Partition Symmetrical Entropy Functions*

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*Chen, Qi; Yeung, Raymond W., Partition Symmetrical Entropy Functions, arXiv:1407.7405v1 [cs.IT]

Outline

- Sneak Peak
- Background:
  - Basic facts from convex analysis
  - Matroids reboot
  - Partitions
  - Group action Reboot
- New Regions Ψ_p and $\overline{\Psi_p^*}$: Points fixed under action of a variable permutation group with set of orbits = partition p
- Symmetrical properties of Γ_N and $\overline{\Gamma_N^*}$: how we get rid of symmetries with Ψ_p and $\overline{\Psi_p^*}$
- For which variable permutation groups $\Psi_p = \overline{\Psi_p^*}$?

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Entropy function

- $\mathcal{N} = \{1, \ldots, n\}$
- For a set of discrete random variables $X_{\mathcal{N}} = \{X_i : i \in \mathcal{N}\}$, we define a function $\mathbf{h} : 2^{\mathcal{N}} \to \mathbb{R}$ by:

$$\mathbf{h}(\mathcal{A}) = H(X_{\mathcal{A}}), \mathcal{A} \subset \mathcal{N}$$

- Where, $H(X_{\emptyset}) = 0$ by convension.
- $\mathcal{H}_n = \mathbb{R}^{\mathcal{N}}$ is the entropy space of n random variables
- A vector $\mathbf{h} \in \mathcal{H}_n$ is called **entropic** if $\mathbf{h}$ is the entropy function of a set of n random variables

Entropy function region

- The region in $\mathcal{H}_n$ of all entropy functions is denoted by Γ_n^*: the entropy function region
- As $\mathbf{h}(\emptyset) = 0, \Gamma_n^* \subset \mathcal{H}_{\mathcal{N}}^0 \triangleq \{\mathbf{h} \in \mathcal{H}_n \mid \mathbf{h}(\emptyset) = 0\}$
- Following polymatroidal axioms are satisfied by entropy function: For any $A, B \subset \mathcal{N}$,

  1 $\mathbf{h}(\emptyset) = 0$

  2 $\mathbf{h}(\mathcal{A}) \leq \mathbf{h}(\mathcal{B})$, if $\mathcal{A} \subset \mathcal{B}$

  3 For $\mathcal{A} \subset \mathcal{N}$ and $i, j \in \mathcal{N} - \mathcal{A}$ then
  $\mathbf{h}(\mathcal{A} \cup \{i\}) + \mathbf{h}(\mathcal{A} \cup \{j\}) \geq \mathbf{h}(\mathcal{A} \cup \{i, j\}) + \mathbf{h}(\mathcal{A})$

- Region enclosed by above inequalities is called the polymatroidal region/Shannon outer bound denoted as $\Gamma_n \subset \mathcal{H}_n^0$
- Above are the Shannon-type (elemental) information inequalities

Entropy function region

- $\Gamma_n^* \subset \Gamma_n$
- $\Gamma_2^* = \Gamma_2$, $\Gamma_3 * \subsetneq \Gamma_3$ but $\overline{\Gamma_3^*} = \Gamma_3$, since the boundary of Γ_3 contains non-entropic polymatroids
- Vectors in the closure of entropy function region are called **almost entropic**
- All polymatroids are almost entropic when $n = 3$
- For $n \geq 4, \overline{\Gamma_n^*} \subsetneq \Gamma_n$
- For any $\mathbf{h} \in \overline{\Gamma_n^*}$ and $\mathbf{h} \notin \Gamma_n^*$, $\mathbf{h}$ must be on the boundary of $\overline{\Gamma_n^*}$ [F. Matúš, 2007]
- There are in fact infinitely many linear non-Shannon type inequalities [F. Matúš, 2007]

Partition Symmetrical Entropy Functions

- A partition p of $\mathcal{N}$ is a set of nonempty subsets $\{\mathcal{N}_1, \ldots, \mathcal{N}_t\}$ of $\mathcal{N}$ such that distinct blocks $\mathcal{N}_i$ and $\mathcal{N}_j$ are disjoint and $\bigcup_{i=1}^{t} \mathcal{N}_i = \mathcal{N}$
- Induces a permutation group Σ_p which obeys the partition i.e. keeps elements within each block $\mathcal{N}_i$ in the same block
- The group Σ_p can be thought to act on entropy space $\mathcal{H}_n$: For $\sigma \in \Sigma_p$ and for $\mathbf{h} \in \mathcal{H}_n$, $\sigma(\mathbf{h}) = \mathbf{h}(\sigma(\mathcal{A})), \mathcal{A} \subset \mathcal{N}$
- It also induces an action on power set of $\mathcal{H}_n$, i.e. For any $T \subset \mathcal{H}_n$, $\sigma(T) = \{\sigma(\mathbf{h}) : \mathbf{h} \in T\}$
- The set of fixed points of $\mathcal{H}_n$ under this action is $S_p = \{\mathbf{h} \in \mathcal{H}_n : \sigma\mathbf{h} = \mathbf{h}\ \forall \sigma \in \Sigma_p\}$
- $\mathbf{h} \in S_p \Leftrightarrow \mathbf{h}(\mathcal{A}) = \mathbf{h}(\mathcal{B})$ whenever $|\mathcal{A} \cap \mathcal{N}_i| = |\mathcal{B} \cap \mathcal{N}_i|\ \forall i = 0, \ldots, t$

New Regions

- S_p is a subspace of $\mathcal{H}_n$ called the p-symmetrical subspace
- Polymatroids or entropy functions that in S_p are said to be p-symmetrical.
- Region of p-symmetrical polymatroids is denoted by Ψ_p
- Region of p-symmetrical entropy functions is denoted by Ψ_p^*

Results

Theorem 1. $\overline{\Psi_p^*} = \Psi_p$ if and only if p is a 1-partition or a 2-partition with one of its blocks being a singleton when $n \geq 4$

Proof idea:

- **If part:** In aforementioned two cases, entreme rays of Ψ_p can be freely expanded to uniform matroids which are almost entropic.
- **Only if part:** For other partitions p, regions Ψ_p contains polymatroids which when freely expanded give Vámos

Results

Theorem 2: Each facet of Ψ_p corresponds to a p-orbit of facets of Γ_n

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Convex Cones

- A convex set $C \in \mathbb{R}^d$ is called a convex cone if for any $\mathbf{c} \in C$ and $a \geq 0$ we have $a\mathbf{c} \in C$
- *pointed* $\Rightarrow$ doesn't contain a line
- *polyhedral* $\Rightarrow$ intersection of finite set of closed halfspaces
- Γ_n is pointed and polyhedral
- $\overline{\Gamma_n^*}$ is pointed but not polyhedral [F. Matúš, 2007]
- A hyperplane P is called a supporting hyperplane of a convex set C if one of its corresponding closed halfspace $P^+ \supset C$ and $dist(P, C) = 0$
- A *face* of a convex cone $C \subset \mathbb{R}^d$ is the cone C itself or $C \cap P$, where P is a supporting hyperplane of C. A face that is not C or the origin is called a proper face

Convex Cones

- A *maximum proper face* of a convex cone is a proper face which is not contained by any other proper face
- Similarly define *minimum proper face*
- An *extreme ray* R of a convex cone C is a subset of C and for any $\mathbf{r} \in R$ such that $\mathbf{r} = \mathbf{c}_1 + \mathbf{c}_2$ and $\mathbf{c}_1, \mathbf{c}_2 \in C$, we have $\mathbf{c}_1, \mathbf{c}_2 \in R$ where $\mathbf{c}_1 = a\mathbf{r}$ and $\mathbf{c}_2 = (1-a)\mathbf{r}$
- For a polyhedral cone C with $dimC = d'$, where $d' \geq 2$, the maximum proper faces called *facets* are $(d'-1)$ dimensional faces while the minimum proper faces are 1-dimensional faces that are same as extreme rays

Convex Cones

- Any non-origin face of a convex cone is the convex combination of some extreme rays of the cone.
- The intersection of any collection of faces of a convex cone is a face of the cone.
- Any face of a convex cone that is not the cone itself is the intersection of some facets.
- For $\mathbf{c} \in \mathbb{R}^d$, let $B(\mathbf{c}, r)$ be an open ball of radius r centered at $\mathbf{c}$
- For a set $A \subset \mathbb{R}^d$, *relative interior of* A is the set
  $ri(A) \triangleq \{\mathbf{x} \in \mathbb{R}^d \mid \exists \epsilon > 0, B(\mathbf{x}, \epsilon) \cap aff(A) \subset A\}$
  while *relative boundary* is the set
  $rb(A) \triangleq \overline{A} \setminus ri(A)$
- Relative boundary of a polyhedral cone is the union of all its facets

Γ_n

2 types of elemental inequalities give 2 types of facets:

1) $E(i) \triangleq \{\mathbf{h} \in \gamma_n \mid \mathbf{h}(\mathcal{N} \setminus \{i\}) = \mathbf{h}(\mathcal{N})\}$;

2) $E(ij, \mathcal{K}) \triangleq \{\mathbf{h} \in \Gamma_n : \mathbf{h}(\mathcal{K} \cup \{i\}) + \mathbf{h}(\mathcal{K} \cup \{j\}) = \mathbf{h}(\mathcal{K}) + \mathbf{h}(\mathcal{K} \cup \{i, j\})\}$, for distinct $i, j \in \mathcal{N}. \mathcal{K} \subset \mathcal{N} \setminus \{i, j\}$

