Solving Rota's Conjecture*

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ASPITRG

*Jim Geelen, Bert Gerards and Geoff Whittle, Solving Rota's conjecture, Notices of the American Mathematical Society, Aug 17, 2014: 736–743

Outline

- Background
- What is Rota's conjecture?
- Graph Minors and WQO theorem for graphs
- Graph minors Structure theorem

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Matroid

- A *matroid* consists of a pair $E, \mathcal{I}$ where E is a finite set and $\mathcal{I}$ is a collection of subsets of E
- E is called thge *ground set*
- Subsets in $\mathcal{I}$ are called *independent sets*
- $(E, \mathcal{I})$ obey certain axioms:

  I1 $\emptyset \in \mathcal{I}$

  I2 Subsets of independent sets are independent

  I3 For each $X \subseteq E$, the maximal independent sets of X have the same size

Cryptomorphic axiom systems

- $(E, \mathcal{B})$: where $\mathcal{B}$ is collection of all maximal independent sets of E
- (E, r): $r : 2^E \to \mathbb{Z}_{\geq 0}$ is called the rank function
- These generalize the idea of bases of vector space and linear rank

Column matroid

- Let $\mathbf{A}$ be a matrix over field $\mathbb{F}$
- Let E be the set of column indices of A
- Let $\mathcal{I}$ be the collection of subsets of E that index linearly independent sets of columns
- The *column matroid* of A, denoted as $M(A)$ is $(E, \mathcal{I})$

$\mathbb{F}$-representability

- A matroid is called $\mathbb{F}$-representable if it is column matroid of some matrix over $\mathbb{F}$
- **Conjecture** The proportion of n element matroids that are representable is vanishingly small as $n \to \infty$

Whitney's Problem(s)

H. Whitney, On the abstract properties of linear independence, Amer. J. Math. 57 (1935), 509–533

Are following problems *decidable*?*

(P1) Is the given matroid representable over *any* field?

YES!

P. Vámos, A necessary and sufficient condition for a matroid to be linear, In Möbius algebras (Proc. Conf. Univ. Waterloo, 1971), 162–169. University of Waterloo,Waterloo.

(P2) Is the given matroid representable over *given finite field*?

Rota's Conjencture!

*A decision problem is decidable if there exists a finite terminating algorithm to solve it

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Matroid Duality

- *Dual* of a matroid M is a matroid denoted as M^* whose bases are compelments of those of M
- **Theorem** A matroid M is $\mathbf{F}$-representable if and only if M^* is $\mathbb{F}$-representable

Deletion and Contraction

- Let C and D be sets of elements in a matroid $M = (E, \mathcal{I})$
- The matroid obtained by deleting D is defined as $(E - D, \{I \subseteq E - D : I \in \mathcal{I}\})$
- *Contraction* is dual operation of deletion
- $(M/C) = (M^* \setminus C)^*$

Minors

- Minors of a matroid M are matroids of type $M \setminus D/C$
- A minor is called *proper* if $D \cup C$ is nonempty
- The class of $\mathbb{F}$-representable matroids is closed under deletion and duality
- If a matroid is $\mathbb{F}$-representable, so are all its minors
- Hence the class of $\mathbb{F}$-representable matroids is closed under taking minors

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Excluded Minors

- Naturally we start look for matroids that are not $\mathbb{F}$-representable but all their proper minors are
- We call these excluded minors
- **Conjencture(Rota)**: For each finite field $\mathbb{F}$, there are, up to isomorphism, only finitely many excluded minors for the class of $\mathbb{F}$-representable matroids

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- (P2) is decidable if this is true

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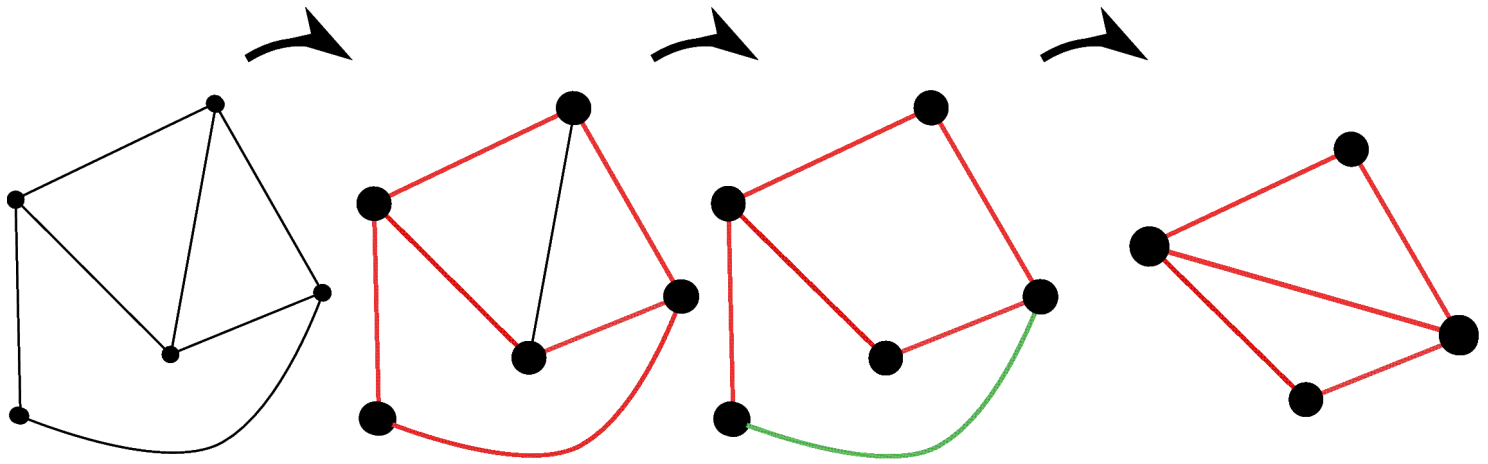
Graph Minors

- A *minor* of a graph G is a graph that is obtained from a subgraph of G by *contracting* some edges



Characterization of planar graphs

- Planar graphs are graphs that can be embedded in a plane i.e.they can be drawn in such a way that no edges cross each other

- Planar graphs are closed under taking minors

- **Kuratowski's Theorem**: A graph is not planar iff it has a minor isomorphic to $K_{3,3}$ or K_5
  (K. Kuratowski, Sur le problème des courbes gauches en topologie, Fund. Math. 15 (1930), 271283.)

Generalized Kuratowski Theorem

- Robertson and Seymour generalized Kuratowski's theorem from plane to arbitrary surfaces

- **Theorem**:For any given surface, there are only finitely many excluded minors for the class of graphs that embed in the surface.
  N. Robertson and P. D. Seymour, Graph Minors. VIII. A Kuratowski theorem for general surfaces, J. Combin. Theory Ser. B 48 (1990), 255-288.

