Exploiting Symmetry in Computing Polyhedral Bounds on Network Coding Rate Regions

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Outline

1. Motivation and Background
   - Explicit Expressions for MSNC Rate Regions
   - Symmetry Can Help in Polyhedral Computation
   - Symmetry in Network Coding
   - Classes of Polyhedral Symmetries
2. Symmetries of polyhedral Bounds and NSG
   - NSG and Rate Region Symmetries
3. Convex Hull method (CHM)
   - Basic ideas behind CHM
4. Symmetry Exploiting CHM (symCHM)
   - Symmetry Exploitation in symCHM
   - Comparison of CHM and symCHM

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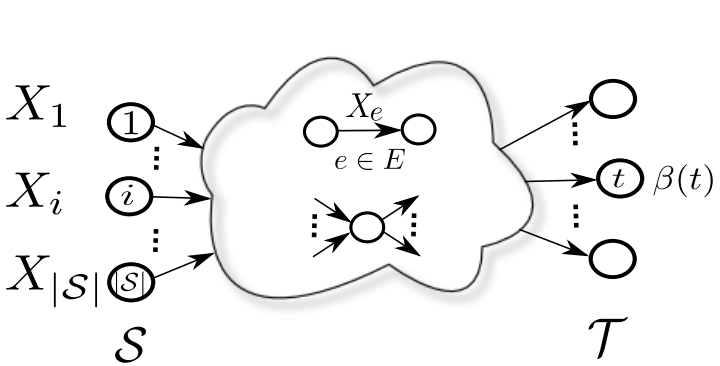
Problems of Interest

MSNC

- Multisource Network Coding
- Described by tuple $(\mathcal{G}, \mathcal{S}, \mathcal{T}, \beta)$



- $\beta : \mathcal{T} \to 2^{\mathcal{S}} \setminus \emptyset$ allows arbitrary subset demands

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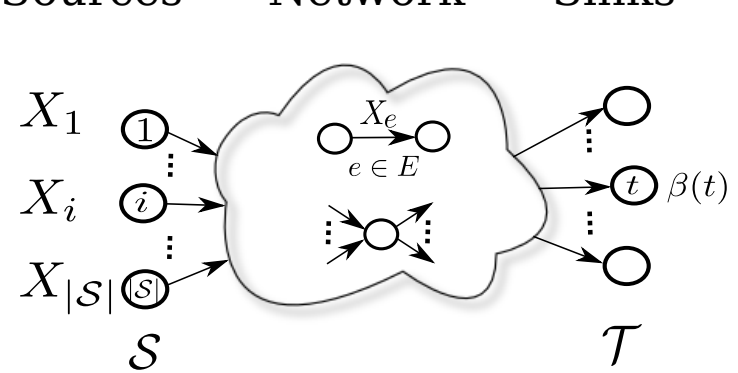

- $\beta : \mathcal{T} \to 2^{\mathcal{S}} \setminus \emptyset$ allows arbitrary subset demands

- $\mathcal{X}_n \triangleq$ RVs associated with a MSNC instance
- $n \triangleq |\mathcal{S}| + |E|$
- $h : 2^{\mathcal{X}_n} \setminus \emptyset \to \mathbb{R}_{\geq 0}$ is the entropy function
- $\omega_s, s \in \mathcal{S}$ and $R_e, e \in E$ are the source and edge rates

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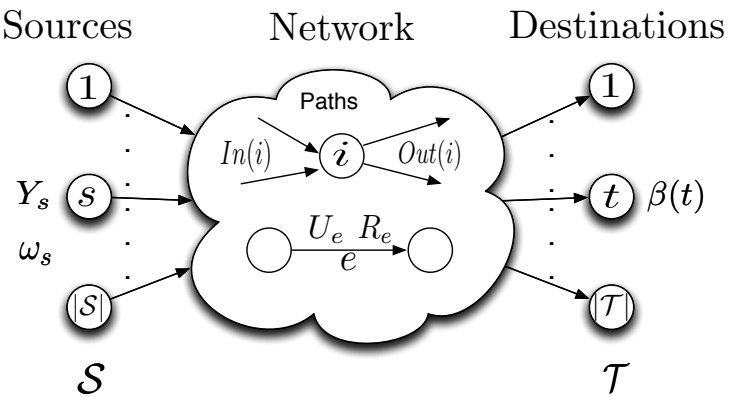


