Polyhedral Description Conversion up to Symmetries*

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[1] *D. Bremner, M. D. Sikiric, and A. Schurmann. Polyhedral representation conversion up to symmetries. CoRR, abs/math/0702239, 2007
[2] Thomas Rehn. Polyhedral Description Conversion up to Symmetries. Diploma thesis (mathematics), Otto von Guericke University Magdeburg, November 2010
[3] Abstract Algebra: Theory and Applications by Thomas Judson. Available online.

Outline:Part-I

- Groups
- Properties of Groups
- Permutations/Symmetry Group
- Cosets and Lagrange's Theorem
- Isomorphisms and Caylay's Theorem
- Group Actions and Orbits
- Fixed point sets and stabilizers
- Face lattices of polyhedra
- Combinatorial automorphism group of polyhedra

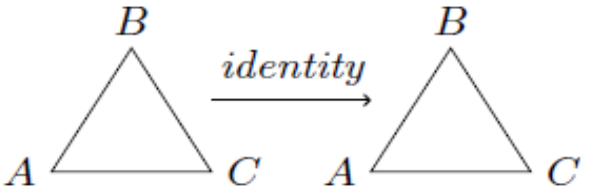
Outline:Part-II

- Representation conversion problems
- Adjacency decomposition method
- Neighbors of extreme rays
- Support cone
- Reduced support cone
- Enumeration of G-inequivalent extreme rays of reduced support cone
- Example

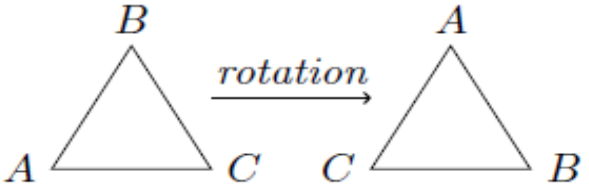
A simple example

- Let $S = \{1, 2, 3\}$
- $\pi : S \to S$ given as: $\pi(1) = 2, \pi(2) = 1, \pi(3) = 3.$
- This is a bijective map.
- For any set S bijective map like $\pi : S \to S$ is called **permutation** of S.

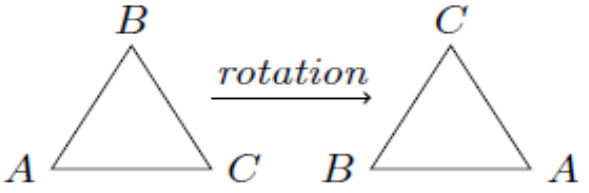
Rigid Motions of an equilateral triangle



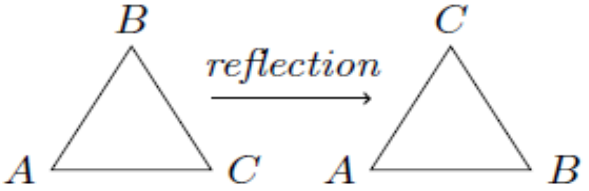
$$id = \begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix}$$



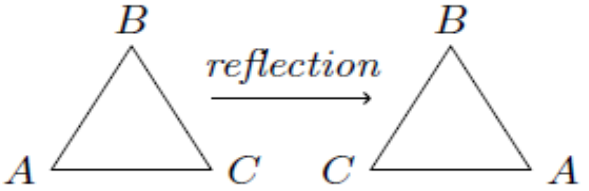
$$\rho_1 = \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix}$$



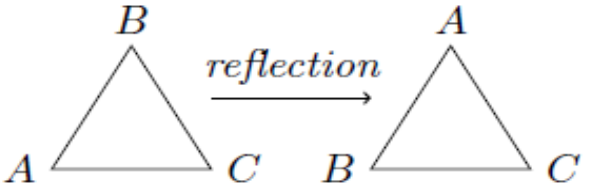
$$\rho_2 = \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}$$



$$\mu_1 = \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix}$$



$$\mu_2 = \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$$



$$\mu_3 = \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}$$

Figure Credits: Judson, Thomas W. Abstract Algebra: Theory and Applications. Boston, MA: PWS Pub., 1994. Print.

Rigid Motions and Symmetry

- A **rigid motion** of that 2-D geometric figure is a map from the plane to itself preserving the arrangement of sides and vertices as well as its distances and angles
- Rigid motions of a 2-D geometric figure is a way of characterizing its symmetries

The Caylay Table for symmetries of equilateral triangle

$\circ$	id	ρ_1	ρ_2	μ_1	μ_2	μ_3
id	id	ρ_1	ρ_2	μ_1	μ_2	μ_3
ρ_1	ρ_1	ρ_2	id	μ_3	μ_1	μ_2
ρ_2	ρ_2	id	ρ_1	μ_2	μ_3	μ_1
μ_1	μ_1	μ_2	μ_3	id	ρ_1	ρ_2
μ_2	μ_2	μ_3	μ_1	ρ_2	id	ρ_1
μ_3	μ_3	μ_1	μ_2	ρ_1	ρ_2	id

Figure Credits: Judson, Thomas W. Abstract Algebra: Theory and Applications. Boston, MA: PWS Pub., 1994. Print.

Group

- A **binary operation** or **law of composition** on set G is a function $G \times G \to G$ that assigns every pair $(a, b) \in G$ a unique element $a \circ b$, or ab in G is called composition of a and b.

- A **group** $(G, \circ)$ is a set G together with a law of composition $(a, b) \mapsto a \circ b$ satisfying axioms:
  1. operation is **associative** i.e.
  $\forall a, b, c \in G,\ (a \circ b) \circ c = a \circ (b \circ c)$
  2. $\exists$ **identity element** s.t. $\forall a \in G, a \circ e = e \circ a = a$
  3. $\exists$ **inverse element** a^{-1}, $\forall a \in G$ s.t. $a \circ a^{-1} = e$

Group

- **Order** of group G is $|G|$. Group is **finite** if its order is finite, else it is **infinite**
- If the composition operation is **commutative**

Examples of groups

- Set of integers under addition as operation is called $\mathbb{Z}$, Z forms a group under ordinary addition
- Set of integers *modulo* n is called $\mathbb{Z}_n$ and is also a group under *modulo* n addition
- Denote $\mathbb{M}_2$ to be set of all 2×2 matrices under matrix multiplication
- **General linear group:** $GL_2(\mathbb{R}) \subset \mathbb{M}_2$ and has all invertible 2×2 matrices under matrix multiplication.