- The set of all facets of Γ_n is $\mathcal{E}_n$
- Members of $\mathcal{E}_n$ are denoted as $E(\mathcal{I}, \mathcal{K}) : E(i) = E(\{i\}, \emptyset), E(ij, \mathcal{K}) = E(\{i, j\}, \mathcal{K})$

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Matroid

- A matroid is an ordered pair $\{\mathcal{N}, \mathbf{r}\}$, with the *ground set* $\mathcal{N}$ and *rank function* $\mathbf{r}$ satisfying conditions that: for any $\mathcal{A}, \mathcal{B} \subset \mathcal{N}$,
  - $0 \leq \mathbf{r}(\mathcal{A}) \leq |\mathcal{A}|$ and $\mathbf{r}(A) \in \mathbb{Z}$
  - $\mathbf{r}(\mathcal{A}) \leq \mathbf{r}(\mathcal{B})$ if $\mathcal{A} \subseteq \mathcal{B}$
  - $\mathbf{r}(\mathcal{A}) + \mathbf{r}(\mathcal{B}) \geq \mathbf{r}(\mathcal{A} \cup \mathcal{B}) + \mathbf{r}(\mathcal{A} \cap \mathcal{B})$
- The rank function of a matroid $M = \{\mathcal{N}, \mathbf{r}\}$ can be restricted to power set of $\mathcal{N}' \subset \mathcal{N}$ to create *submatroid* $N = \{\mathcal{N}', \mathbf{r}'\}$
- A singleton with rank 0 are called a *loop*
- Matroid = Polymatroids $\cap$ (cardinality bound, integrality)
- Hence any matroid rank function is inside Γ_n
- If a matroid is representable over $\mathbb{F}_q$, it is entropic

Free Expansion

Let $\mathbf{h}$ be an integer valued polymatroid. Consider a set $\mathcal{M}$ with cardinality $m \triangleq \sum_{i \in \mathcal{N}} \mathbf{h}(\{i\})$ and any mapping $\phi : \mathcal{N} \to 2^{\mathcal{M}}$ such that $\phi(i)$ has the cardinality $\mathbf{h}(\{i\}), i \in \mathcal{N}$ and $\phi(i) \cap \phi(j) = \emptyset$ for $i \neq j$. Then the *free expansion* $\mathbf{g} \in \Gamma_m$ of $\mathbf{h}$ by ϕ is defined by:

$$\mathbf{g}(\mathcal{A}) = \min_{\mathcal{B} \subset \mathcal{N}} (\mathbf{h}(\mathcal{B}) + |\mathcal{A} \setminus \phi(\mathcal{B})|), \mathcal{A} \subset \mathcal{M}$$

- It is said that $\mathbf{g}$ *factors* to $\mathbf{h}$ under ϕ or $\mathbf{h}$ is a factor of $\mathbf{g}$
- $\mathbf{g}$ so created is an integer valued polymatroid and in fact a matroid [Nguyen, 1978]
- Each singleton i in $\mathcal{N}$ is split into $\mathbf{h}(\{i\})$ independent parts in $\mathcal{M}$

Free Expansion

Claim: For any set $\mathcal{A} \subset \mathcal{M}$, if $\mathcal{A} = \phi(\mathcal{B}')$ then $\mathbf{g}(\mathcal{A}) = \mathbf{h}(\mathcal{B}')$

- Consider any $\mathcal{A} \subset \mathcal{M}$ s.t. $\mathcal{A} = \phi(\mathcal{B}')$ for some $\mathcal{B}' \subset \mathcal{N}$
- Then for all $\mathcal{B} \subset \mathcal{N}$,

$$\begin{aligned}
\mathbf{h}(\mathcal{B}) + |\mathcal{A} \setminus \phi(\mathcal{B})| &= \mathbf{h}(\mathcal{B}) + |\phi(\mathcal{B}') \setminus \phi(\mathcal{B})| \\
&= \mathbf{h}(\mathcal{B}) + |\phi(\mathcal{B}' \setminus \mathcal{B})| \\
&= \mathbf{h}(\mathcal{B}) + \sum_{i \in \mathcal{B}' \setminus \mathcal{B}} \mathbf{h}(\{i\}) \\
&\geq \mathbf{h}(\mathcal{B}) + \mathbf{h}(\mathcal{B}' \setminus \mathcal{B}) \\
&\geq \mathbf{h}(\mathcal{B} \cup \mathcal{B}') \\
&\geq \mathbf{h}(\mathcal{B}') \qquad \blacksquare
\end{aligned}$$

We can define $\mathbf{h}(\mathcal{B}) = \mathbf{g}(\phi(\mathcal{B})), \mathcal{B} \subset \mathcal{N}$ to be the *inverse* of free expansion

Free expansion as a tool to prove that a polymatroid is almost entropic

Theorem [F. Matúš, 2007] An integer valued polymatroid is entropic iff its free expansion is entropic

Uniform matroid

- A uniform matroid $U_{m,n}$ is a matroid with ground set $\mathcal{N}$ and rank function $U_{m,n}(\mathcal{A}) = min\{m, |\mathcal{A}|\}$, $\forall A \subset \mathcal{N}$
- Every uniform matroid is representable over some field. Hence, almost entropic.
- Uniform matroid + loops is still almost entropic
- Factors of uniform matroid are almost entropic

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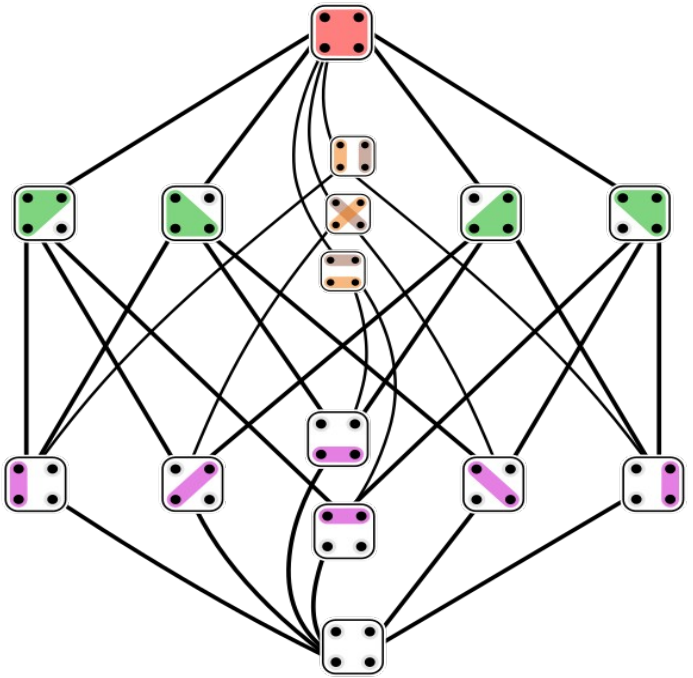
Partitions

- For a set $\mathcal{S}$ and an index set $\mathcal{I}$, a collection $p = \{\mathcal{S}_i, i \in \mathcal{I}\}$ of disjoint subsets of $\mathcal{S}$ such that $\mathcal{S} = \cup_{i \in \mathcal{I}} \mathcal{S}_i$ and $\mathcal{S}_i \neq \emptyset$ is called **partition** of set $\mathcal{S}$



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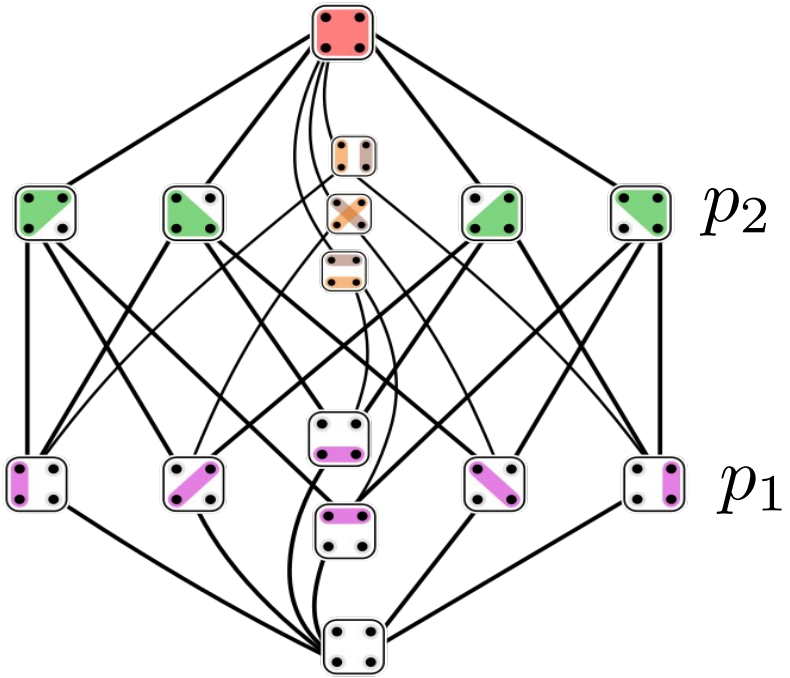