A partial order on set of non-isomorphic graphs

- Let $\mathcal{G}$ be set of all non-isomorphic finite undirected graphs
- If G can be obtained from H by taking minor, we say $G \leq H$
- $'\leq$' is a partial order on $\mathcal{G}$ since it is:

  P1 Reflexive
  Every graph is minor of itself

  P2 Transitive
  Every minor of G is itself a minor of G

  P3 Antisymmetric
  If G and H are minors of each other, then they must be isomorphic

A quasi-order on set of all graphs

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Well-founded quasi order

- A quasi order R on X is Well-founded if

WQO1 There are no infinite descending chains
(infinite sequences of type $x_1 > x_2 > x_3, \ldots, x_i \in X$)

WQO2 There are no infinite antichains
(infinite subsets with pairwise incomparable elements
$A \subseteq X$ s.t. neither $x_i \leq x_j$ nor $x_j \leq x_i$, $\forall x_i, x_j \in A$)

- Set of all graphs:

WQO1 is true since starting with a finite graph and deleting /contracting edges must end at empty graph

WQO2 the non-trivial part

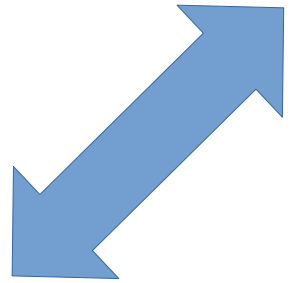
Why is WQO so important?

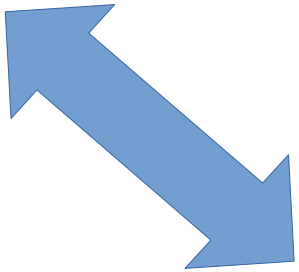
A2

In any infinite set of graphs
$\exists$ a pair x, y that is comparable
under minor relation $\leq$

A1

No infinite antichains





A3

Let S be an infinite subset of $\mathcal{G}$
and M be the minor minimal subset
of S i.e. $\forall x \in M$ if $y \leq x \Rightarrow y \notin S$,
Then M is finite

Why is WQO so important?

A2 In any infinite set of graphs $\exists$ a pair x, y that is comparable under minor relation $\leq$

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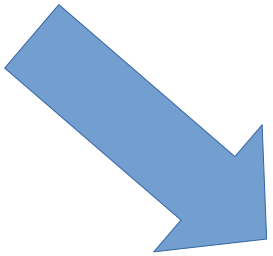
M is an antichain for any S. If not, $\exists x \in M$ s.t. $y \leq x$ and $y \in M$, thus contradicting minor minimality of x



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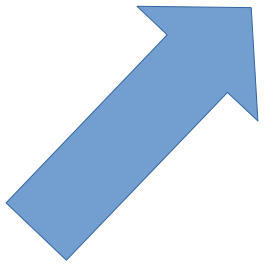

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Obvious

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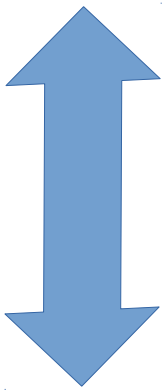

Upwards: Every element in $S \setminus M$ is comparable to at least one element of M

Downwards: If M was to be infinite, , $y \in M$ s.t. $y \leq x$, i.e. $x \in M$ has a minor that is also in M $\Rightarrow\Leftarrow$

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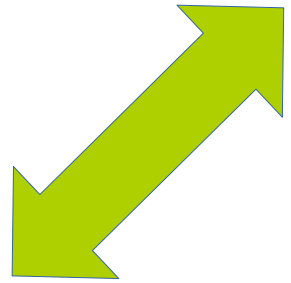

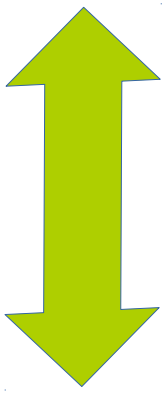

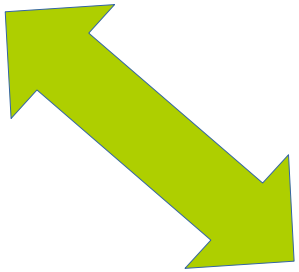

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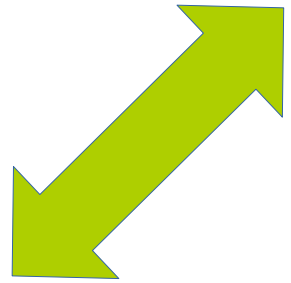