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Properties of Groups

- Unique identity element
- Unique inverse for every element
- $\forall a \in G, (a^{-1})^{-1} = a$
- $(ab)^{-1} = b^{-1}a^{-1} \quad \forall a, b \in G$
- $\forall a, b \in G \; ax = b$ and $xa = b$ have unique solution $x \in G$

Properties of Groups

- **Left and right cancellation laws:** $ba = ca \Rightarrow b = c$ and $ab = ac \Rightarrow b = c$
- Exponent notation(esp. for multiplication operator) for repetitive operations, inverse operation etc.
    1. $g^m g^n = g^{m+n}$
    2. $(g^m)^n = g^{mn}$
    3. $(gh)^n = (h^{-1}g^{-1})^{-n} \forall n \in \mathbb{Z}$
- For abelian G, $(gh)^n = g^n h^n$

Subgroups

- A **subgroup** H of G is is a subset closed under operation of group G
- **Trivial** if $H = \{e\}$
- **Proper** if $H \subsetneq G$
- Denoted as $H \leq G$ (general) and $H < G$(proper)

Examples of subgroups

1. $2\mathbb{Z} \leq \mathbb{Z}$ under addition
2. Nonzero integers $\subset \mathbb{R}$ under multiplication,
3. $SL_2(\mathbb{R})$: the group of 2×2 invertible matrices with determinent 1 is subgroup of $GL_2(\mathbb{R})$ under matrix multiplication

Cyclic Subgroups and Cyclic Groups

- One element of group determines entire subset of group/subgroup then that subgroup is called cyclic subgroup.
- The subgroup generated by $a \in G$ is denoted by $< a >$
- Elements in this subgroup are $a \circ a, a \circ a \circ a, ...$
- For any $a \in G$, $< a >$ is called **cyclic subgroup** of G.
- If $\exists a \in G : G =< a >$ we call G a **cyclic group** with generator a

Examples

- Set of integer powers of 2 is a cyclic subgroup of group of positive rationals $\mathbb{Q}^+$ under ordinary multiplication
- $3\mathbb{Z}$ is cyclic subgroup of $\mathbb{Z}$, $3\mathbb{Z} =< 3 >$

Properties of Cyclic Groups

- cyclic $\Rightarrow$ *abelian*
- Let $G =< a >$ and G has order n. then $a^k = e$ iff $k \bmod n = 0$
- Every subgroup of cyclic group is cyclic
- If $b \in G, b = a^k$ where $G =< a >$ is cyclic of order n, then b is generator of a cyclic subgroup of G of order $\frac{n}{gcd(k,n)}$
- Hence if a to be generator of cyclic group G of n elements then $a = b^k$ for some other gererator b of G for k that is relatively prime to n

Permutation Group/Symmetry Group

- Group of permutations of elements of set X is called S_X
- When $X = \{1, 2, 3..., n\}$ it is denoted by S_n: **permutation group on n letters**
- The underlying set for S_n is the **set of ways of permuting** $\{1, 2, 3, ..., n\}$

Disjoint Cycle Notation

- We can refer to a permutation σ as
  $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$
- Or we can use the **disjoint cycle notation** as$(\ 4321)$
- (4321) $4 \mapsto 3, 3 \mapsto 2, 2 \mapsto 1, 1 \mapsto 4$
- σ above is described with with just one cycle.
- As a minimal representation, we need cycles to be disjoint
- $\therefore$ (4321) is trivially valid
- Some permutations may consist of more than one cycle

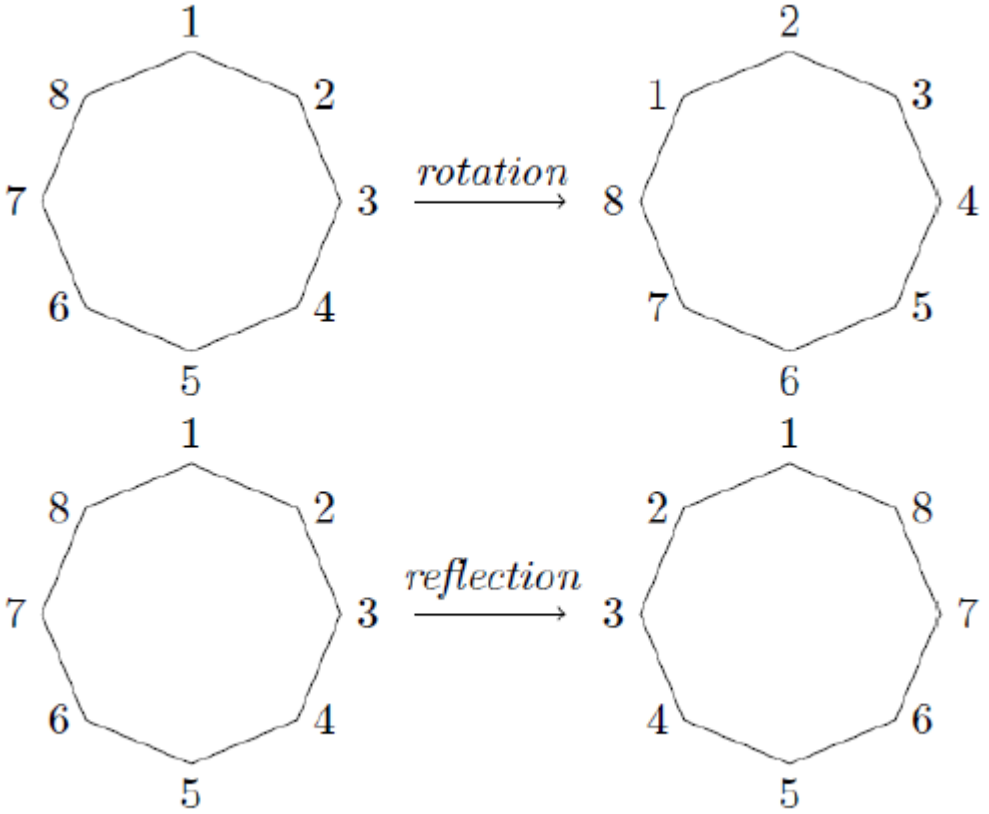
Transposition

- Example: (135)(27) is minimal because cycles are disjoint but (135)(347) reduces to (13475)
- **Transposition** is disjoint cycle representation with each cycle having only 2 elements
- number of cycles in transposition representation of id is even(intuitive)
- a permutation is **even** if it can be represented as even number of transpositions otherwise it is **odd**
- Set of all even permuatations A_n is a subgroup of S_n and its order is $n!/2$ = number of odd permutations