$p_1 \leq p_2$

p_1 is *refinement* of p_2

p_1 is *finer than* p_2

p_2 is *coarser than* p_1

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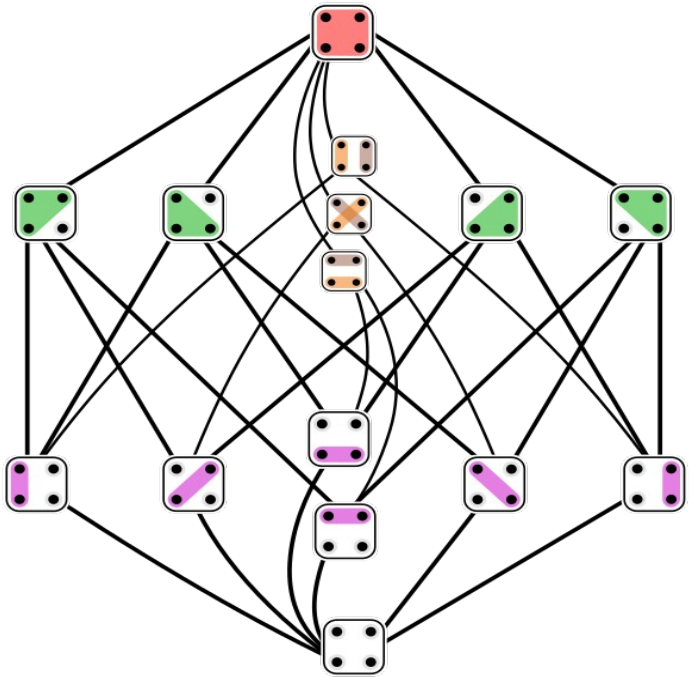

Number of ways to partition a set of n labeled elements: Bell Numbers

```
                1
              1   2
            2   3   5
          5   7   10  15
        15  20  27  37  52
      52  67  87  114 151 203
    203 255 322 409 523 674 877
```

OEIS A000110

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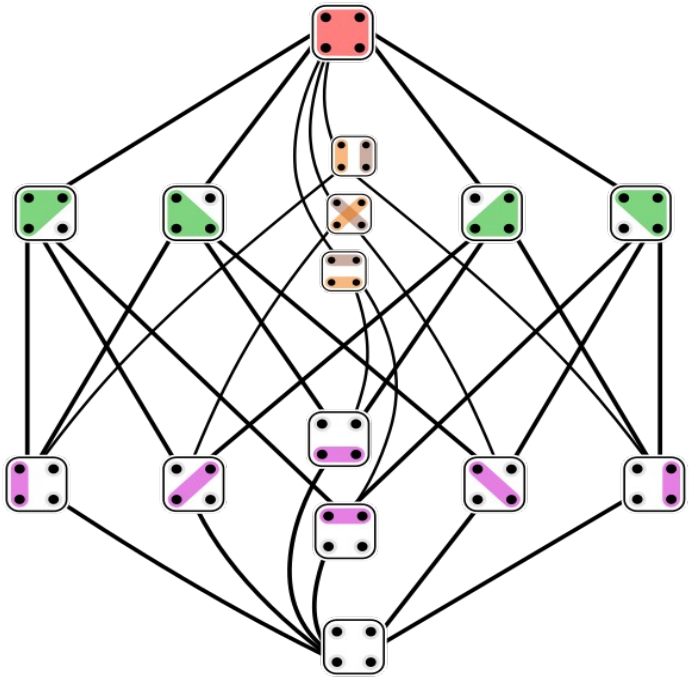
Partitions

- For a set $\mathcal{S}$ and an index set $\mathcal{I}$, a collection $p = \{\mathcal{S}_i, i \in \mathcal{I}\}$ of disjoint subsets of $\mathcal{S}$ such that $\mathcal{S} = \cup_{i \in \mathcal{I}} \mathcal{S}_i$ and $\mathcal{S}_i \neq \emptyset$ is called **partition** of set $\mathcal{S}$



Number of ways to k-partition a set of n labeled elements:
Stirling Numbers of the second kind
$S_2(n,k), n \geq 1, 1 \leq k \leq n$

```
                1
              1   1
            1   3   1
          1   7   6   1
        1   15  25  10  1
      1   31  90  65  15  1
    1   63  301 350 140 21  1
```

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Partitions

- Denote the set of all t-partitions of $\mathcal{N}$ by $\mathcal{P}_{t,n}$
- Denote the set of all partitions of $\mathcal{N}$ by $\mathcal{P}_n$
- For $p \in \mathcal{P}_{t,n}$ and $\mathcal{A} \subset \mathcal{N}$, we call $\lambda_{\mathcal{A},p} \triangleq (|\mathcal{A} \cap \mathcal{N}_1|, \ldots, |\mathcal{A} \cap \mathcal{N}_t|)$ the *partition vector* of $\mathcal{A}$ under p
- Denote ith entry of this vector by $\lambda_{\mathcal{A},p}(i) = |\mathcal{A} \cap \mathcal{N}|$
- Partitions p and p' are equivalent if λ_p can be obtained by permuting $\lambda_{p'}$
- For a partition p with vector λ_p we consider non-decreasing rearrangement of λ_p to be the representative of the equivalence class

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Permutation groups from orbits(partition)

- A *permutation* of $\mathcal{N}$ is a bijection $\sigma : \mathcal{N} \to \mathcal{N}$
- The set Σ_n of all permutations of $\mathcal{N}$ with composition as operator is a group of order $n!$
- Given a partition $p \in \mathcal{P}_n$ we can define a subgroup of Σ_n

$$\Sigma_p = \{\sigma \in \Sigma_n : \sigma(j) \in \mathcal{N}_i, j \in \mathcal{N}_i, i = 1, \ldots, t\}$$

- This group keeps members of a block within a same block
- Order=$\prod_{i=1}^{t} n_i!$

Action of Σ_p on $\mathcal{H}_n$

- Brief reboot: A group Σ acts on a set $\mathcal{S}$ if there exists a function $\Sigma \times \mathcal{S} \to \mathcal{S}$ called an action, denoted by $(\sigma, s) \mapsto \sigma s$, such that:

  1) $(\sigma_1 \sigma_2)s = \sigma_1(\sigma_2 s)\ \forall \sigma_1, \sigma_2 \in \Sigma$ and $s \in \mathcal{S}$
  2) $1s = s\ \forall s \in \mathcal{S}$, where 1 is identity of Σ

- For any $\sigma \in \Sigma_n$ we define $\sigma' : \mathcal{H}_n \to \mathcal{H}_n$ as:

$$\sigma'(\mathbf{h})(\mathcal{A}) = \mathbf{h}(\sigma(\mathcal{A})),\ \mathcal{A} \subset \mathcal{N}$$

- $\sigma : 2^{\mathcal{N}} \to 2^{\mathcal{N}}$, defined as $\sigma(\mathcal{A}) = \{\sigma(i) : i \in \mathcal{A}\}$ is a bijection and hence permutes $2^{\mathcal{N}}$

- $\sigma \xrightarrow{\text{induced action}}$ permutation of $2^{\mathcal{N}} \xrightarrow{\text{induced action}}$ permutation of $\mathcal{H}_n$

Orbits and fixed set under action

- If a group Σ acts on $\mathcal{S}$, then for all $s \in \mathcal{S}$, orbit of s under action of Σ is:

$$\mathcal{O}_\Sigma(s) = \{\sigma s : \sigma \in \Sigma\}$$

- The set of orbits is a partition of $\mathcal{S}$
- Based on the induced action of Σ_p on $\mathcal{H}_n$ we can define orbits of $\mathcal{H}_n$ as $\mathcal{O}_{\Sigma_p}(\mathbf{h})$ or $\mathcal{O}_p$ or a *p-orbit*
- If $\mathbf{h}_1, \mathbf{h}_2$ are in same p-orbit they are p-equivalent
- The fixed set of $\mathcal{T} \subset \mathcal{S}$ under action of of Σ is:

$$\text{fix}_\Sigma(\mathcal{T}) = \{s \in \mathcal{T} : \sigma s = s, \forall \sigma \in \Sigma\}$$

Orbits and fixed set under action

- For $p = \{\mathcal{N}_1, \ldots, \mathcal{N}_t\}$, fixed space of $\mathcal{H}_n$ is:

$$\text{fix}_p(\mathcal{H}_n) = \{\mathbf{h} \in \mathcal{H}_n : \mathbf{h}(\mathcal{A}) = \mathbf{h}(\mathcal{B}) \text{ if } \lambda_{\mathcal{A},p} = \lambda_{\mathcal{B},p}\}$$

- Notice the use of word space
- $\text{fix}_p(\mathcal{H}_n)$ is $\mathcal{H}_n \cap \{\text{linear equalities }\} \Rightarrow$ subspace of $\mathcal{H}_n$
- For a vector to remain fixed, the subsets of $\mathcal{N}$ that can be mapped to each other under induced action on $2^{\mathcal{N}}$ must have same entropy
- Subsets $\mathcal{S}$ of $\mathcal{N}$ with same $\boldsymbol{\lambda}_{\mathcal{S},p}$ are exactly the sets with same entropy
- Call $\text{fix}_p(\mathcal{H}_n)$ the *p-symmetrical subspace*

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Set-wise orbits and invariance

- $\sigma \xrightarrow[\text{action}]{\text{induced}} \text{perm } 2^{\mathcal{N}} \xrightarrow[\text{action}]{\text{induced}} \text{perm } \mathcal{H}_n \xrightarrow[\text{action}]{\text{induced}} \text{perm } 2^{\mathcal{H}_n}$
- *Setwise orbit* of $\mathcal{T} \subset \mathcal{S}$ under action of Σ is the set of subsets:

$$\mathcal{O}(\mathcal{T}) = \{\sigma\mathcal{T} : \sigma \in \Sigma\}$$

- Call orbits of elements of $\mathcal{S}$ the *pointwise orbits*
- We shall denote both setwise and pointwise orbits of $\mathcal{H}_n$ as p-orbits and sets belonging to same p-orbit as p-equivalent
- If a group Σ acts on a set $\mathcal{S}$, a subset $\mathcal{T} \subset \mathcal{S}$ is called *invariant* if:

$$\mathcal{T} = \sigma(\mathcal{T}) \text{ for any } \sigma \in \Sigma$$

Set-wise orbits and invariance

1. If $\mathcal{T}$ is invariant, then $\mathcal{T} = \sigma(\mathcal{T}) = \{\sigma(s) : s \in \mathcal{T}\} = \cup_{s\in\mathcal{T}}\sigma(s)$
   i.e. it is union of pointwise orbits

