}_{n_k})^t$$

Example

Consider the LP given by the utility vector $\mathbf{c}(1,1)^t$ and the inequality system:

$$\begin{aligned} x_1 + x_2 &\leq 2 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Part-II

- Orbits
- The set of Fixed Points
- Solution in the subspace of fixed points via Orbit-Stabilizer theorem aka dimensionality reduction

The set of Fixed Points

Definition: Given a subset $Y \subseteq \mathbb{R}^n$ and a group $G \subseteq S_n$ acting on Y, the set of *fixed points* of Y wrt G is given by,

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The set of Fixed Points

- Let $\mathbf{P}_g$ be the permutation matrix corresponding to $g \in G$ which is a symmetry of $\mathbb{R}^n$
- For every $\mathbf{x} \in Fix_g(\mathbb{R}^n)$, $\mathbf{P}_g\mathbf{x} = \mathbf{x}$
- Hence, $Fix_g(\mathbb{R}^n)$ is the eigenspace $Eig_1(g)$ corresponding to eigenvalue 1
- $Fix_G(\mathbb{R}^n)$ is the inetersection of all those eigenspaces
- **Claim**: $Fix_G(\mathbb{R}^n)$ wrt group $G \in S_n$ is a subspace of $\mathbb{R}^n$

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- **Claim**: $Fix_G(\mathbb{R}^n)$ wrt group $G \in S_n$ is a subspace of $\mathbb{R}^n$ Because eigenspaces are subspaces!
- Let $G \subseteq S_n$ be a symmetry group of LP Λ
- **Claim**: The line $l := \{r\mathbf{c} \mid r \in \mathbb{R}\}$ is a subspace of $Fix_G(\mathbb{R}^n)$

$dim(Fix_G(\mathbb{R}^n))$ and number of orbits

Lemma Let $O \subseteq B$, $G \subseteq S_n$ a group acting transitively (only one orbit) on O and $V := \langle O \rangle$ and $Fix_G(V) = V \cap Fix_G(\mathbb{R}^n)$. Then,

$$Fix_G(V) = \langle \sum_{\mathbf{e}_i \in O} \mathbf{e}_i \rangle$$

i.e. $Fix_G(V)$ is one-dimensional

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Proof:

- WLOG, let $O = \{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$
- $\mathbf{v} := \sum_{\mathbf{e}_i \in O} \mathbf{e}_i = (1, \ldots, 1, 0, \ldots, 0)^t \in V$ is fixed by G
- $\mathbf{v} \in V \cap Fix_G(\mathbb{R}^n) = Fix_G(V) \Rightarrow \langle v \rangle \subseteq Fix_G(V)$
- $\Rightarrow dim(Fix_G(V)) \geq 1$

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Proof:

- Conversely, consider $(m-1)$-dimensional subspaces $W_j \subseteq V, W_j := \langle O \setminus \{\mathbf{e}_j\} \rangle$
- Let $dim(Fix_G(V)) > 1$, then $dim(W_j \cap Fix_G(V)) = dim(W_j) + dim(Fix_G(V)) - dim(V)$ (Follows from Grassmann formula: $dim(A) + dim(B) = dim(A \cap B) + dim(A + B)$)

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Proof:

- $dim(W_j) + dim(Fix_G(V)) - dim(V) = dim(Fix_G(V)) - 1 \geq 1$
- In perticular, $\exists \mathbf{0} \neq \mathbf{w} := \sum_{i=1}^{m-1} \mathbf{a}_i \mathbf{e}_i \in (W_m \cap Fix_G(V))$ where, for atleast one $l \in \{1, \ldots, m-1\}$ s.t. $\mathbf{a}_l \neq 0$
- Since there is only one orbit, $\exists g \in G$ which maps $\mathbf{e}_l$ to $\mathbf{e}_m$
- Since $\mathbf{w} \in Fix_G(V)$, $\mathbf{w}$ is fixed by g

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Proof:

- w=w^g $= \sum_{i=1}^{m-1} \mathbf{a}_{i^{(g^{-1})}} e_i + \mathbf{a}_l \mathbf{e}_m \notin W_m$
- contradiction!
- $\therefore w \neq W_m \cap Fix_G(V)$. Consequently, $dim(Fix_G(V)) \leq 1$