Dihedral Groups

- The dihedral group denoted by D_n is a group associated with a regular n-gon.
- **Rotations** We denote $\frac{360^\circ}{n}$ rotation of an n-gon by r, $2 \times \frac{360^\circ}{n}$ rotation by r^2 and so on. Hence rotation r generates all other rotations
- **Reflections** We define a reflection of n-gon by the vertex it leaves fixed: odd number of vertices:
- $r^n = id, s^2 = id, srs = r^{-1}$
- The group of rigid motions of a cube is S_4

Example: D_8


1
2
3
4
5
6
7
8
rotation
2
3
4
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6
7
8
1
1
2
3
4
5
6
7
8
reflection
1
8
7
6
5
4
3
2

Cosets

- For any $g \in G$ and H $\leq G$ the set $gH = \{gh : h \in H\}$ is called **left coset** of H with **representative** g
- Similarly, $Hg = \{hg : h\}$ is the right coset
- **Example:** Let $H \leq \mathbb{Z}_6$ and $H = \{0, 3\}$ then 6 elements of G give following 6 cosets:

1. $0 + H = \{0, 3\}$
2. $1 + H = \{1, 4\}$
3. $2 + H = \{2, 5\}$
4. $3 + H = \{0, 3\}$
5. $4 + H = \{1, 4\}$
6. $5 + H = \{2, 5\}$

Cosets

- Only 3 of these cosets are unique. They are:
  $\{0,3\},\{1,4\},\{2,5\}$
- Further,$\{0,3\} \cup \{1,4\} \cup \{2,5\} = \mathbb{Z}_6$
- This leads to following theorem:
  For any $H \leq G$, left cosets of H in G partition G
  OR
  The group G is disjoint union of left cosets of H in G
- Number of left cosets=number of right cosets

Cosets

- For $H \leq G$ ***Index*** of H in G is number of left(right) cosets of H in G denoted by $[G : H]$
- ***Theorem(Lagrange):*** Let G be a finite group and let $H \leq G$. Then $|G|/|H| = [G : H]$ is the number of distinct cosets of H in G. Hence, if H is a subgroup of G then, number of elements in H must divide number of elements in G.
- **Converse is not true**

Double Cosets

- Generalize the notion of cosets
- Let H and K be subgroups of group G.
- An (H, K)-***double coset*** is a set of the form HxK for some $x \in G$. Here HxK is defied as:

  $HxK = \{hxk | h \in H and k \in K\}$

- $(H, \{id\})$-double cosets are the right cosets of H and $\{id\}, H$-double cosets are the left cosets of H
- Set of all (H, K)-double cosets is denoted as $H \backslash G / K$

Double Cosets

- $H\backslash G/K$ partitions G
- (H, K)-double cosets for given H, K need not be of same cardinality
- **Example:** If G is S_3, $H =< (1,2) >$ and $K =< (1,3) >$, then the two resulting (H, K)-double cosets are $\{e, (1,2), (1,3), (1,3,2)\}$ and {(2,3),(1,2,3)}

Isomorphisms

- Two groups $(G, \circ)$ and $(H, \cdot)$ are ***isomorphic*** $(G \cong H)$ if $\exists$ bijective map $\phi : G \to H$ preserving group operation. ϕ is then called an ***isomorphism***.
- **Example:** $\mathbb{Z}_4 \cong \langle i \rangle$
- ***Theorem(Caylay):*** Every group is isomorphic to a group of permutations(err, but we don't know the isomorphism ϕ)
- Calay's theorems it important to find an isomorphism of the group G that we want to study to a group that we know a lot about(S_n or matrix groups)
- **Example:** $\mathbb{Z}_3$ is isomorphic with $G = \{(0), (012), (021)\}$

Automorphisms

- If instead ϕ is $\phi : G \to G$ and its bijective, we call it ***automorphism***
- **Example:** $\phi : \mathbb{R} \to \mathbb{R}$ s.t. $\phi(x) = e^x$ is an automorphism between $(R, +)$ and itself

Group Actions

- **A left action** of G on X is a map $G \times X \mapsto X$ given by $(g, x) \mapsto gx$ where,

  1. $ex = x \ \ \forall x \in X$
  2. $(g_1 g_2)x = g_1(g_2 x), \forall x \in X$ and $\forall g_1, g_2 \in G$

- If conditions 1 and 2 are satisfied we call X as the ***G-set*** (That is same as saying G 'can act' on X)

- Every group trivially acts on every set through the map $(g, x) \mapsto x$

- **Example:** $GL_2(\mathbb{R})$ acting on $X = \mathbb{R}^2$ via matrix-vector multiplication. We can verify that condition 1 is satisfied with $e = I_2$ and condition 2 is satisfied because matrix multiplication is associative.

G-equivalence

- **G-equivalence:** If G acts on a set X and $x, y \in X$, then x is said to be G-***equivalent*** to y if $\exists g \in G$ s.t. $gx = y$. We denote G-***equivalence*** of x and y as $x \sim_G y$
- One can check that G-***equivalence*** is an equivalence relation by verifying its reflexive, symmetry and transitive properties

Orbits

- Each G-equivalent partition of X so formed is called ***orbit*** of X under G.
- We denote each orbit by a representative. e.g. orbit containing X is called $\mathcal{O}_x$
- **Example:** $G = \{(1), (123), (132), (45), (123)(45), (132)(45)\}$ and $X = \{1, 2, 3, 4, 5\}$
- X is a G-set
- The orbits are:

  1. $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}_3 = \{1, 2, 3\}$ and
  2. $\mathcal{O}_4 = \mathcal{O}_5 = \{4, 5\}$