2. Subsets of $\mathrm{fix}_\Sigma(\mathcal{S})$ are setwise invariant

- Call $T \subset \mathcal{H}_n$ p-invariant if it is invariant under action of Σ_p
- $\mathbf{h} \in \mathcal{H}_n$ is p-invariant $\iff$ $\mathbf{h} \in S_p$
- $T \subset \mathcal{S}_p \implies T$ is p-invariant
- Γ_n, Γ_n^* and $\overline{\Gamma_n^*}$ are all p-invariant for any $p \in \mathcal{P}_n$
  i.e. they map to themselves under induced action

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Dimension of fixed space

- For $p \in \mathcal{P}_n$ the p-symmetrical subspace is:

$$S_p = \{\mathbf{h} \in \mathcal{H}_n : \mathbf{h}(\mathcal{A}) = \mathbf{h}(\mathcal{B}) \text{ if } \boldsymbol{\lambda}_{\mathcal{A},p} = \boldsymbol{\lambda}_{\mathcal{B},p}\}$$

- $\dim S_p =$ Number of different $\boldsymbol{\lambda}$ vectors
- For any $\mathcal{A} \subset \mathcal{N}, \boldsymbol{\lambda}_{\mathcal{A},p}(i) = |\mathcal{A} \cap \mathcal{N}_i|$ can be $0, 1, \ldots, n_i$
- $(n_i + 1)$ possible values for each $i = 1, \ldots, t$
- $\dim S_p = n_p \triangleq \prod_{i=1}^{t}(n_i + 1)$
- S_p is isomorphic to $\mathbb{R}^{n_p}$
- How many sets $\mathcal{A} \subset \mathcal{N}$ with same $\boldsymbol{\lambda}_{\mathcal{A},p}$? $\prod_{i=1}^{t} \binom{n_i}{k_i}$

Dimension of fixed space

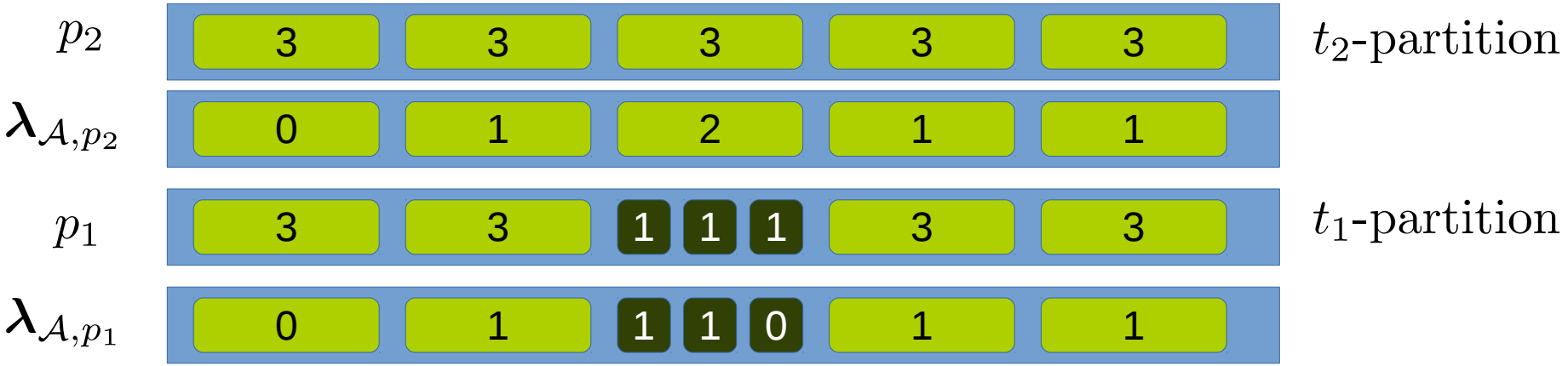
- For $p \in \mathcal{P}_n$ the p-symmetrical subspace is:

$$S_p = \{\mathbf{h} \in \mathcal{H}_n : \mathbf{h}(\mathcal{A}) = \mathbf{h}(\mathcal{B}) \text{ if } \boldsymbol{\lambda}_{\mathcal{A},p} = \boldsymbol{\lambda}_{\mathcal{B},p}\}$$

- Identify each co-ordinate of S_p with its $\boldsymbol{\lambda}$ vector
- So for every $\mathbf{h} \in S_p$, let $\mathbf{s}(\mathbf{h}, p) = (s_{k_1,\ldots,k_t})_{k_1,\ldots,k_t \in \mathcal{N}_p}$ (where $\mathcal{N}_p = \{(k_1, \ldots, k_t) : k_i \in \{0, \ldots, n_i\}, i \in 1, \ldots, t\}$)

Partition refinement and containment relationships of fixed spaces

- Consider $p_1, p_2 \in \mathcal{P}_n$ such that $p_1 \leq p_2$ i.e. p_1 is refinement of p_2
- $p_1 = \{\mathcal{N}_1^{(1)}, \ldots, \mathcal{N}_{t_1}^{(1)}\}$, $p_2 = \{\mathcal{N}_1^{(2)}, \ldots, \mathcal{N}_{t_2}^{(2)}\}$



$$\lambda_{\mathcal{A},p_2}(i) = |\mathcal{A} \cap \mathcal{N}_i^{(2)}| = |\mathcal{A} \cap (\cup_{j \in \mathcal{J}_i} \mathcal{N}_j^{(1)})| = \sum_{j \in \mathcal{J}_i} |\mathcal{A} \cap \mathcal{N}_j^{(1)}| = \sum_{j \in \mathcal{J}_i} \lambda_{\mathcal{A},p_1}(j)$$

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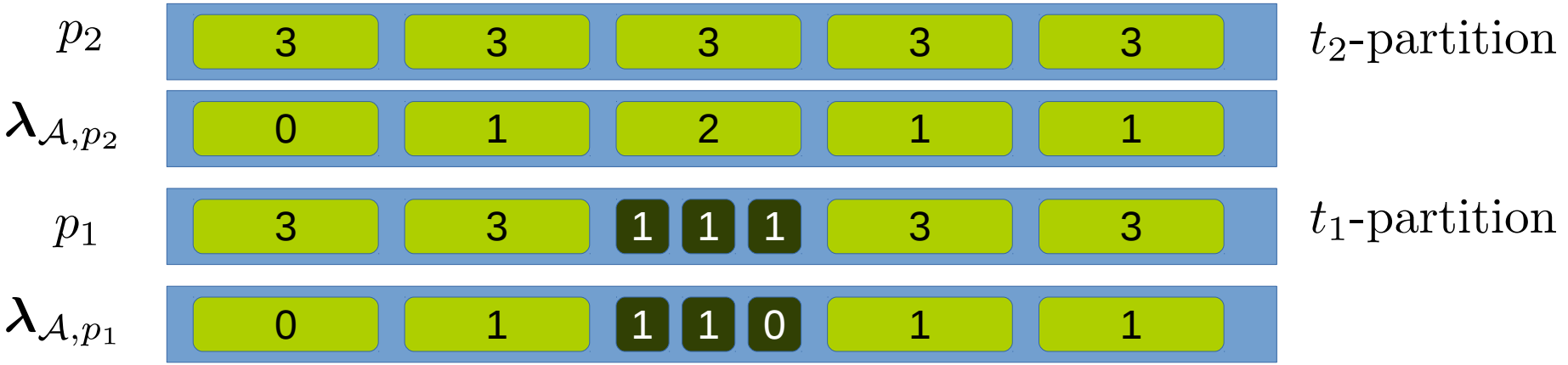

$$\mathcal{A}, \mathcal{B} \text{ have same } \boldsymbol{\lambda}_{\cdot,p_1} \implies \mathcal{A}, \mathcal{B} \text{ have same } \boldsymbol{\lambda}_{\cdot,p_2}$$

Partition refinement and containment relationships of fixed spaces

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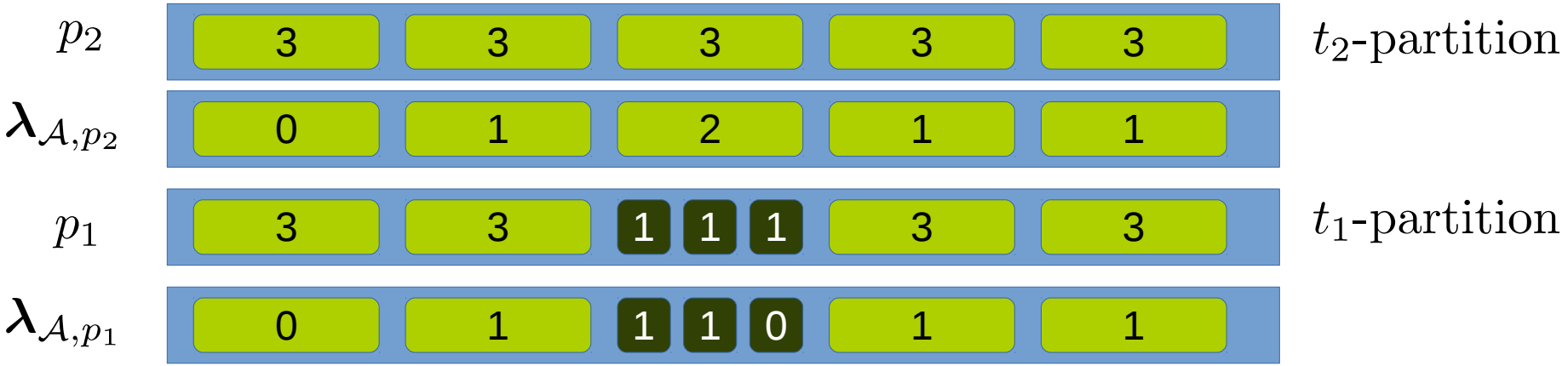