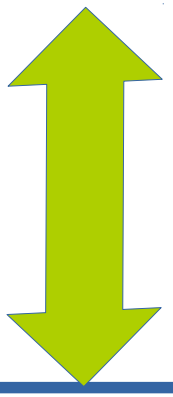

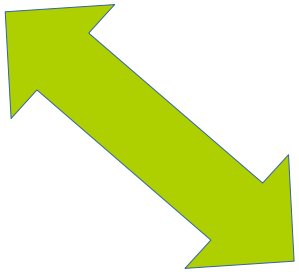

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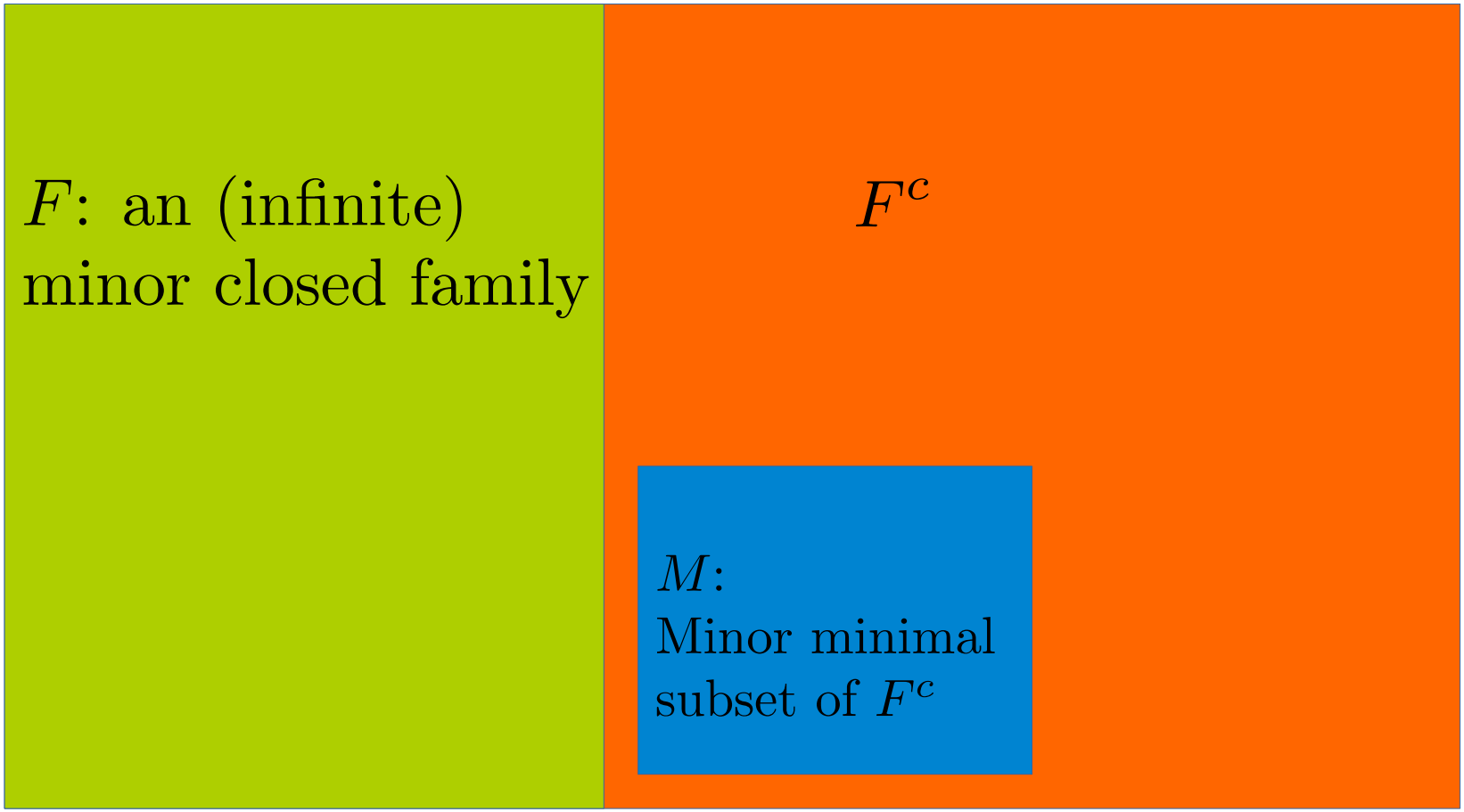
WQO and minor closed families of graphs



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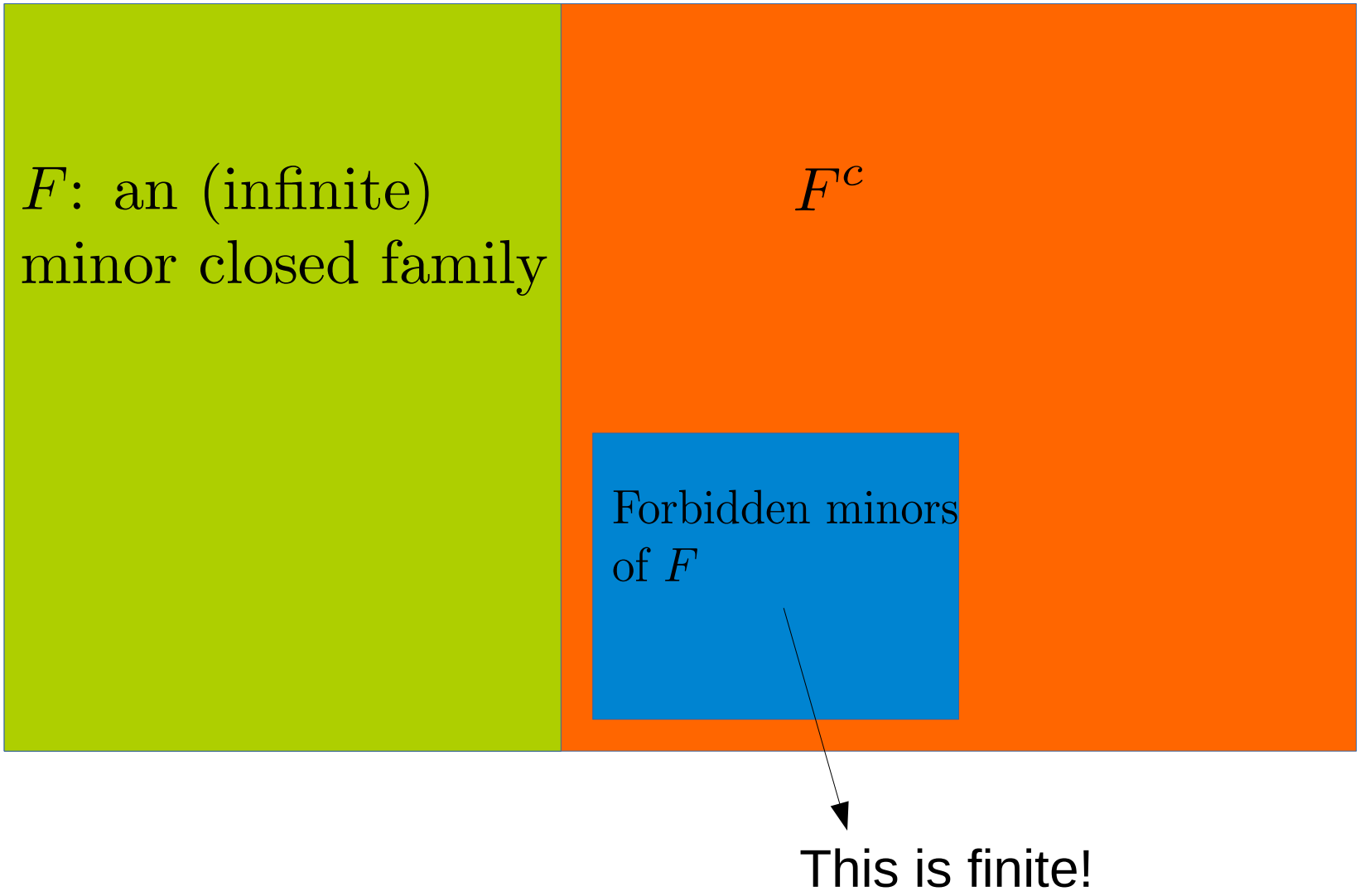
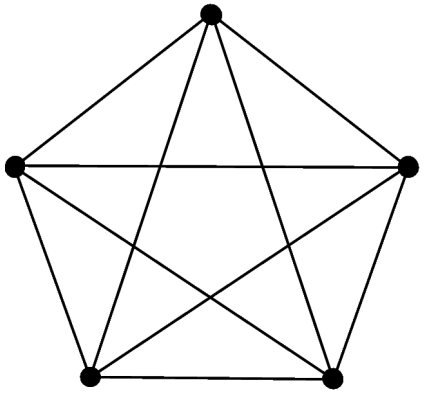


A further generalization: WQO Theorem

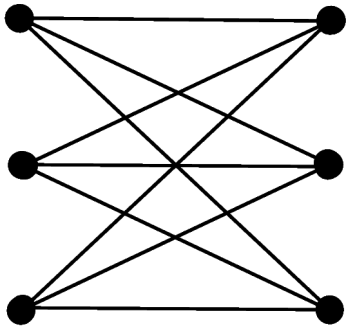
Theorem: Each minor-closed class of graphs has only finitely many excluded minors
N. Robertson and P. D. Seymour, Graph Minors. XX. Wagners Conjecture, J. Combin. Theory Ser. B 92 (2004), 325-357
Alternatively,