Fixed point sets

- ***Fixed point set*** X_g of $g \in G$ is the set $\{x | x \in X_g$ if $x \in X$ and $gx = x\}$
- **Example:** $X = \{1, 2, 3, 4, 5, 6\}$ and suppose G is permutation group given by permutations $\{(1), (12)(3456), (35)(46), (12)(3654)\}$
- Fixed point sets in X are:

  1. $X_{(1)} = X$
  2. $X_{(35)(46)} = \{1, 2\}$
  3. $X_{(12)(3456)} = X_{(12)(3654)} = \phi$

Stabilizer Subgroup

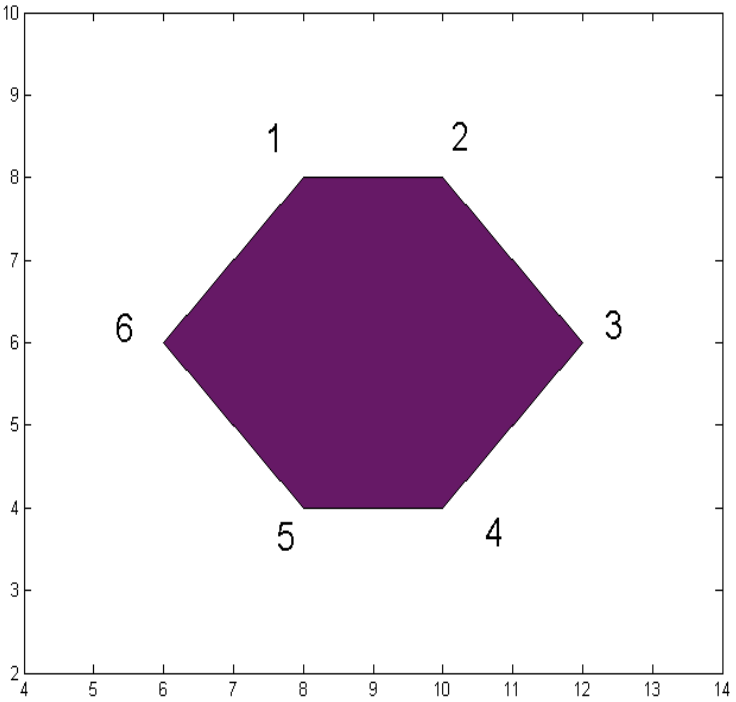
- Subgroup of G, denoted as G_x that fixes given x called ***stabilizer subgroup***
- **Example:** $X = \{1, 2, 3, 4, 5, 6\}$ and suppose G is permutation group given by permutations $\{(1), (12)(3456), (35)(46), (12)(3654)\}$
- The stabilizer subgroups are:

  1. $G_1 = G_2 = \{(1), (35)(46)\}$
  2. $G_3 = G_4 = G_5 = G_6 = \{(1)\}$

Groups acting on polyhedra

- We define groups that act on the set of extreme rays of a polyhedral cone
- Such an action would rearrange the face lattice of the polyhedron
- Specifcally, we require that group action must keep the inclusion ralationships of the of face lattice intact

Example $\mathcal{P}$

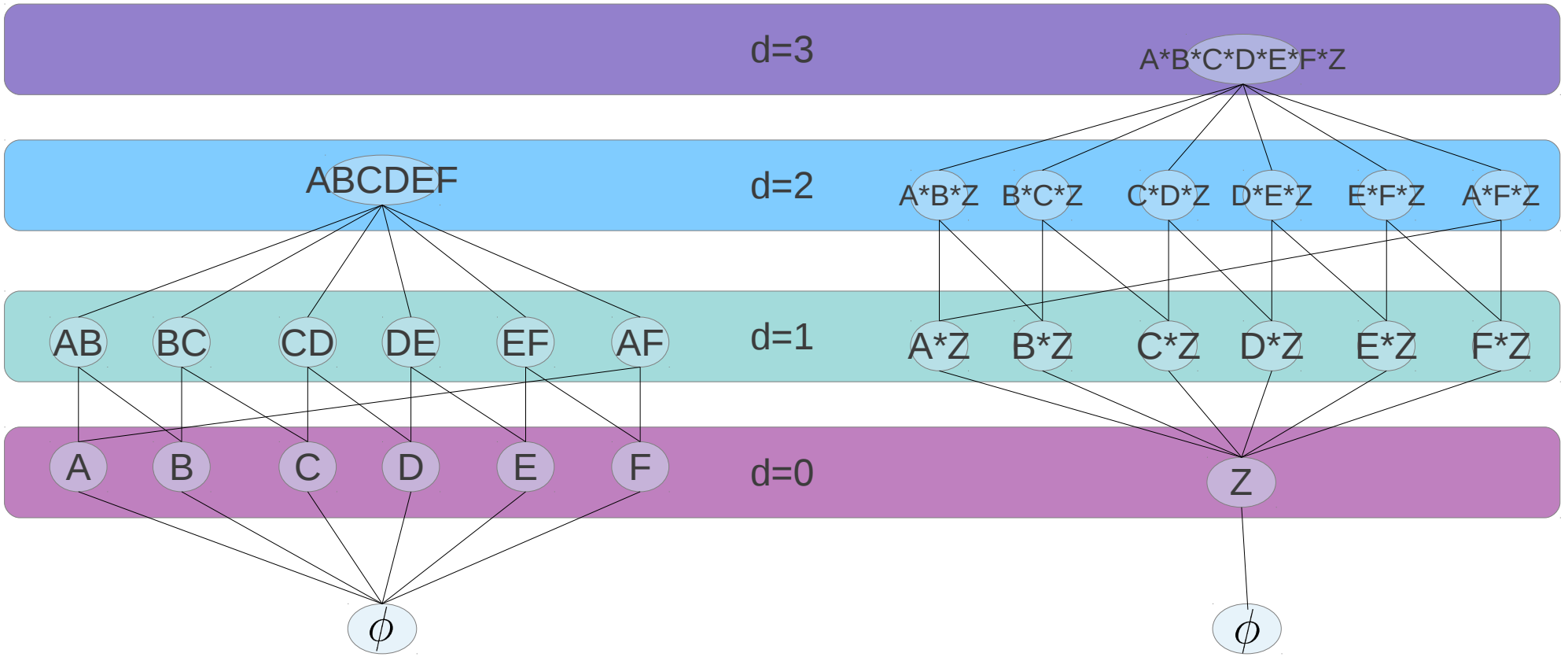


$$x_1 + x_2 \leq 18$$
$$x_1 - x_2 \leq 6$$
$$x_2 \leq 8$$
$$-x_1 + x_2 \leq 0$$
$$-x_1 - x_2 \leq -12$$
$$-x_2 \leq -4$$