$$\mathbf{h} \in S_{p_2} \implies \mathbf{h} \in S_{p_1}$$

Partition refinement and containment relationships of fixed spaces

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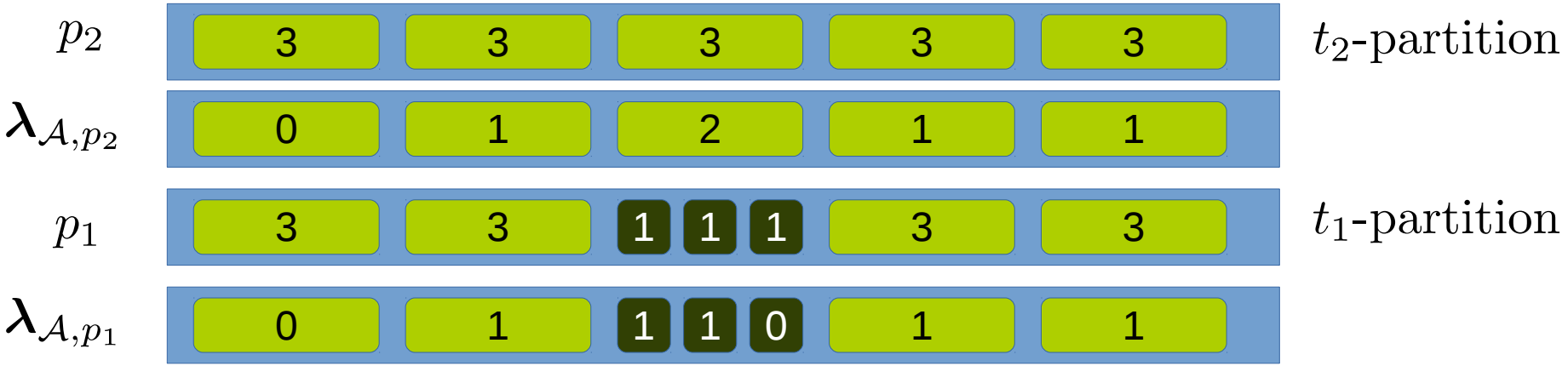
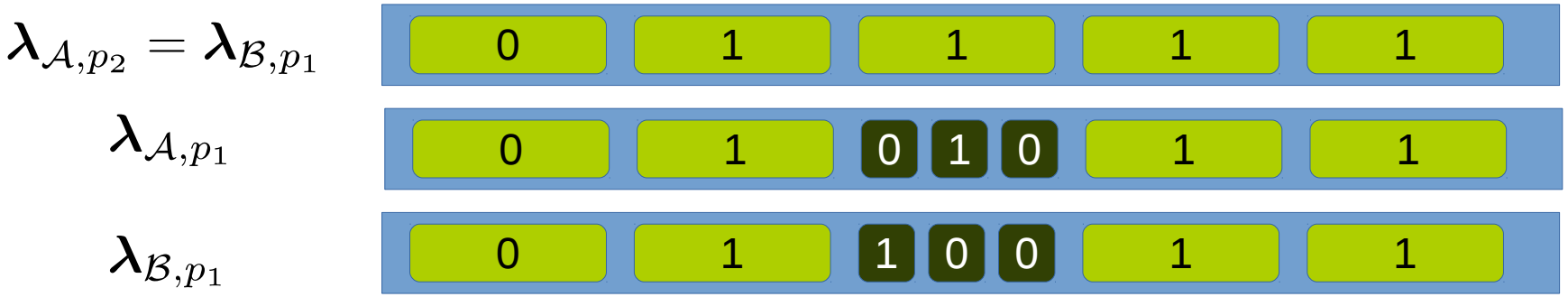


$$\mathbf{h} \in S_{p_2} \implies \mathbf{h} \in S_{p_1}$$

- **Claim**: $p_1 \leq p_2 \implies S_{p_2} \subset S_{p_1}$

Partition refinement and containment relationships of fixed spaces

- **Claim**: $p_1 < p_2 \implies S_{p_2} \subsetneq S_{p_1}$
- $p_1 = \{\mathcal{N}_1^{(1)}, \ldots, \mathcal{N}_{t_1}^{(1)}\}$, $p_2 = \{\mathcal{N}_1^{(2)}, \ldots, \mathcal{N}_{t_2}^{(2)}\}$



$$\mathbf{h}(\mathcal{A}) = \mathbf{h}(\mathcal{B}) \text{ in } p_2, \mathbf{h}(\mathcal{A}) \neq \mathbf{h}(\mathcal{B}) \text{ in } p_1$$

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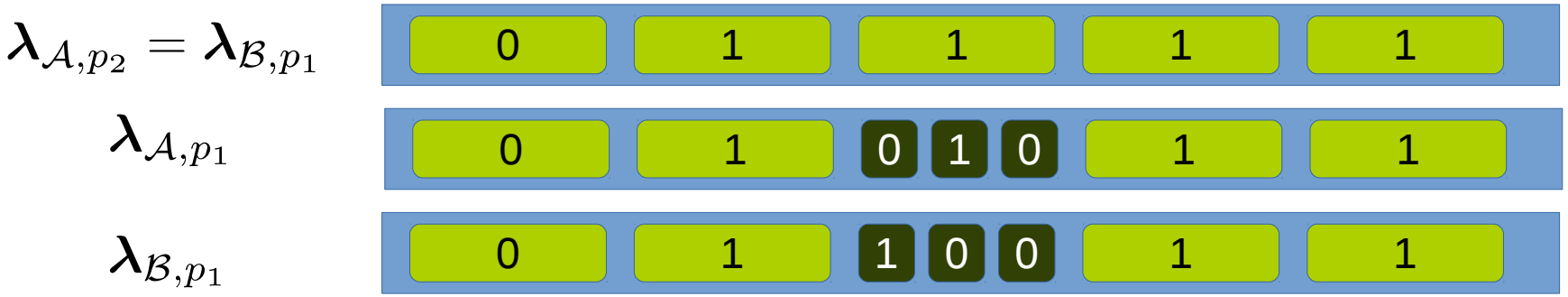

$$\mathbf{h}(\mathcal{A}) = \mathbf{h}(\mathcal{B}) \text{ in } p_2, \mathbf{h}(\mathcal{A}) \neq \mathbf{h}(\mathcal{B}) \text{ in } p_1$$

$$QED$$

New Regions

- The *p-symmetrical entropy function region*:

$$\Psi_p^* \triangleq \mathrm{fix}_p(\Gamma_n^*) = \Gamma_n^* \cap S_p$$

- The *p-symmetrical polymatroidal region*:

$$\Psi_p \triangleq \mathrm{fix}_p(\Gamma_n) = \Gamma_N \cap S_p$$

- $\Psi_p, \Psi_p^* \subset S_p^0 \triangleq \{\mathbf{h} \in S_p : \mathbf{h}(\emptyset) = 0\}$

Equivalence classes of partitions and regions

- $\mathbf{n}_p$ is the vector of sizes of blocks of p in non-decreasing manner
- We consider two partitions with same $\mathbf{n}_p$ to be equivalent
- Assume an order on $\mathcal{N}$
- Define $\mathcal{P}^*_{t,n}$ to be the set of those partitions in $\mathcal{P}_{t,n}$ for which $\mathcal{N}_i$ and $\mathcal{N}_j, i < j$ are two blocks of the partition then $k < l$ for any $k \in \mathcal{N}_i, l \in \mathcal{N}_j$
- $\mathcal{P}^*_{t,n}$ is the set of representative partitions
- **Claim**: The regions Ψ_p and Ψ^*_p are isomorphic for equivalent partitions

Symmetrical Properties of Γ_n and Γ_n^*

- p-equivalent faces and equivalent facets of Γ_n
- Orbits of facets of Γ_N (How many?)
- Orbits of facets of Γ_N and facets of Ψ_p
- Average entropy function and relation to barycenters of the orbits

Lemma 1: For $p \in \mathcal{P}_n$, let $T_1, T_2 \subset \mathcal{H}_n$ be p-equivalent. Then T_1 is a face (facet) of Γ_n iff T_2 is a face(facet) of Γ_n

Proof:

- For each face F of Γ_n, $F = \Gamma_n \cap P$ for some supporting hyperplane P
- Let $T_1 = \Gamma_n \cap P$
- For any $\sigma \in \Sigma_p, \sigma(T_1) = \sigma(P \cap \Gamma_n) = \sigma(P) \cap \sigma(\Gamma_n) = \sigma(P) \cap \Gamma_n$
- $\sigma(P)$ is also a supporting hyperplane of Γ_n
- Opposite direction: Use σ^{-1}

Implication: Facet maps to facet

Lemma 2: For $p \in \mathcal{P}_n$ and $E_i \triangleq E(\mathcal{I}_i, \mathcal{K}_i), i = 1, 2$, the following statements are equivalent:

1) E_1 and E_2 are p-equivalent;

2) there exists $\sigma \in \Sigma_p$ such that $\mathcal{I}_1 = \sigma(\mathcal{I}_2)$ and $\mathcal{K}_1 = \sigma(\mathcal{K}_2)$;