Part 3

- Prove a general relationship between number of orbits of the set of standard basis vectors

  and the dimension of subspace fixed points
- Equivalence classes among feasible points of an LP based on their utility value
- Prove that for every feasible point there is a fixed point with same utility value
- How to formulate smaller LP given a large symmetric LP and its symmetry group

Reboot

Definition: Given a subset $Y \subseteq \mathbb{R}^n$ and a group $G \subseteq S_n$ acting on Y, the set of *fixed points* of Y wrt G is given by,

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$dim(Fix_G(\mathbb{R}^n))$ and number of orbits

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i.e. $Fix_G(V)$ is one-dimensional
Recall:

- $G \leq S_n$ acts on set of n labels by permuting them. It acts on $\mathbb{R}^n$ via linear transformations (left multiplication) with permutation matrices
- cost\utility vector $\mathbf{c}$ is fixed by $G \Rightarrow l := \{r\mathbf{c} \mid r \in \mathbb{R}\}$ is in $Fix_G(\mathbb{R}^n)$
- G can be thought to be acting on the set of standard bases B by permuting the labels of bases

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$dim(Fix_G(\mathbb{R}^n))$ and number of orbits

Theorem Let k be the number of orbits of B under G and V_i be the subspace spanned by elements of orbit O_i of B. Then the following statements hold:

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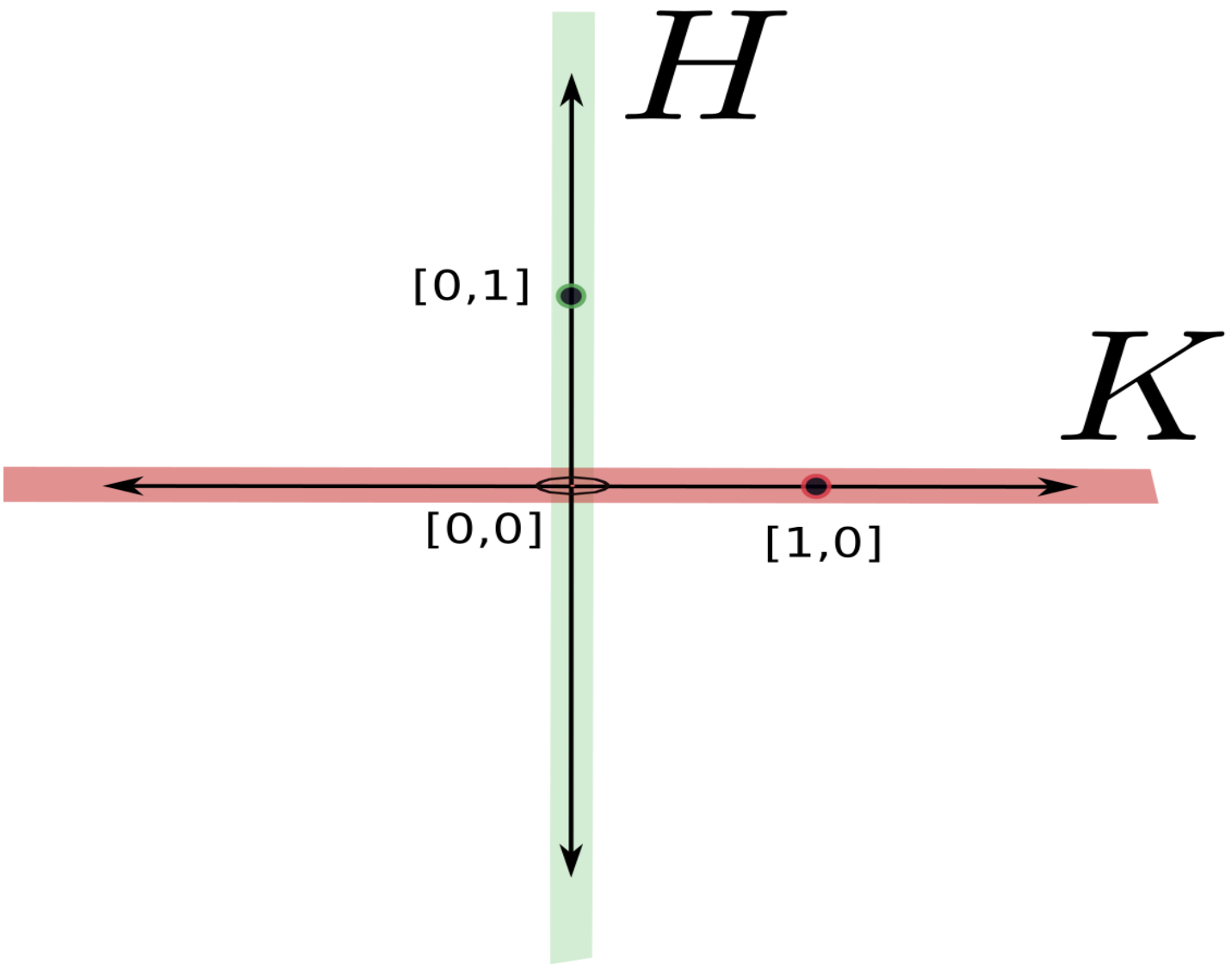
Direct sums