After Homogenization: $\mathcal{P}'$

$$
\begin{aligned}
x_1 + x_2 - 18x_3 &\leq 0 \\
x_1 - x_2 - 6x_3 &\leq 0 \\
x_2 - 8x_3 &\leq 0 \\
-x_1 + x_2 &\leq 0 \\
-x_1 - x_2 + 12x_3 &\leq 0 \\
-x_2 + 4x_3 &\leq 0 \\
-x_3 &\leq 0
\end{aligned}
$$

Hasse Diagrams of $\mathcal{P}$ and $\mathcal{P}'$



Hasse Diagram of $\mathcal{P}$

Hasse Diagram of $\mathcal{P}'$

Combinatorial Automorphism Group of $\mathcal{P}$

- Let there be two polyhedra $\mathcal{P}$ and$\mathcal{Q}$
- Call f a face lattice isomorphism between two face lattices $L(\mathcal{P}), L(\mathcal{Q})$ if f is a bijection of the faces of $\mathcal{P}$ to the faces of $\mathcal{Q}$ such that for all faces F, F' of $\mathcal{P}$,

$$F \subseteq F' \iff f(F) \subseteq f(F')$$

- A face lattice isomorphism between $L(\mathcal{P})$ and $L(\mathcal{P})$ is called a face lattice automorphism

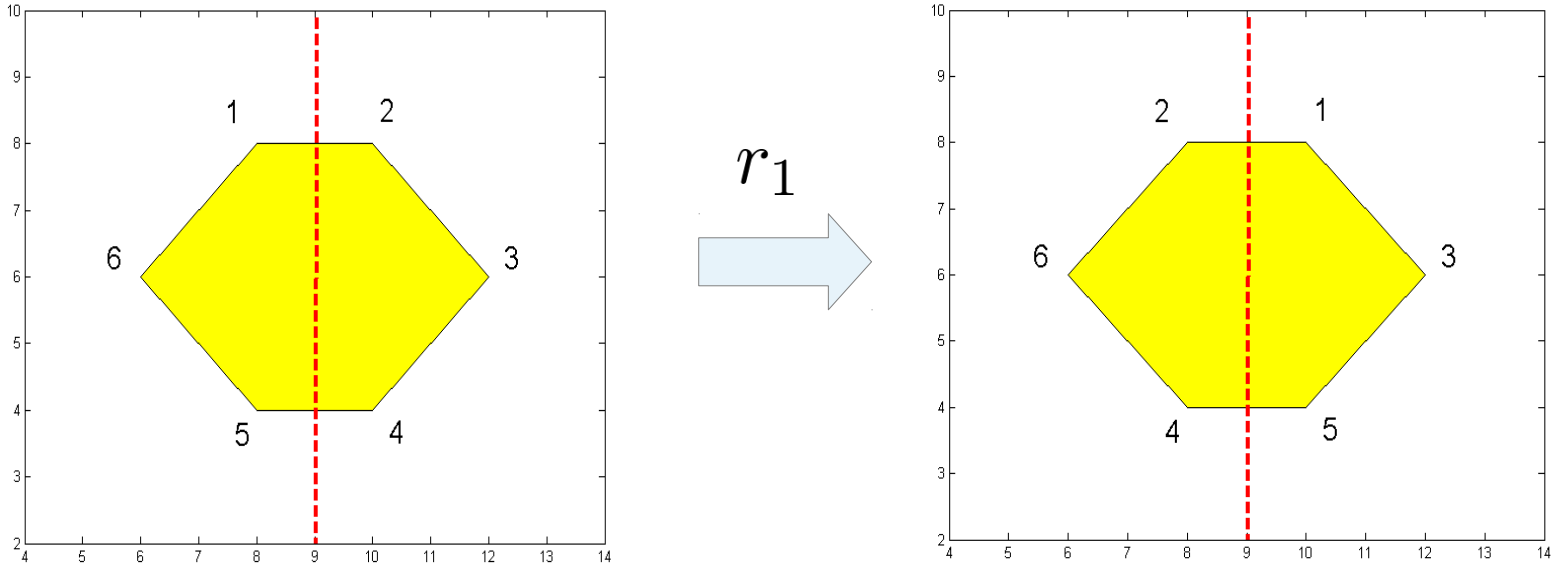
Combinatorial Automorphism Group of $\mathcal{P}$

- A **combinatorial symmetry** of of a polyhedron $\mathcal{P}$ is an automorphism f of its face lattice $L(\mathcal{P})$
- Alternatively, every automorphism of the face lattice can be represented as permutation of the set of extreme rays
- The group of permutations of rays that each member of the group correspond to some automorphism $\mathcal{P}$ is called $Aut(\mathcal{P})$
- $Aut(P) \leq S_n$ where n is the number of extreme rays of $\mathcal{P}$

A subgroup of combinatorial automorphism group

- The combinatorial autmorhism group $Aut(\mathcal{P})$ of the example $\mathcal{P}$ is D_6, the dihedral group corresponding to any hexagon
- Lets say we know a subgroup $G \leq Aut(\mathcal{P})$
- $G = \langle r_1, r_2 \rangle$
- $r_1 = (15)(24)$
- $(12)(45)(36)$