3) $\boldsymbol{\lambda}_{\mathcal{I}_1,p} = \boldsymbol{\lambda}_{\mathcal{I}_2,p}$ and $\boldsymbol{\lambda}_{\mathcal{K}_1,p} = \boldsymbol{\lambda}_{\mathcal{K}_2,p}$

Proof: **Claim**: (1) $\iff$ (2)

- If E_1 and E_2 are equivalent, $\exists\, \sigma \in \Sigma_p$ s.t. $\sigma(E_1) = E_2$
- **Case** $|\mathcal{I}_1| = 1$ i.e. $E_1 = E(i)$ and $\mathcal{I}_1 = \{i\}$ for some $i \in \mathcal{N}$

$$
\begin{aligned}
E_2 &= \{\mathbf{h} \in \Gamma_n : \sigma^{-1}(\mathbf{h})(\mathcal{N} \setminus \{i\})\} \\
&= \{\mathbf{h} \in \Gamma_n : \mathbf{h}(\sigma^{-1}(\mathcal{N})) = \mathbf{h}(\sigma^{-1}(\mathcal{N} \setminus \{i\}))\} \\
&= \{\mathbf{h} \in \Gamma_n : \mathbf{h}(\mathcal{N}) = \mathbf{h}(\mathcal{N} \setminus \sigma^{-1}(\{i\}))\}
\end{aligned}
$$

Hence E_2 is $E(\sigma^{-1}(\{i\}), \emptyset)$

Lemma 2: For $p \in \mathcal{O}_n$ and $E_i \triangleq E(\mathcal{I}_i, \mathcal{K}_i), i = 1, 2$, the following statements are equivalent:

1) E_1 and E_2 are p-equivalent;

2) there exists $\sigma \in \Sigma_p$ such that $\mathcal{I}_1 = \sigma(\mathcal{I}_2)$ and $\mathcal{K}_1 = \sigma(\mathcal{K}_2)$;

3) $\boldsymbol{\lambda}_{\mathcal{I}_1,p} = \boldsymbol{\lambda}_{\mathcal{I}_2,p}$ and $\boldsymbol{\lambda}_{\mathcal{K}_1,p} = \boldsymbol{\lambda}_{\mathcal{K}_2,p}$

Proof: **Claim**: (1) $\iff$ (2)

- If E_1 and E_2 are equivalent, $\exists\, \sigma \in \Sigma_p$ s.t. $\sigma(E_1) = E_2$
- **Case** $|\mathcal{I}_1| = 2$ i.e. $E_1 = E(ij, \mathcal{K})$ and $\mathcal{I}_1 = \{i, j\}$,$i, j \in \mathcal{N}$

$$
\begin{aligned}
E_2 = \{\mathbf{h} \in \Gamma_n : \sigma^{-1}(\mathbf{h})(\{i\} \cup \mathcal{K}_1) + \sigma^{-1}(\mathbf{h})(\{j\} \cup \mathcal{K}_1) \\
= \sigma^{-1}(\mathbf{h})(\mathcal{K}_1) + \sigma^{-1}(\mathbf{h})(\{ij\} \cup \mathcal{K}_1)\} \\
= \{\mathbf{h} \in \Gamma_n : \mathbf{h}(\sigma^{-1}(\{i\}) \cup \sigma^{-1}(\mathcal{K}_1)) + \mathbf{h}(\sigma^{-1}(\{j\} \cup \sigma^{-1}(\mathcal{K}_1))) \\
= \mathbf{h}(\sigma^{-1}(\mathcal{K}_1)) + \mathbf{h}(\sigma^{-1}(\{ij\}) \cup \sigma^{-1}(\mathcal{K}_1))
\end{aligned}
$$

Hence E_2 is $E(\sigma^{-1}(\{ij\}), \sigma^{-1}(\mathcal{K}_1))$

Lemma 2: For $p \in \mathcal{O}_n$ and $E_i \triangleq E(\mathcal{I}_i, \mathcal{K}_i), i = 1, 2$, the following statements are equivalent:

1) E_1 and E_2 are p-equivalent;

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Proof: **Claim**: (2) $\iff$ (3)

- (2) $\implies$ $\forall$ blocks $\mathcal{N}_l$ of p, $|\mathcal{I}_1 \cap \mathcal{N}_l| = |\sigma(\mathcal{I}_2 \cap \mathcal{N}_l) = |\mathcal{I}_2 \cap \mathcal{N}_l|$ (σ is a bijection that respects p) $\implies$ (3)

- (3) $\implies$ $\exists \sigma \in \Sigma_p$ such that $\mathcal{I}_1 \cap \mathcal{N}_l = \sigma(\mathcal{I}_2 \cap \mathcal{N}_l)$ and $\mathcal{K}_1 \cap \mathcal{N}_l = \sigma(\mathcal{K}_2 \cap \mathcal{N}_l)$ $\implies$ (2)

$$\text{Lemma 2} \implies |\mathfrak{R}_p| = |\mathfrak{E}_p|$$

where, $\mathfrak{R}_p$ is set of all distinct $\boldsymbol{\lambda}_{\mathcal{I},p}, \boldsymbol{\lambda}_{\mathcal{K},p}$ and $\mathfrak{E}_p = \{\mathcal{E}_p(\boldsymbol{\lambda}) : \boldsymbol{\lambda} \in \mathfrak{R}_p\}$

Example

$$p = [n]$$

Type 1 facets

$$E(i) = E(\{i\}, \emptyset)$$
$$[\boldsymbol{\lambda}_{\mathcal{I},p}, \boldsymbol{\lambda}_{\mathcal{K},p}] = [(1), (0)]$$

Type 2 facets

$$[\boldsymbol{\lambda}_{\mathcal{I},p}, \boldsymbol{\lambda}_{\mathcal{K},p}] = [(2), (k)]$$
$$\text{for } i, j \in \mathcal{N}, \mathcal{K} \subset \mathcal{N} \setminus \{i, j\}$$
$$\text{where } k \triangleq |K|$$

Everybody in same orbit!

One orbit per $k \in \{0, 1, \ldots, n-2\}$

$$\mathfrak{R}_{[n]} = \{[(1), (0)]\} \cup \{[(2), (k)] : k = 0, \ldots, n-2\}$$
$$\mathfrak{E}_{[n]} = \{\mathcal{E}_{[n]}(\boldsymbol{\lambda}) : \boldsymbol{\lambda} \in \mathfrak{R}_{[n]}\}$$

Lemma 3: If facets E_1 and E_2 are equivalent then $E_1 \cap S_p = E_2 \cap S_p$

Proof:

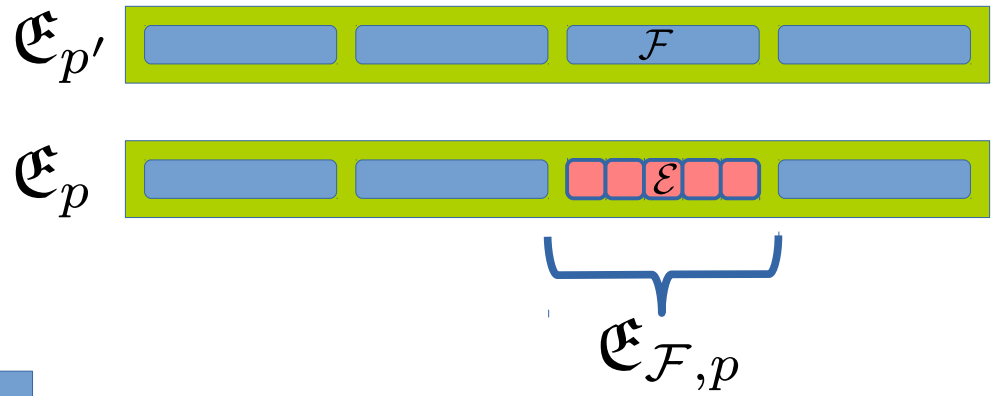
- $E_1 \equiv E_2 \implies \exists \sigma \in \Sigma_p$ s.t. $E_2 = \sigma(E_1)$.
- $E_2 \cap S_p = \sigma(E_1) \cap S_p = \sigma(E_1) \cap \sigma(S_p) = \sigma(E_1 \cap S_p) = E_1 \cap S_p$

Lemma 4: $E_1 \not\equiv E_2 \implies E_1 \cap S_p \not\subset E_2 \cap S_p$ and $\mathrm{E}_2 \cap S_p \not\subset E_1 \cap S_p$

Sketch of proof:

Lemma 10:
[Partition refinement leads to refinement of orbits of facets]
For $p, p' \in \mathcal{P}_n$ if $p \leq p'$, then $\mathfrak{E}_p \leq \mathfrak{E}_{p'}$

Isolation : $p \leq p'$



Lemma 14: $p, p' \in \mathcal{P}_n$ and $p \leq p'$. For any $\mathcal{E}_{p'}(\boldsymbol{\lambda}') \in \mathfrak{E}_{p'}$ with $\boldsymbol{\lambda}' \in \mathfrak{R}_{p'}$, for each $\boldsymbol{\lambda} \in \mathfrak{R}_p$, $\mathcal{E}_p(\boldsymbol{\lambda}) \in \mathfrak{E}_{\mathcal{E}_{p'}(\boldsymbol{\lambda}'),p}$ has an isolation in $\mathfrak{E}_{\mathcal{E}_{p'}(\boldsymbol{\lambda}'),p}$

$\mathbf{i} \in \Psi_p$ is *isolation* of $\mathcal{E} \in \mathfrak{E}_{\mathcal{F},p}$ if for any $E \in \mathcal{E}, \mathbf{i} \notin E \cap S_p$ but $\forall E' \in \mathcal{F} \setminus \mathcal{E}, \mathbf{i} \in E' \cap S_p$