- **Definition**: Let V be a vector space and H and K are subspaces of V s.t. $H \cap K = \{\mathbf{0}\}$. The *direct sum* of H and K is the set of vectors

$$H \oplus K = \{\mathbf{u} + \mathbf{v} \mid \mathbf{u} \in H \text{ and } \mathbf{v} \in K\}$$

Direct sums



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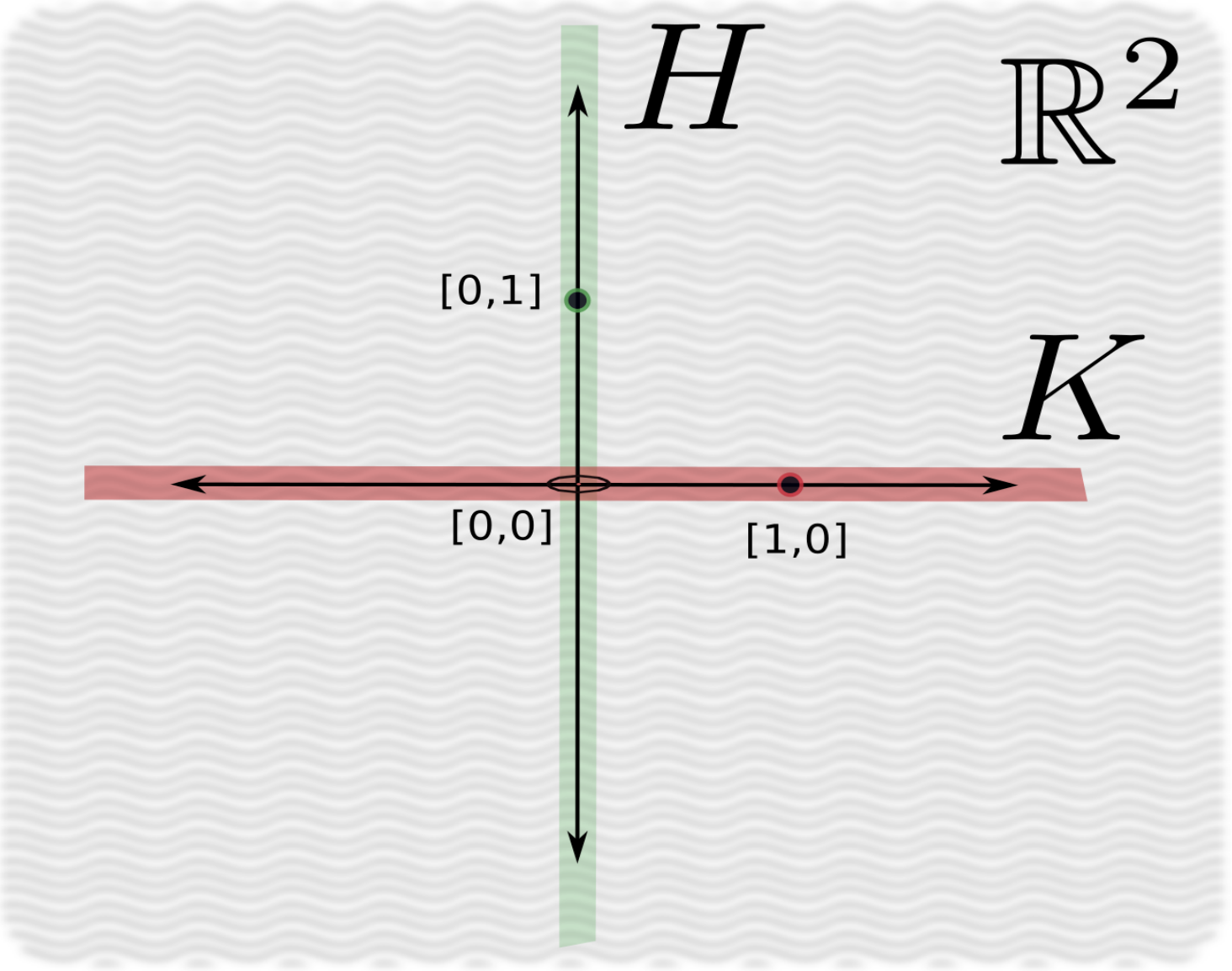