- In each infinite set of graphs, there are two graphs, one is isomorphisc to minor of other
- There are countably many distinct minor closed classes of graphs

Excluded minors for planarity

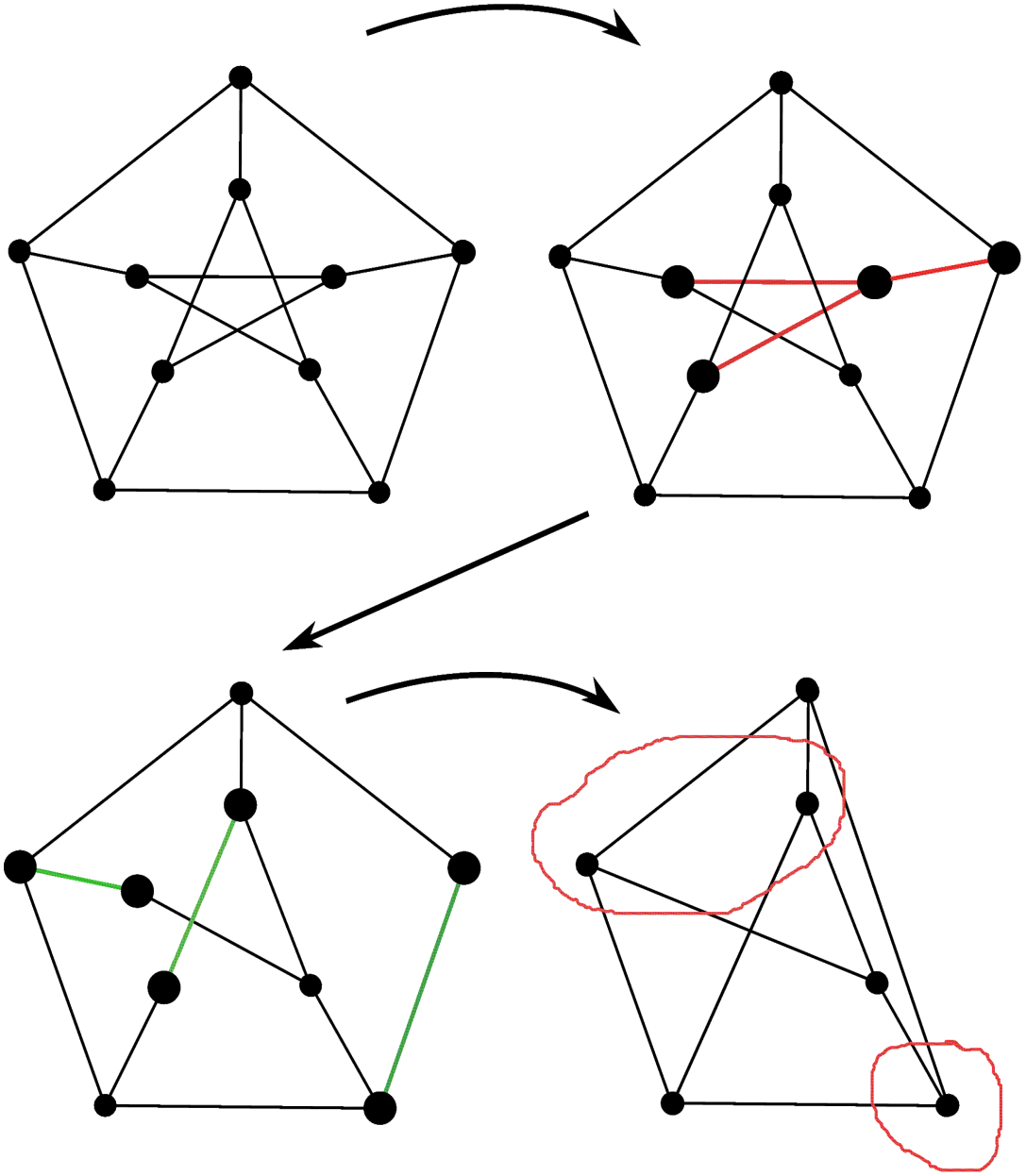


K_5

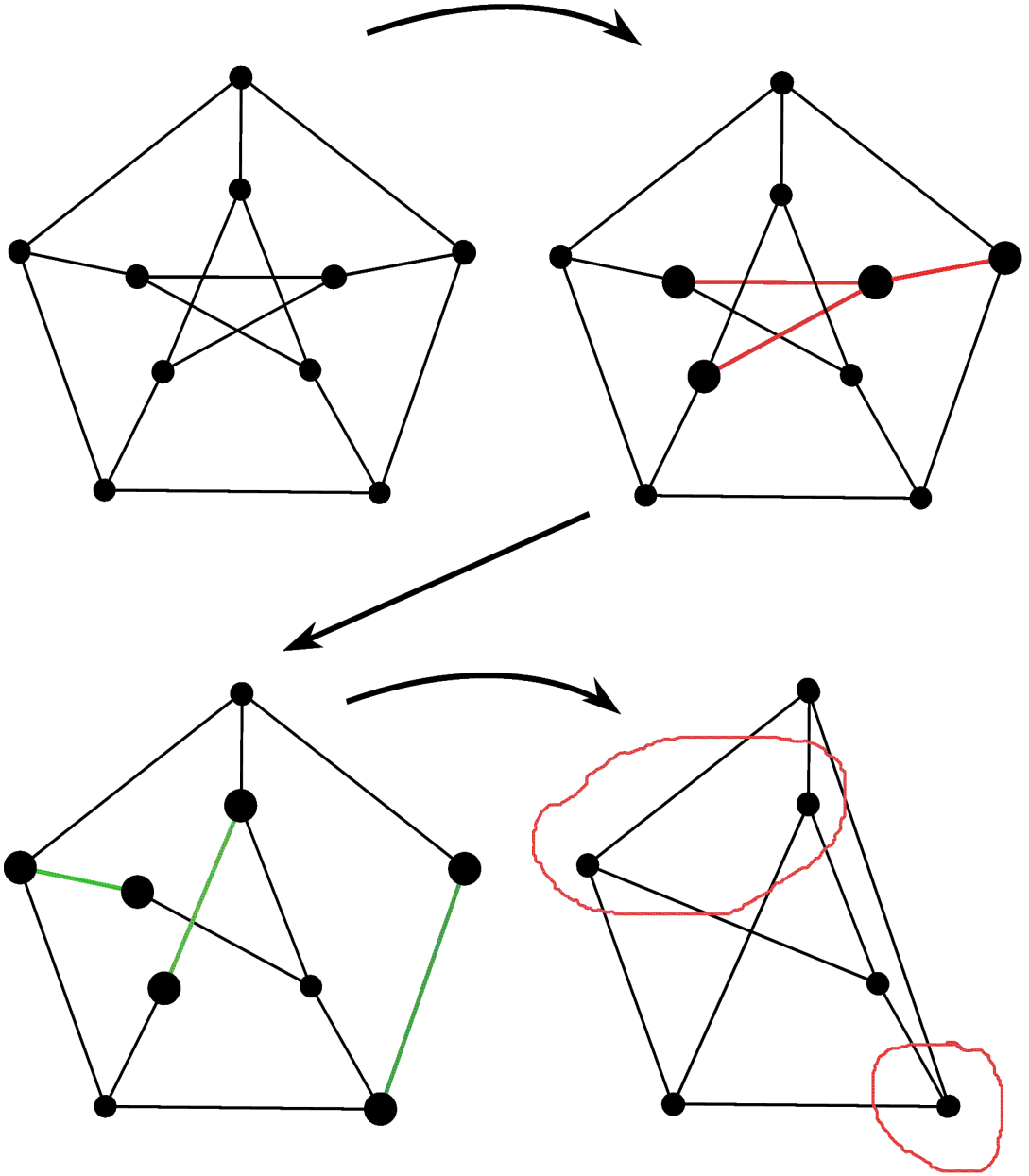


$K_{3,3}$

Example: Peterson graph

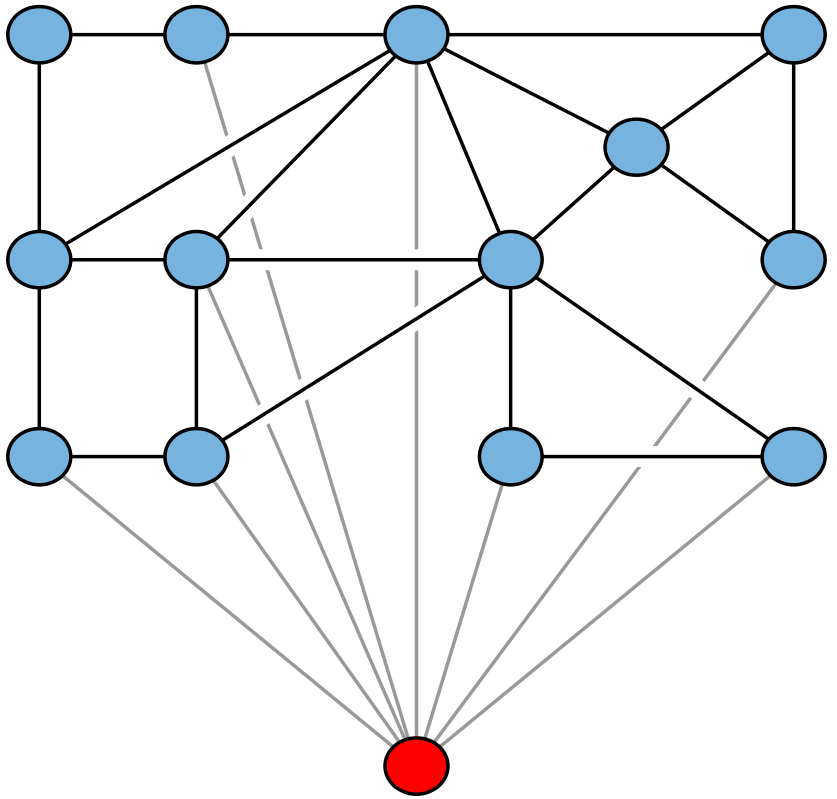


Example: Peterson graph Not Planar



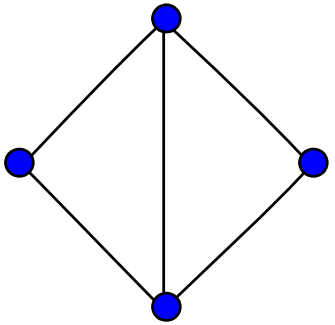