- Prove Lemma 4 by induction on the size of covering partition p'
- In the base case, $p' = p_0$ and $p = \{\mathcal{N}\}$. For any $\boldsymbol{\lambda} \in \mathfrak{R}_n, \mathcal{E}_n(\boldsymbol{\lambda})$ has isolation $U_{k,n}$ in $\mathfrak{E}_n$
- Assume true for $p' \in \mathcal{P}_{t-1,n}$. Let $p \leq p', p \in \mathcal{P}_{t,n}$

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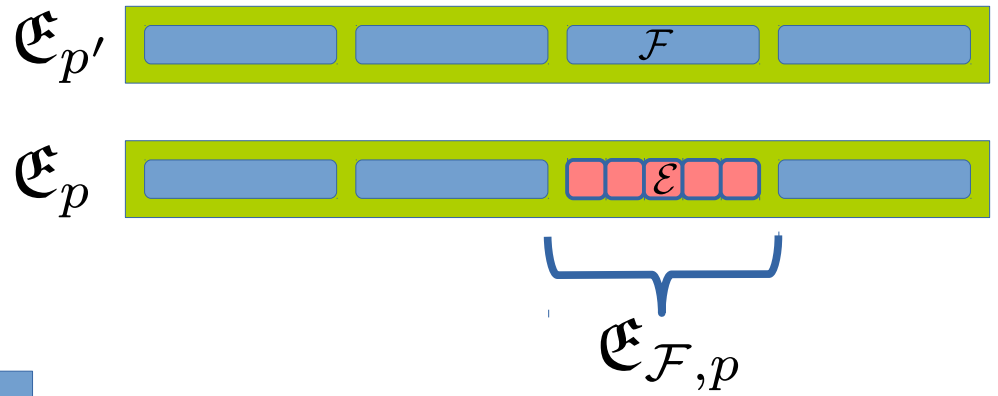

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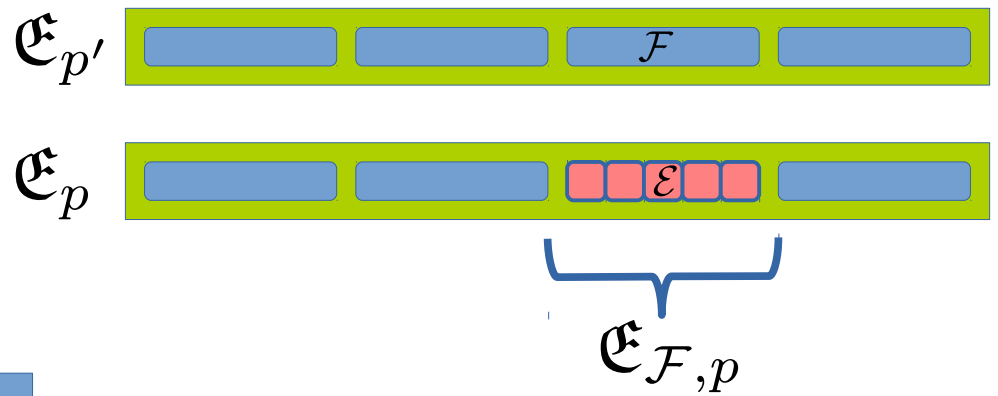

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Case I:

$E_1 \overset{\text{p}}{\not\equiv} E_2$ and $E_1 \overset{\text{p'}}{\not\equiv} E_2$

- Induction hyopthes $\implies E_1 \cap S_{p'} \not\subset E_2 \cap S_{p'}$
- $p \leq p' \implies S_{p'} \cap S_p$
- Now $E_1 \cap S_p \subset E_2 \cap S_p \implies E_1 \cap S_p \cap S_{p'} \subset E_2 \cap S_p \cap S_{p'} \implies \Rightarrow\Leftarrow$
- Similarly prove $E_2 \cap S_p \not\subset E_1 \cap S_p$

Lemma 4: $E_1 \not\equiv E_2 \implies E_1 \cap S_p \not\subset E_2 \cap S_p$ and $\mathrm{E}_2 \cap S_p \not\subset E_1 \cap S_p$

Sketch of proof:

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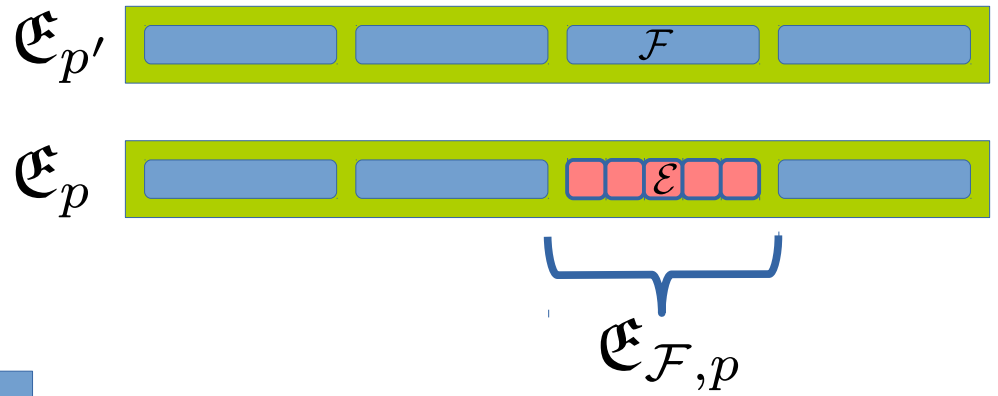

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Case I:

$E_1 \overset{\mathrm{p}}{\not\equiv} E_2$ and $E_1 \overset{\mathrm{p'}}{\equiv} E_2$

- Follows from Lemma 14

- $\mathcal{G}_p \triangleq$ set of all facets of Ψ_p

Lemma 5: For any facet $G \in \mathcal{G}_p$ of Ψ_p, $\exists$ facet $E \in \mathcal{E}_n$ s.t. $G \subset E$

Proof:

Claim: $rb(\Psi_p) = rb(\Gamma_n)$

- Assume $rb(\Psi_p) \not\subset rb(\Gamma_n) \implies \exists\, \mathbf{h} \in rb(\Psi_p)$ s.t. $\mathbf{h} \notin rb(\Gamma_n)$
  (since $\Gamma_n = ri(\Gamma_n) \dot{\cup} rb(\Gamma_n)$ and $\Psi_p \subset \Gamma_n$)

$\implies$ $\mathbf{h} \in ri(\Gamma_n)$

$\implies$ $\exists \epsilon > 0$ s.t. $B(\mathbf{h}, \epsilon) \cap S_p^0 \subset B(\mathbf{h}, \epsilon) \cap \mathcal{H}_n^0 \subset ri(\Gamma_n) \subset \Gamma_n$

$\implies$ $B(\mathbf{h}, \epsilon) \cap S_p^0 \subset \Gamma_n \cap S_p^0 = \Psi_p$

- But $\mathbf{h} \in rb(\Psi_p) \subset rb(\Psi_p) \implies B(\mathbf{h}, \epsilon) \cap S_p^0 \not\subset \Psi_p \Rightarrow\Leftarrow$

- $\mathcal{G}_p \triangleq$ set of all facets of Ψ_p

Lemma 5: For any facet $G \in \mathcal{G}_p$ of Ψ_p, $\exists$ facet $E \in \mathcal{E}_n$ s.t. $G \subset E$

Proof:

- $rb(\Psi_p) \subset rb(\Gamma_n)$

Claim: For a facet $G \in \mathcal{G}_p$ and any $E \in \mathcal{E}_n$, $G \cap E$ is a face of G
Proof:

- $rb(\Gamma_n) = \cup_{E \in \mathcal{E}_n} E$ and $rb(\Psi_p) = \cup_{G \in \mathcal{G}_p} G$
- $rb(\Psi_p) \subset rb(\Gamma_n) \implies G \subset rb(\Gamma_n) \; \forall \; G \in \mathcal{G}_n$
- P is supporting hyperplane of Γ_n s.t. $E = P \cap \Gamma_n$
- $\mathbf{0} \in G \cap P$ and $P^+ \supset \Gamma_n \supset G \implies P$ is supporting hyperplane of G
- $G \subset \Gamma_n \implies G \cap P = (G \cap \Gamma_n) \cap P = G \cap (\Gamma_n \cap P) = G \cap E \blacksquare$

- $\mathcal{G}_p \triangleq$ set of all facets of Ψ_p

Lemma 5: For any facet $G \in \mathcal{G}_p$ of Ψ_p, $\exists$ facet $E \in \mathcal{E}_n$ s.t. $G \subset E$

Proof:

- $rb(\Psi_p) \subset rb(\Gamma_n)$
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Claim: For any $G \in \mathcal{G}_p, \exists E \in \mathcal{E}_n$ s.t. $G \cap E = G$
Proof:

- Assume $\forall E \in \mathcal{E}_n, G \cap E \subsetneq G \implies G \cap E$ is proper face of G and $G \cap E \subset rb(G)$
- So there are points in G that are not in $G \cap E$
- Specifically, these points are $\mathbf{h} \in ri(G)$

$\implies \mathbf{h} \notin \cup_{E \in \mathcal{E}_n}(G \cap E) = G \cap (\cup_{E \in \mathcal{E}_n} E) = G \cap rb(\Gamma_n) = G \Rightarrow\Leftarrow$

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- $\exists\, E$ s.t. $G \cap E = G$ or $G \subset E$