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$dim(Fix_G(\mathbb{R}^n))$ and number of orbits

Proof of (i) :

- The set of orbits $\mathcal{O} = \{O_1, \ldots, O_k\}$ is a partition of the basis B of $\mathbb{R}^n$ and we have: $_i \cap V_j = \{\mathbf{0}\}$ for $i \neq j$
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- $\mathbb{R}^n = \bigoplus_{i=1}^{k} V_i$
- $\therefore Fix_G(\mathbb{R}^n) = \mathbb{R}^n \cap Fix_G(\mathbb{R}^n) = \left(\bigoplus_{i=1}^{k} V_i\right) \cap Fix_G(\mathbb{R}^n)$
- Hence, any $\mathbf{v} \in Fix_G(\mathbb{R}^n)$ can be represented uniquely as $\mathbf{v} = \sum_{i=1}^{k} \mathbf{v}_i$ where $\mathbf{v}_i \in V_i$

$dim(Fix_G(\mathbb{R}^n))$ and number of orbits

Proof of (i) :

- **Claim**: $Fix_G(\mathbb{R}^n) \subseteq \bigoplus_{i=1}^{k} Fix_G(V_i)$
- Recall that any $\mathbf{v} \in Fix_G(\mathbb{R}^n)$ can be represented uniquely as $\mathbf{v} = \sum_{i=1}^{k} \mathbf{v}_i$ where $\mathbf{v}_i \in V_i$
- For any $g \in G$

$$\sum_{i=1}^{k} \mathbf{v}_i = \mathbf{v} = \mathbf{v}^g = \sum_{i=1}^{k} \mathbf{v}_i^g$$

- Uniquenses $\Rightarrow$ g maps each $\mathbf{v}_i$ to a certain $\mathbf{v}_j \in V_j$

- But every subspace $V_i, i \in [k]$ is invariant under $G \Rightarrow \mathbf{v}_i^g = \mathbf{v}_i$
- $(\bigoplus_{i=1}^{k} V_i) \cap Fix_G(\mathbb{R}^n) \subseteq \bigoplus_{i=1}^{k} (V_i \cap Fix_G(\mathbb{R}^n))$ ■

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Proof of (ii) :

- We already proved that:
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  i.e. $Fix_G(V)$ is one-dimensional

$$\dim \mathrm{Fix}_G(V_i) = 1$$

$dim(Fix_G(\mathbb{R}^n))$ and number of orbits

Proof of (ii) :

- We already proved that:
  Lemma Let $O \subseteq B$, $G \subseteq S_n$ a group acting transitively (only one orbit) on O and $V := \langle O \rangle$ and $Fix_G(V) = V \cap Fix_G(\mathbb{R}^n)$. Then,

$$Fix_G(V) = \langle \sum_{\mathbf{e}_i \in O} \mathbf{e}_i \rangle$$

i.e. $Fix_G(V)$ is one-dimensional

$$\dim(\mathrm{Fix}_G(V_i) = 1)$$

$$dim(Fix_G(\mathbb{R}^n)) = dim\left(\bigoplus_{i=1}^{k} Fix_G(V_i)\right) = \sum_{i=1}^{k} dim(Fix_G(V_i)) = k$$

Part 3

- Prove a general relationship between number of orbits of the set of standard basis vectors

  and the dimension of subspace fixed points
- Equivalence classes among feasible points of an LP based on their utility value
- Prove that for every feasible point there is a fixed point with same utility value
- How to formulate smaller LP given a large symmetric LP and its symmetry group