Other minor closed families with forbidden minor characterizations

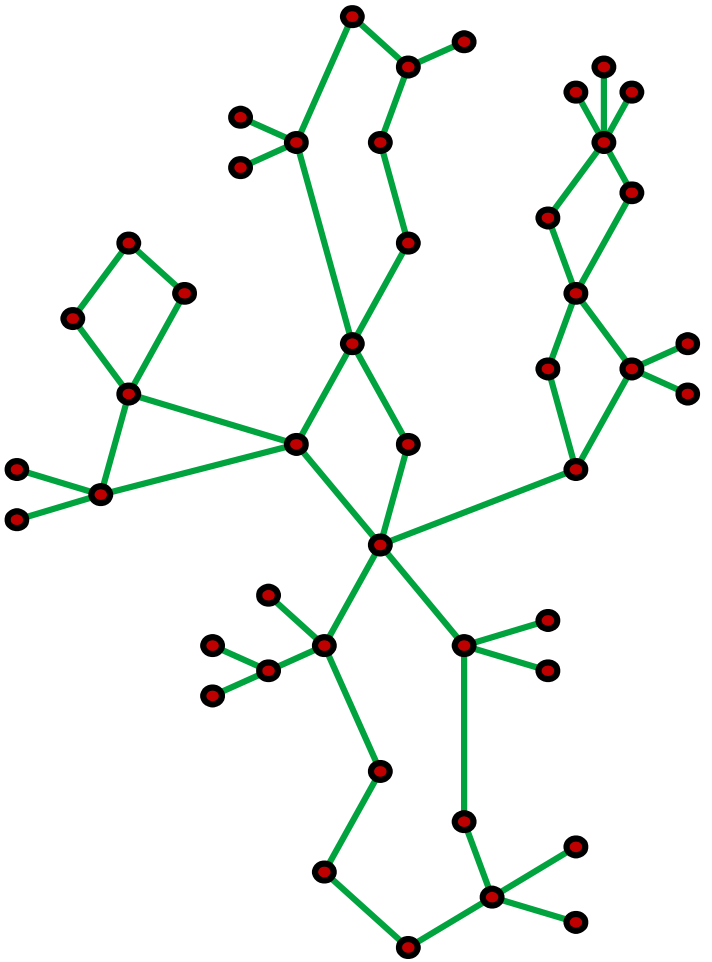
Apex Graphs:
Graphs that can be made planar by removal of a single vertex
Finite (unknown) excluded minors



Other minor closed families with forbidden minor characterizations

Graph Unions* of Cactus graphs:
Graph unions of graphs in which
any two simple cycles contain
at most 1 common vertex
One Excluded minor: Dimond graph