A closer look at G



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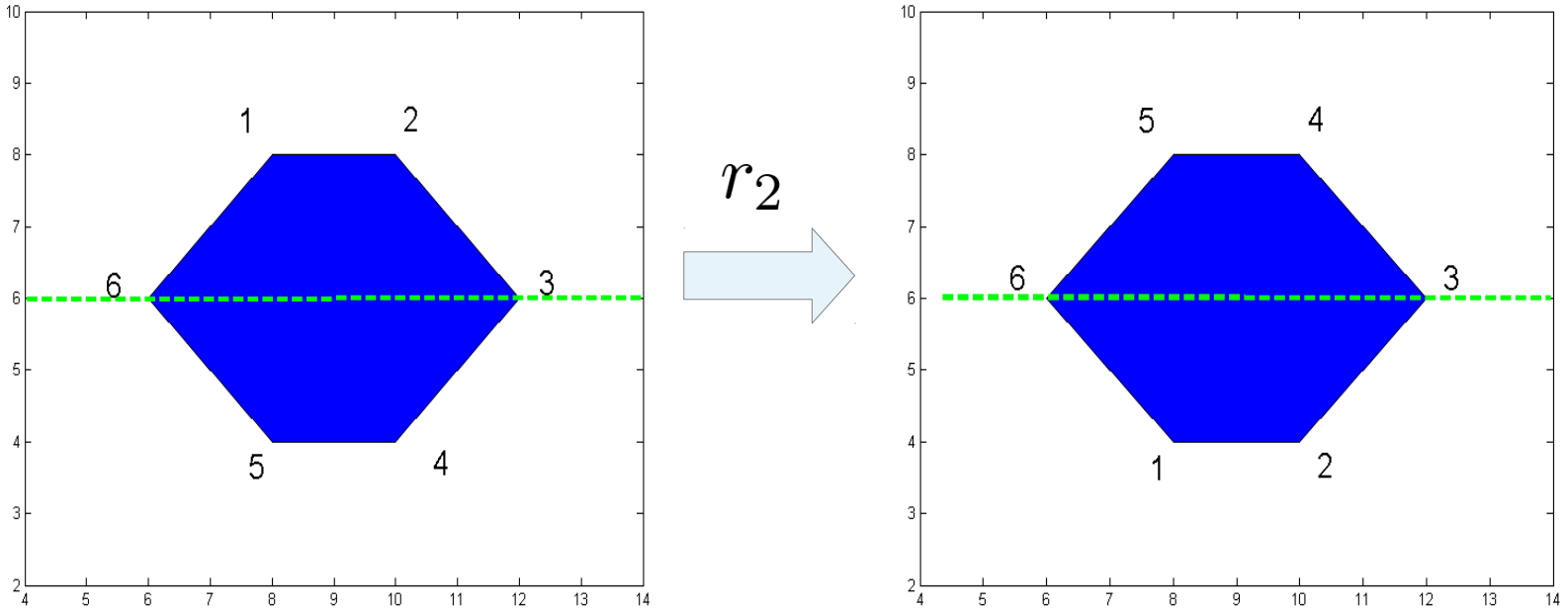

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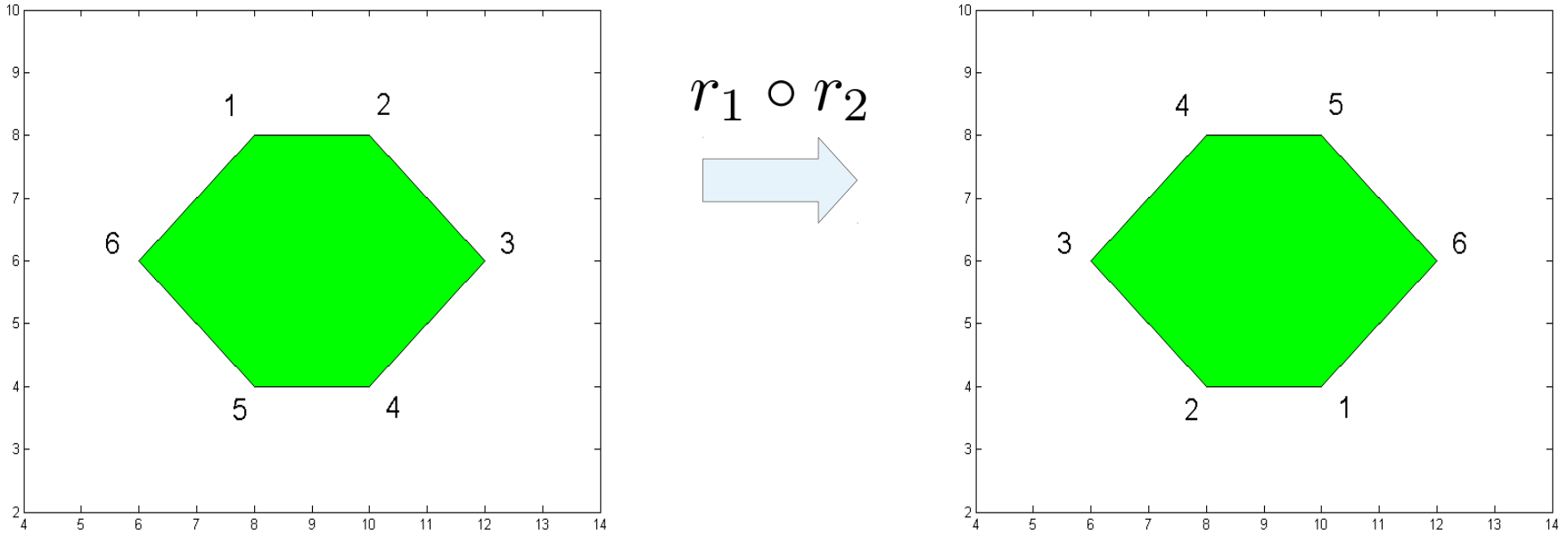
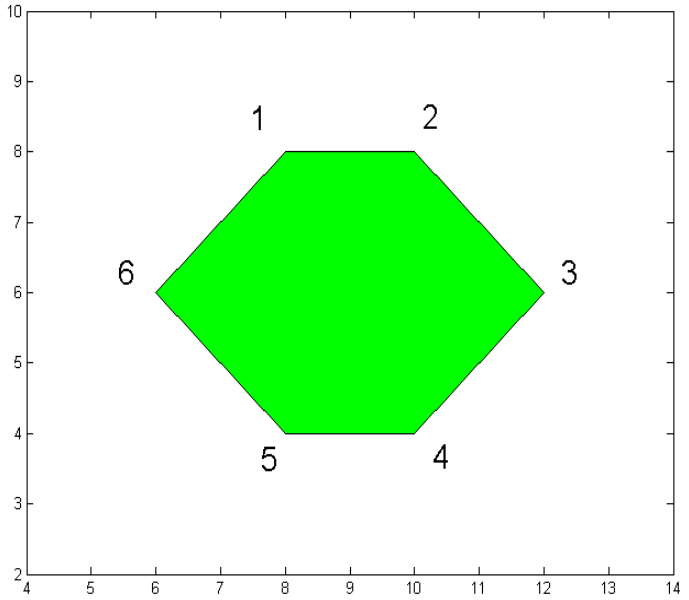
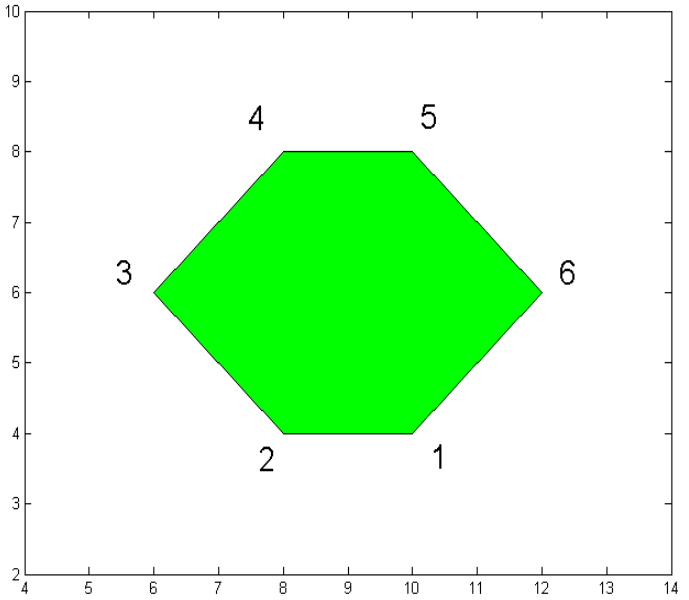