Example

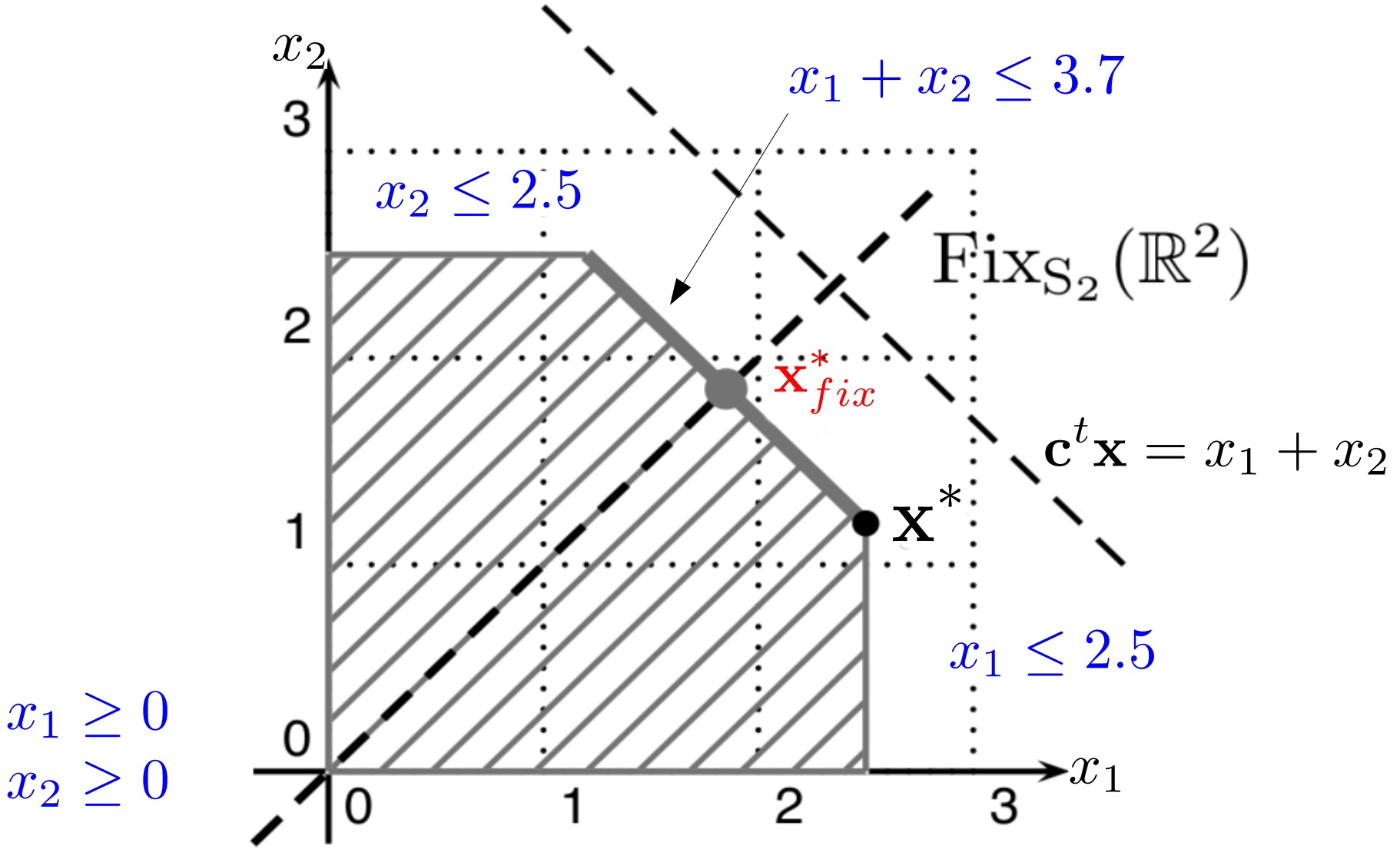
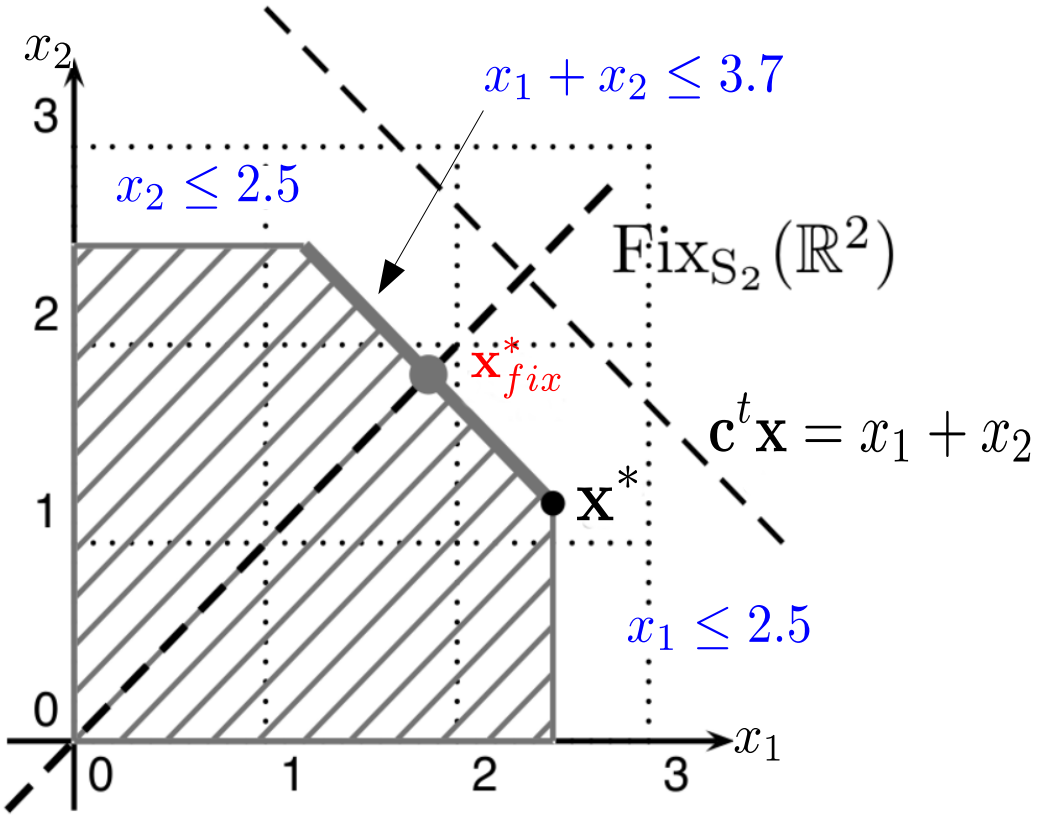