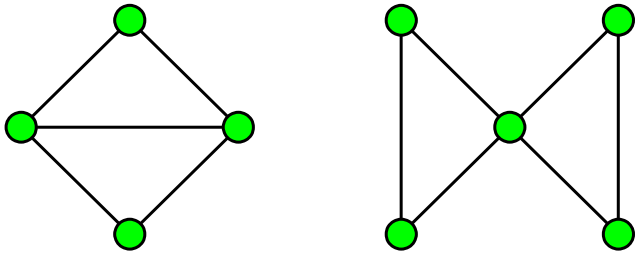


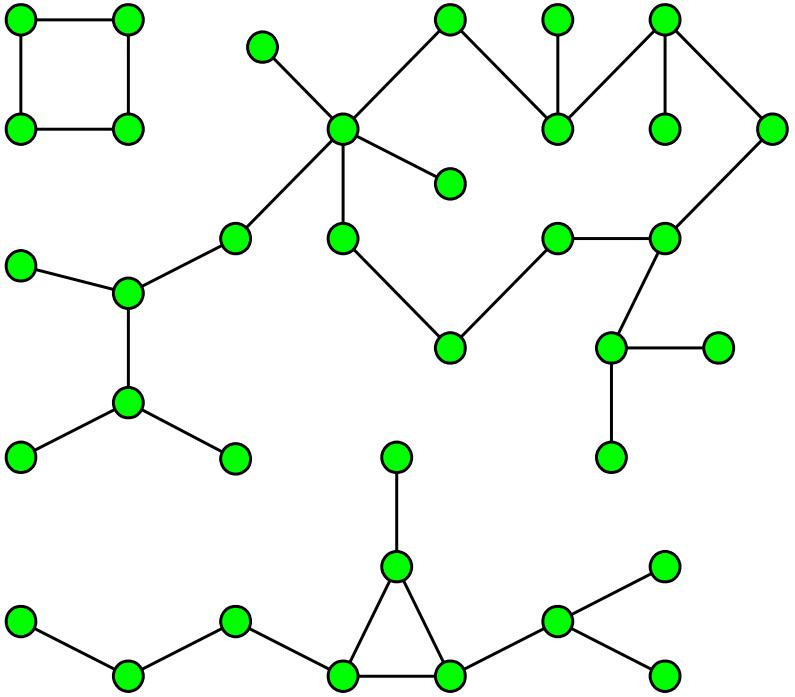


* A binary operation corresponding to disjoint union of vertex sets and edge sets

Other minor closed families with forbidden minor characterizations

Pseudoforests*:
Graphs having at most one cycle
in every connected component
Two Excluded minors:
1) Dimond graph 2) Butterfly graph





* A sub-class of unions of cactus graphs

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Graph Minors Structure Theorem

- The set of excluded minors M form an antichain under the well quasi order
- Let $F = (H_1, H_2, H_3, \ldots)$
- A graph is said to be H_1-free if has no minor isomorphicto H_1
- $(H_2, H_3, \ldots)$ must be H_1-free (having no minor isomorphic to H_1)
- Let $EX(H_1)$ be the set of all graphs not having a minor isomorphic to H_1
- If H_1 does not embed into a surface S, graphs that embed into surface S must be in $EX(H_1)$. Call this set P.
- Graph minors structure theorem provides a means to construct graphs in $EX(H_1)$ from P

Part-2 Outline

- Graphs to matroids: Forbidden minor characterization of graphic matroids
- Matroid WQO theorem
- Projectively Inequivalent Representations
- Bifurcation
- Representability under circuit hyperplane relaxation

Note: forbidden minor characterization perfect graphs

- Minors (in the sense we discussed so far) yield a relation on set of all graphs
- So do induced subgraphs
- Perfect graphs are characterized by *forbidden induced minors*
- Forbidden minors*: Odd holes (simple cycles of length not less than 5) and their complements

Chudnovsky, Maria; Cornuéjols, Gérard; Liu, Xinming; Seymour, Paul; Vušković Kristina (2005). "Recognizing Berge graphs". Combinatorica 25 (2): 143–186

Note: WQO for Digraphs?

- What relation one should use?
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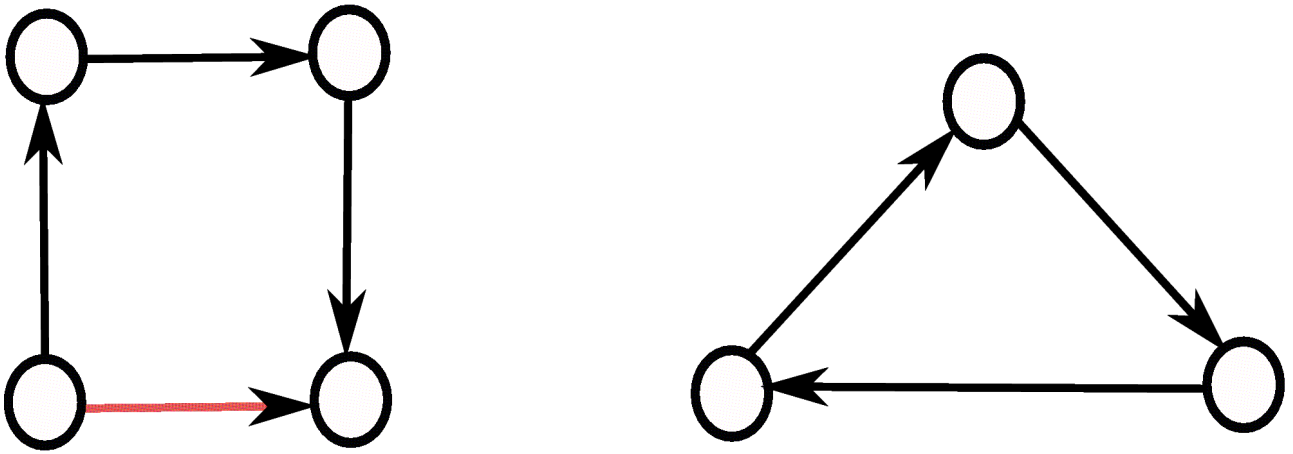