A closer look at G



$r_1 \circ r_2$

$r_2 \circ r_1$



Caylay table for G

$\circ$	id	r_1	r_2	f
r_1	id	r_1	r_2	f
r_1	id	id	f	r_2
r_2	r_2	f	id	id
f	f	r_2	r_1	id

Caylay table for G

$\circ$	id	r_1	r_2	f
r_1	id	r_1	r_2	f
r_1	id	id	f	r_2
r_2	r_2	f	id	id
f	f	r_2	r_1	id

G is abelian!

The Representation Conversion Problem

- WLOG, polyhedra can be expressed in two equivalent ways:
  - (1) The halfspace/inequality representation (H-rep)
  - (2) The extreme ray representation (V-rep)
- *(Prob 1)* (1)---->(2): Extreme ray enumeration
- *(Prob 2)* (2)---->(1): Facet enumeration
- $(Prob1) \leq_p (Prob2)$ and $(Prob2) \leq_p (Prob1)$
- Hence we can try solving *(Prob 2) only*
- Additionally, WLOG we can assume that input polyhedra are actually polyhedral cones

Known exact algorithms

- (PM) Pivoting Methods: Use simplex method as tool Roughly speaking, traverse a directed graph with LP bases as vertices and for every pair of vertices, the existence of reverse simplex pivot creating a directed edge. Recover extreme rays of the polyhedral cone via an onto mapping from set of bases to the set of rays.
- (IM) Incremental Methods: Builds the set of facets of polyhedra formed by successively larger set of input extreme rays
- (DM) Decomposition Methods: Decompose *(Prob 2)* into set of smaller (in dimension) *(Prob1)s* or (Prob2)s and solve them using (PM) or (IM)

The adjacency decomposition method

- Defines a notion of adjacency of rays (neighborhood)
- Maintains a set of G-inequivalent rays of input cone
- For every extreme ray in this set, the problem of finding its neighbors is posed as (Prob 2)
- Symmetry is exploited by keeping track of only the G-inequivalent neighbors

Neighborhood of rays

- Two rays r, r' of a d-dimensional cone $P \subset \mathbb{R}^d$ are **neighbors** if there is a 2-face F of P such that $r, r' \in F$ and $r \neq r'$
- Let $r \in P$ be a ray of d-dimensional cone $P = P(A, 0)$. An inequality $\langle a_i, x \rangle \leq 0$ is **active** in r if it is strictly fulfilled in r, i.e. $\langle a_i, r \rangle = 0$.
- Otherwise it is called **inactive**
- **Axis** of a polyhedral cone $P(A, 0)$ is $a_P := \sum_{i=1}^{m} a_i$
- For a full-dimensional P, $\exists\, r$ is in the interior, $\langle a_P, r \rangle < 0$, hence $a_P \neq 0$

Neighborhood of rays

- Same concept as adjacency of rays in double description method
- Let $Z(r)$ and $Z(r')$ be indices of rows of A tight at r and r' respectively
- **Theorem:** (Fukuda) r and r' are adjacent extreme rays $\Leftrightarrow rank(A_{Z(r)\cap Z(r')}) = d-2$
- Roughly speaking, the common 2-face here is $\overline{F} = \{x : A_{Z(r)\cup Z(r')x=0}\} \cap P$
- $dim(\overline{F}) = nullity(A_{Z(r)\cup Z(r')}) = d - rank$ $= d-(d-2) = 2$

Support Cone

- For an extreme ray $r \in P$, support cone $C(P, r)$ of r is

$$C(P, r) = \{x \in \mathbb{R}^d : \langle a_j, x \rangle \leq 0 \ \ \forall j \in I\}$$

- I is the index set of inequalities active at r
- Clearly, $P \subseteq C(P, r)$, but, $C(P, r)$ is not pointed ($r\mathbb{R} \subseteq C(P, r)$)
- We make it pointed by considering its intersection with any hyperplane H
- This creates cone C_H which is pointed and has much of the same combinatorial structure as $C(P, r)$

Reduced Support Cone

- Let $H_P = \{x \in \mathbb{R}^d : \langle a_P, x \rangle = 0\}$
- $\overline{C}(P, r) = C(P, r) \cap H_P$
- Clearly, if $C(P, r) \in \mathbb{R}^d$ then, $\overline{C}(P, r) \in \mathbb{R}^{d-1}$
  (Simply substitute H_p in $C(P, r)$ to eliminate a variable)

Extreme rays of $\overline{C}(P, r)$

- **Lemma:** Let r, r' be two neighborly rays of the cone $P = P(A, 0)$. Then after scaling r', the difference $r' - r$ is a ray of the reduced support cone $\overline{C}(P, r)$
  Proof:

- For an extreme ray r of $P(A, r)$, if a_i are columns of A^T, $\sum_i \langle a_i, r\rangle = \langle a_P, r\rangle < 0$. Same holds for r'

- WLOG, $\sum_{i=1}^{m} \langle a_i, r' - r\rangle = \langle a_P, r' - r\rangle = 0$
  (since, $\langle a_P, r' - r\rangle = \langle a_P, r'\rangle - \langle a_P, r\rangle$ and
  $\langle a_P, \alpha r'\rangle = \alpha \langle a_P, r'\rangle$ for any non-negative scalar α)

- We just showed that $r' - r \in H_P$. Now we show that $r' - r \in C(P, r)$

Proof contd...