$\Longrightarrow$ Lemma 5

Lemma 6: For any $E \in \mathcal{E}_n$, $E \cap S_p$ is a facet of Ψ_n

Proof Sketch:

- **Claim:**$E \cap S_p$ is a face of Ψ_p
  (Use the same supporting hyperplane P s.t. $E = \Gamma_n \cap P$)

Claim:The face $E \cap S_p$ is in fact a facet

- Let $E \cap S_p$ be not a facet of $\Psi_p \implies \exists\, G \supsetneq (E \cap S_p)$
- Lemma 5 $\implies$ $\exists$ facet $E' \in \mathcal{E}_n$ s.t. $G \subset E' \implies E' \cap S_p \supset G$ (since $G \subset S_p$)
- $E' \cap S_p \supset G \supsetneq E \cap S_p$ so far
- Lemma 4 $\implies E \overset{\text{p}}{\equiv} E' \implies E' \cap S_p \supset G \supsetneq E \cap S_p (= E' \cap S_p ?)$

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QED

Theorem 2: Every p-orbit of facets of Γ_n corresponds to a facet i.e. mapping $\omega_p : E \mapsto E \cap S_p$ satisfies

1) $\omega_p(E_1) = \omega_p(E_2) \iff E_1 \overset{\text{p}}{\equiv} E_2$

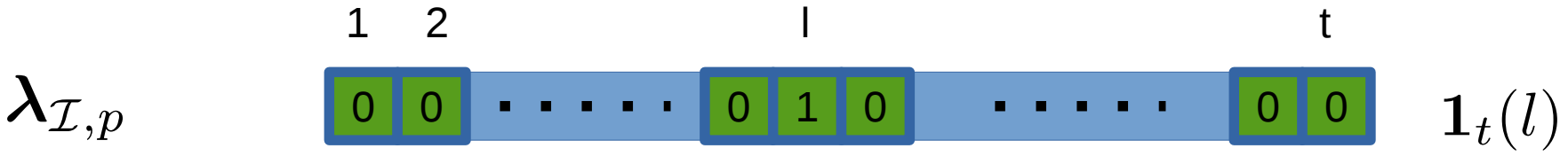
2) ω_p is a surjection from $\mathcal{E}_n$ onto $\mathcal{G}_p$

Proof:

- Lemma 3 and Lemma 4 $\implies$ 1)
- Lemma 6 $\implies$ $\omega_p : \mathcal{E}_n \to \mathcal{G}_p$
- Let $G \in \mathcal{G}_p$
- Lemma 5 $\implies$ $\exists$ facet E of Γ_n s.t. $G \subset E \implies G \subset E \cap S_p \subset E$ (since $G \subset S_p$)
- Lemma 6 $\implies$ $E \cap S_p$ is a facet of $\Psi_p \implies G = E \cap S_p$
- Since every $\omega_p^{-1}(\cdot)$ is nonempty, ω_p is surjective

Types of facets of Ψ_p

- We shall classify facets of Ψ_p based on their *parent* facet in Γ_n
- **Type I**: Facets whose parents are of type $E(\mathcal{I}, \mathcal{K}) = E(\{i\}, \emptyset)$



- There are t possible values $\boldsymbol{\lambda}_{I,p}$ can take
- Lemma 3 $\implies$ $E \cap S_p$ is same for all $E \in \mathcal{E}_p([\mathbf{1}_t(l), \mathbf{0}_t])$

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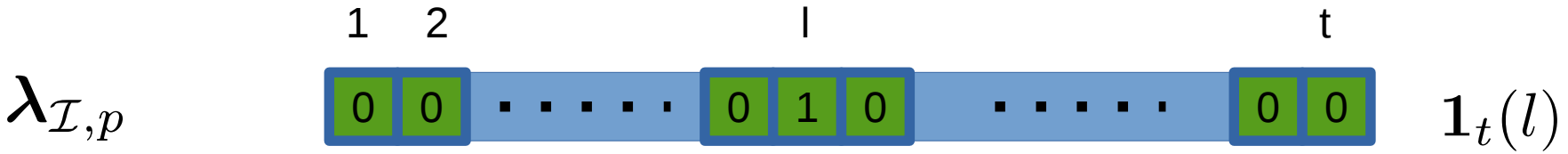

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$$E \cap S_p = \{\mathbf{h} \in \Psi_p : \mathbf{s}_{\boldsymbol{\lambda}_p} = \mathbf{s}_{\boldsymbol{\lambda}_{\mathcal{N}\setminus\{i\},p}}\}$$
$$= \{\mathbf{h} \in \Psi_p : \mathbf{s}_{n_1,\ldots,n_t} = \mathbf{s}_{n_1,\ldots,n_{l-1},\ldots,n_t}\}$$

$\downarrow$

Co-ordinate of $\mathbb{R}^{n_p}$ representing partitions with footprint = some permutation of $\boldsymbol{\lambda}_p$

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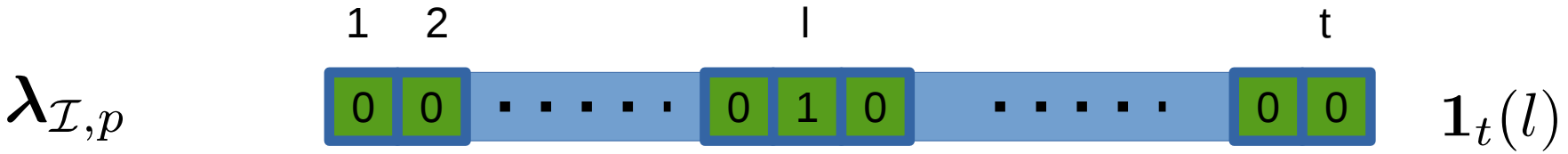

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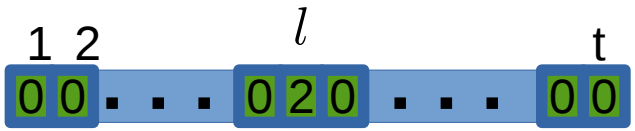
II-i) $\boldsymbol{\lambda}_{\mathcal{I},p} = \mathbf{1}_t(l_1, l_2)$

Possible values of $\boldsymbol{\lambda}_{\mathcal{K},p}$ for each $1 \leq l_1 \leq l_2 \leq t$?

range: $0, \ldots, n_m$ range: $0, \ldots, n_{l_1} - 1$ range= $0, \ldots, n_{l2} - 1$

II-ii) $\boldsymbol{\lambda}_{\mathcal{I},p} = \mathbf{2}_t(l)$



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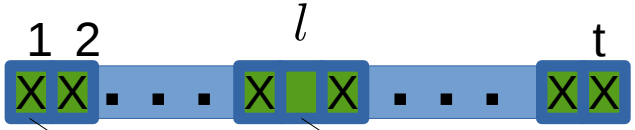

range: $0, \ldots, n_m$ range= $0, \ldots, n_l - 2$

$$n_{l_1} n_{l_2} \times \prod_{m \in \{1,\ldots,t\}, m \neq l_1, l_2} (n_m + 1)$$

$$(n_l - 1) \times \prod_{m \in \{1,\ldots,t\}, m \neq l} (n_m + 1)$$

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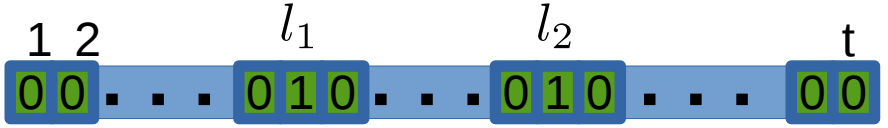

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$$E \cap S_p?$$

$$E \cap S_p = \left\{ \mathbf{h} \in \Psi_p : \begin{array}{r} \mathbf{s}_{\boldsymbol{\lambda}_{\mathcal{K}\cup\{i\},p}} + s_{\boldsymbol{\lambda}_{\mathcal{K}\cup\{j\},p}} \\ = \mathbf{s}_{\boldsymbol{\lambda}_{\mathcal{K},p}} + \mathbf{s}_{\boldsymbol{\lambda}_{\mathcal{K}\cup\{i,j\},p}} \end{array} \right\}$$

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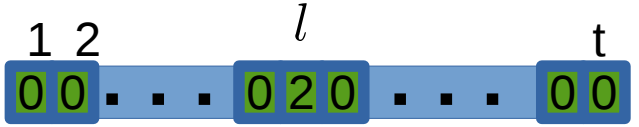

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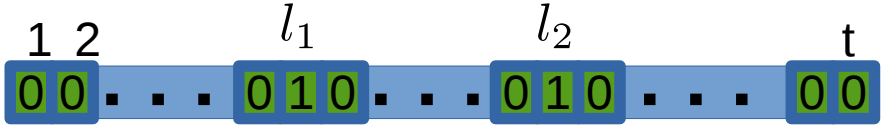

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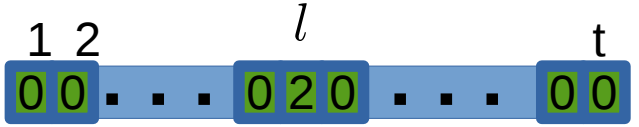

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Number of facets of Ψ_p

$$|\mathfrak{E}_p| = |\mathfrak{R}_p| = \begin{array}{l} t + \left(\displaystyle\sum_{1 \le l_1 \le l_2 \le t} n_{l_1} n_{l_2} \prod_{m \in \{1,\ldots,t\}, m \neq l_1, l_2} (n_m + 1) \right) \\ + \left((n_l - 1) \displaystyle\prod_{m \in \{1,\ldots,t\}, m \neq l} (n_m + 1) \right) \end{array}$$