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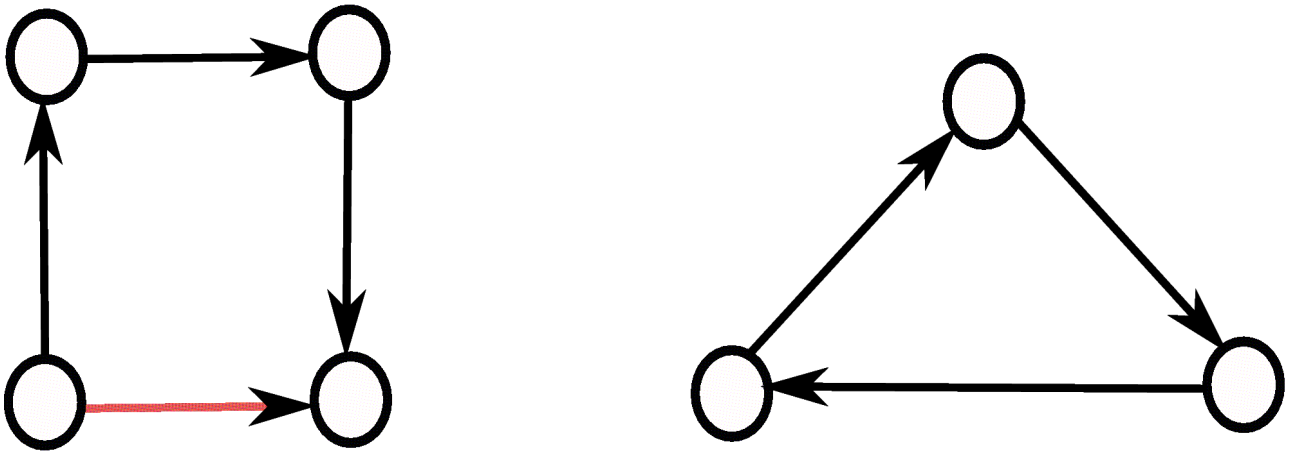
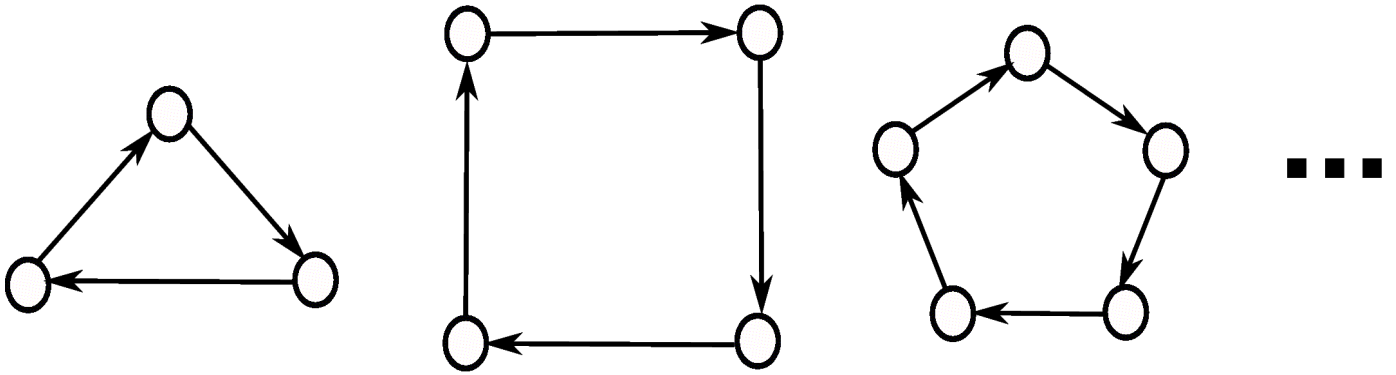


Note: WQO for Digraphs?

- What relation one should use?
  - Edge contraction is absurd
  - WQO fails



Note: WQO for Digraphs?

- How to deal with this problem?
- Consider a subclass: Semi-complete Tournaments*
- Various minor relations
  - Immersion
  - Strongly connected subgraph contraction

*A directed graph obtained from a complete undirected graph by orienting edges

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- Consider a subclass: Semi-complete Tournaments
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  - Immersion: forms WQO on semi-complete tournaments*
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*Maria Chudnovsky and Paul Seymour, “A well-quasi-order for tournaments”, J. Combinatorial Theory, Ser. B, 101 (2011), 47–53

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- How to deal with this problem?
- Consider a subclass: Semi-complete Tournaments
- Various minor relations
  - Immersion: forms a WQO on semi-complete tournaments*
  - Strongly connected subgraph contraction: forms a WQO on semi-complete tournaments*

*Kim, I. & Seymour, P. D. (2012). Tournament Minors. CoRR, abs/1206.3135.

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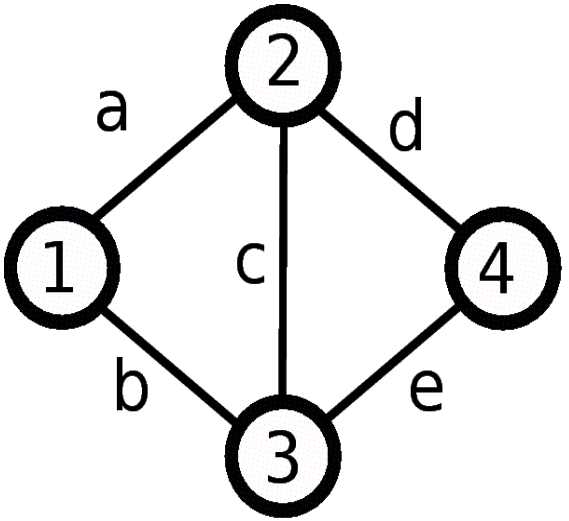
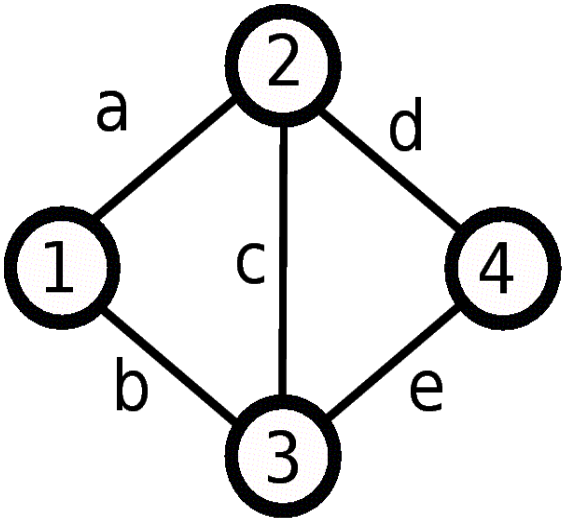
Graphic Matroids



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Graphic Matroids



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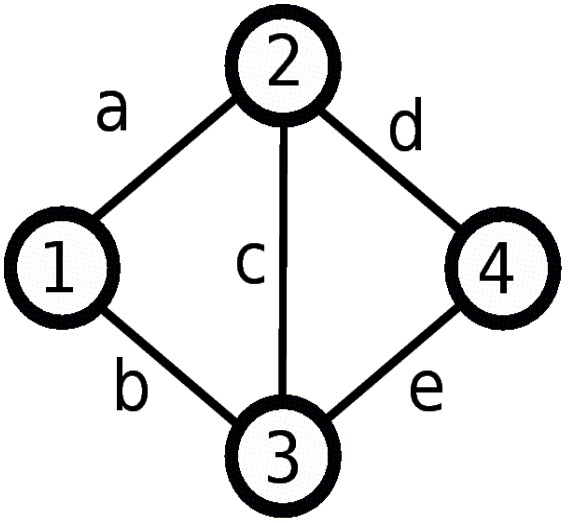
\end{array}
$$