- Let $\langle a_1, x\rangle \leq 0, \ldots, \langle a_{d-1}, x\rangle \leq 0$ be $d-1$ inequalities in $Z(r)$. Additionally, they form H-rep of $C(P, r)$
- Because, r, r' are adjacent, WLOG $\langle a_2, x\rangle \leq 0, \ldots, \langle a_{d-1}, x\rangle \leq 0$ are tight at r' $(Z(r) \cap Z(r'))$
- $C(P, r)$ is $A_{Z(r)} \leq 0$
- At $r' - r$, $A_{Z(r)}(r' - r) = A_{Z(r)}r' - A_{Z(r)}r = A_{Z(r)}r'$
- Of d-1 inequalities in $A_{Z(r)}$, $d-2$ are tight at r'
- The one inequality that is not tight, say, $\langle a_1, x\rangle \leq 0$, holds with strict inequality at r', since it is not active at r' and $r' \in P$.
- Additionally, $d-2$ tight inequalities in $\overline{C}(P, r) \subseteq \mathbb{R}^d$ imply that $r' - r$ is not just a ray of $\overline{C}(P, r)$ but an extreme ray

Moral of the story

- For every extreme ray of r found so far we can enumerate the extreme rays of $C(P, r)$ to find all neightbors of r
- Doing so recursively, we shall traverse the adjacency graph of extreme rays (which is connected, as it turns out)

Computing the neighbors

- Suppose we performed extreme ray enumeration on $\overline{C}(P, r)$
- Every neighbor r' of r can be written as $r + \lambda s$ where s is some extreme ray of $\overline{C}(P, r)$ and $\lambda \in \mathbb{R}^+$
- For an inequality $\langle a_j, x \rangle \leq 0$ of P that is active in r', $\langle a_j, r' \rangle = 0$ i.e. $\langle a_j, r + \lambda s \rangle = 0 \Rightarrow \langle a_j, r \rangle + \lambda \langle a_j, s \rangle = 0$
- Hence $\lambda = -\frac{\langle a_j, r \rangle}{\langle a_j, s \rangle}$
- $\lambda > 0 \Leftrightarrow \langle a_j, r \rangle < 0$ and $\langle a_j, s \rangle > 0$

- Additionally, each inequality of P with $\lambda > 0$ sets an upper bound on λ as follows:
  $r' \in P \Rightarrow \forall k : \ \langle a_k, r' \rangle = \langle a_k, r + \lambda s \rangle \leq 0$
  $\Leftrightarrow \forall k : \ \lambda \leq -\frac{\langle a_k, r \rangle}{\langle a_k, s \rangle}$ where $a_k, s > 0$

Computing the neighbors

- Provided that the $I := \{j : \langle a_j, r\rangle < 0 \text{ and } \langle a_j, s\rangle > 0\} \neq \phi$, we choose λ as

$$\lambda = min_{i \in I} - \frac{\langle a_i, r\rangle}{\langle a_i, s\rangle}$$

- **Claim:** I is always non-empty
  Proof: s is a ray of $\overline{C}(P, r) \Rightarrow \langle a_P, s\rangle = \sum_{i=1}^{m} \langle a_i, s\rangle = 0$
  Since $\overline{C}(P, r)$ is full dimensional in $\mathbb{R}^{d-1}$,
  $\exists$ at least 1 inequality of $\overline{C}(P, r)$ s.t. $\langle a_k, s\rangle < 0$.
  $\therefore$ To make $\sum_{i=1}^{m} \langle a_i, s\rangle = 0$,
  $\exists$ at least 1 inequality s.t. $\langle a_j, s\rangle > 0$
  Clearly, such inequality $\langle a_j, x\rangle \leq 0$ can't be inequality of $\overline{C}(P, r)$ and hence is inactive at r implying $\langle a_j, r\rangle < 0$

Algorithm for computing neighbors

Input: Polyhedral cone $P = P(A, 0)$, ray r
Output: List N of all neighbors of r in P

$N \leftarrow \emptyset$
$P' \leftarrow$ reduced support cone $\bar{C}(P, r)$ of r
compute set of rays R' of P'
forall $s \in R'$ **do**
  $I \leftarrow \{j \ : \ \langle a_j, r\rangle < 0 \text{ and } \langle a_j, s\rangle > 0\}$
  $j \leftarrow \min_{i \in I} -\frac{\langle a_i, r\rangle}{\langle a_i, s\rangle}$
  $r' \leftarrow r - \frac{\langle a_j, r\rangle}{\langle a_j, s\rangle} s$
  $N \leftarrow N \cup \{r'\}$
end

Adjacency Decomposition Method

Input: Polyhedral cone $P = P(A, 0)$, permutation group G
Output: List of G-inequivalent rays R

```
1  r_0 ← one ray of P
2  R ← {r_0}
3  T ← {r_0}                                          // todo-list of rays
4  while T ≠ ∅ do
5  |   choose r ∈ T
6  |   T ← T \ {r}
7  |   P' ← reduced support cone C̄(P, r) of r
8  |   compute set of rays R' of P'
9  |   forall s ∈ R' do
10 |   |   compute the neighboring ray r' of r in direction of s
11 |   |   if r'^G ∩ R = ∅ then
12 |   |   |   R ← R ∪ {r'}
13 |   |   |   T ← T ∪ {r'}
14 |   |   end
15 |   end
16 end
```

Questions