Figure credits: Katrin Herr, R. Bödi, *Symmetries in linear and integer linear programming*, Oberwolfach Report 38/2010

Example

- S_2 itself is the symetry group of this LP
- $Fix_{S_2}(\mathbb{R}^2) = \{r\mathbf{c} \mid \mathbf{r} \in \mathbb{R}\}$



$x_1 \geq 0$
$x_2 \geq 0$

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Orbit-Stabilizer theorem

Theorem:Consider action of group G on finite set X then,

$$|x^G| = \frac{|G|}{|G_x|}$$

In fact there exists a bijection $\gamma : G/G_x \to Orb(x)$ defined as $gG_x \mapsto g * x$ where G/G_x is set of all left(right) cosets of G_x in G and $*$ is the group operation

Tiny Proof: Define map $\phi : G/G_x \to x^G$ as $\phi(gG_x) = g * x$ where $*$ is the group action

$gG_x = hG_x \iff h^{-1}g \in G_x \iff h^{-1}g * x = x \iff g * x = h * x$

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i.e. if two elements of G produce same left cosets with G_x then their cosets map to same element in x^G and vice versa

$\Rightarrow \phi$ is a bijection ■

Barycenter of an orbit

Barycenter of a set of points $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ is $\frac{1}{n} \sum_{i=1}^{n} \mathbf{v}_i$

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Lemma: Given $x \in \mathbb{R}^n$, the barycenter of the orbit $\mathbf{x}^G$ can be written as

$$\frac{1}{|\mathbf{x}^G|} \sum_{\mathbf{y} \in \mathbf{x}^G} \mathbf{y} = \frac{1}{G} \sum_{g \in G} \mathbf{x}^g$$

Proof:

- The set of stabilizers of $\mathbf{x}$ in G is called $G_{\mathbf{x}}$
- $G_{\mathbf{x}}$ is a subgroup of G which means $G = \cap_{g \in G} G_{\mathbf{x}} g$

Barycenter of an orbit

Proof:

- The set of stabilizers of $\mathbf{x}$ in G is called $G_{\mathbf{x}}$
- $G_{\mathbf{x}}$ is a subgroup of G which means $G = \cap_{g \in G} G_{\mathbf{x}} g$
- Let $S = \{s_1, \ldots, s_{|\mathbf{x}^G|}\}$ be representatives of family of cosets $G_{\mathbf{x}} g$
- Because of the bijection between set of cosets of $G_{\mathbf{x}}$ and and $\mathbf{x}^G$, we can write $\sum_{\mathbf{y} \in \mathbf{x}^G} y = \sum_{s \in S} \mathbf{x}^s$
- From OST,

$$\frac{1}{|\mathbf{x}^G|} \sum_{\mathbf{y} \in \mathbf{x}^G} \mathbf{y} = \frac{|G_{\mathbf{x}}|}{|G|} \sum_{s \in S} \mathbf{x}^s = \frac{1}{|G|} \sum_{s \in S} |G_{\mathbf{x}}| \mathbf{x}^s$$

Barycenter of an orbit

Proof:

- Since $\mathbf{x}^g = \mathbf{x}$ for all $g \in G_{\mathbf{x}}$,

$$|G_{\mathbf{x}}|\mathbf{x}^s = \sum_{g \in G_{\mathbf{x}}} (\mathbf{x}^g)^s$$

- $$\frac{1}{|G|} \sum_{s \in S} |G_{\mathbf{x}}|\mathbf{x}^s = \frac{1}{|G|} \sum_{s \in S} \sum_{g \in G_{\mathbf{x}}} (\mathbf{x}^g)^s \frac{1}{|G|} \sum_{s \in S} \sum_{g \in G_{\mathbf{x}}} \mathbf{x}^{(gs)}$$

  Consider G represented as disjoint union:

$$G = \amalg_{s \in S} G_{\mathbf{x}} s$$

- Finally,

$$\frac{1}{|G|} \sum_{s \in S} \sum_{g \in G_{\mathbf{x}}} \mathbf{x}^{(gs)} = \frac{1}{|G|} \sum_{g \in G} \mathbf{x}^g \quad \blacksquare$$

Representative of equivalence class of points having same cost in fixed space

Theorem: Let X be the feasible region of LP λ. If $\mathbf{x} \in \mathbb{R}^n$ is feasible for Λ, there exists a feasible point $\mathbf{x}_{fix}$ in $Fix_G(\mathbb{R}^n)$ with same utility vector as $\mathbf{x}$

Proof: Define

$$\mathbf{x}_{fix} = \frac{1}{|\mathbf{x}^G|} \sum_{\mathbf{y} \in \mathbf{x}^G} \mathbf{y}$$

$\mathbf{x}^G$ is feasible $\Rightarrow$ $\mathbf{x}_{fix}$ is feasible
From Lemma we just proved, and since G acts via linear maps,

$$\mathbf{x}_{fix}^{g'} = \left(\frac{1}{\mathbf{x}^G} \sum_{\mathbf{y} \in \mathbf{x}^G} \mathbf{y} \right)^{g'} = \left(\frac{1}{|G|} \sum_{g \in G} \mathbf{x}^g \right)^{g'} = \frac{1}{|G|} \sum_{g \in G} \mathbf{x}^{gg'} = \frac{1}{|G|} \sum_{\tilde{g} \in G} \mathbf{x}^{\tilde{g}} = \mathbf{x}_{fix}$$

for all $g' \in G$

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Proof: Hence, $\mathbf{x}_{fix}$ is indeed a fixed point i.e. $\mathbf{x}_{fix} \in Fix_G(\mathbb{R}^n)$
Since $\mathbf{c}^t\mathbf{x}^g = \mathbf{c}^t\mathbf{x}$ for all $g \in G$,

$$\mathbf{c}^t\mathbf{x}_{fix} = \frac{1}{|G|}\sum_{g\in G}\mathbf{c}^t\mathbf{x}^g = \frac{1}{|G|}\sum_{g\in G}\mathbf{c}^t\mathbf{x} = \mathbf{c}^t\mathbf{x}$$

■