Graphic Matroids



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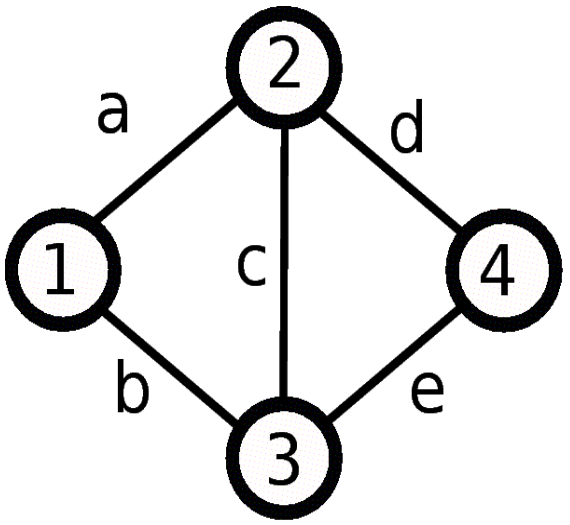
This is both binary and ternary representation

Graphic Matroids



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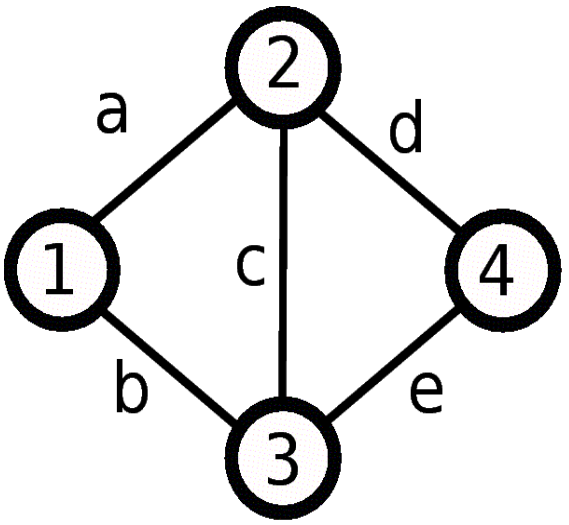
Theorem (Seymour) A matroid is representable over all fields iff it is representable over both $GF(2)$ and $GF(3)$

Seymour, P. D. (1979), Matroid representation over GF(3), Journal of Combinatorial Theory, Series B 26 (2): 159–173

Graphic Matroids



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Theorem Graphic matroids are representable over all fields

Characterization of graphic matroids

Graphic matroids are a minor closed family

Theorem(Whitney): A graph G is planar iff $M(G)^*$ is graphic

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Characterization of graphic matroids

Graphic matroids are a minor closed family

Theorem(Whitney): A graph G is planar iff $M(G)^*$ is graphic

Exclude $M(K_{3,3})^*, M(K_5)^*$!

Theorem: A matroid is graphic if it is binary

Exclude $U_{2,4}$!

Theorem: A matroid is graphic if it is ternary

Exclude F_7 and F_7^*![*]

[*]Excluding $U_{2,4}$ takes care of $U_{2,5}$ and $U_{3,5}$

Characterization of graphic matroids

Graphic matroids are a minor closed family

Theorem(Whitney): A graph G is planar iff $M(G)^*$ is graphic

Theorem(Tutte): The excluded minors for the class of graphic matroids are $U_{2,4}, F_7, (F_7)^*$,$\mathrm{M(K_{3,3})}^*, M(K_5)^*$

W. T. Tutte, Matroids and graphs, Trans. Amer. Math. Soc. 90 (1959), 527–552

Part-2 Outline

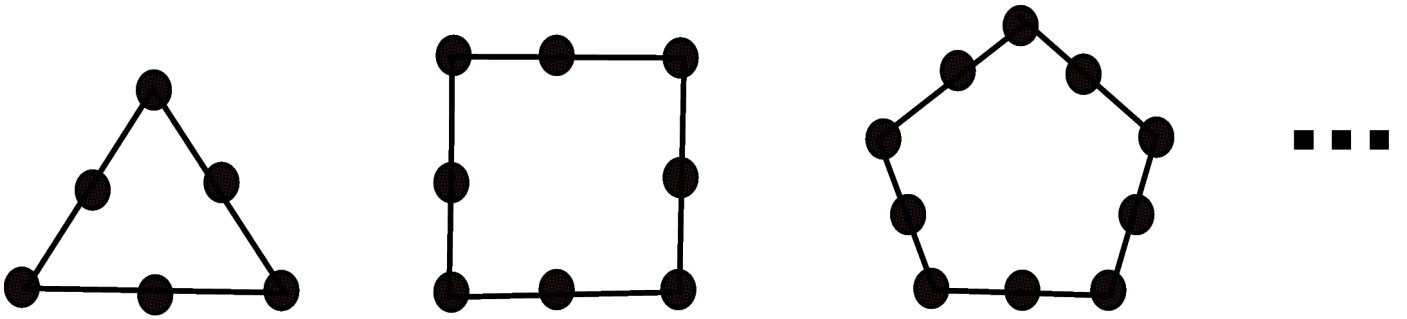
- Graphs to matroids: Forbidden minor characterization of graphic matroids
- Matroid WQO theorem
- Projectively Inequivalent Representations
- Bifurcation
- Representability under circuit hyperplane relaxation

Matroid WQO and rota's conjecture

- A WQO theorem for set of all finite matroids would imply Rota's conjecture is True

Matroid WQO and rota's conjecture

- A WQO theorem for set of all finite matroids would imply Rota's conjecture is True
- Unfortunately WQO theorem doesn't hold for all matroids



Matroid WQO

Theorem(Matroid WQO) For each finite field $\mathbb{F}$ and each minor closed class of $\mathbb{F}$-representable matroids, there are only finitely many $\mathbb{F}$-representable excluded minors

Matroid WQO

Theorem(Matroid WQO) For each finite field $\mathbb{F}$ and each minor closed class of $\mathbb{F}$-representable matroids, there are only finitely many $\mathbb{F}$-representable excluded minors

Matroid WQO does not say anything about Rota's conjecture

Part-2 Outline

- Graphs to matroids: Forbidden minor characterization of graphic matroids
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- Projectively Inequivalent Representations
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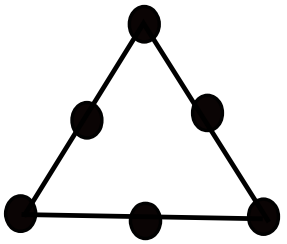
Projective equivalence of matroid representations

- Two matrices A and B are said to be *projectively equivalent* if A can be obtained from B by:
  - Elementary row operations
  - Column scaling

Projective equivalence of matroid representations

Projectively inequivalent representations of 3-whirl over GF(5)



A $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$

B $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{bmatrix}$

C $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 3 \end{bmatrix}$