Entropy function must satisfy:

$(\mathcal{L}_1)$ $\sum_{s\in\mathcal{S}} h_s = h_{\mathcal{S}}$

$(\mathcal{L}_2)^a$ $h_{\mathsf{In}(i)} = h_{\mathsf{In}(i)\cup\mathsf{Out}(i)}$, $i \in V$

$(\mathcal{L}_3)$ $h_e \le R_e$, $\quad h_s \ge \omega_s$

---

[a] $\mathcal{L}_2$ can be partitioned into $\mathcal{L}_2^1, \mathcal{L}_2^2.\mathcal{L}_2^3$ (constraints associated with sources, intermmediate nodes and sinks resp.)

Implicit characterization

Yan, Yeung and Zhang's Characterization

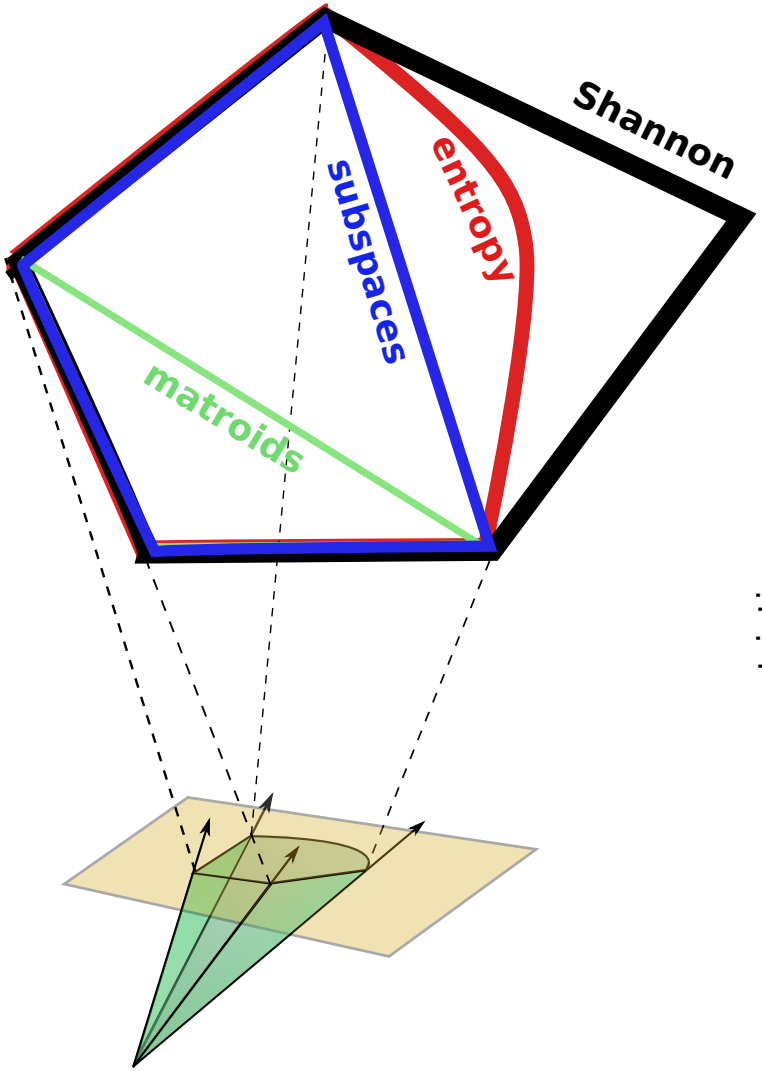
The set of all source rate- edge rate tuples $(\boldsymbol{\omega}, \mathbf{r})$ is given as

$$\mathcal{R}_* = \text{proj}_{\boldsymbol{\omega},\mathbf{r}}(\overline{\text{con}(\Gamma_n^* \cap \mathcal{L}')} \cap \mathcal{L}'') \tag{1}$$

Notation

- $\mathcal{L}' = \mathcal{L}_1 \cup \mathcal{L}_2^1 \cup \mathcal{L}_2^3$, $\mathcal{L}'' = \mathcal{L}_3 \cup \mathcal{L}_2^3$
- $\text{proj}_{\boldsymbol{\omega},\mathbf{r}}(\cdot)$ stands for linear projection onto rate variables
- $\text{con}(\cdot)$ stands for conic hull, $\overline{(\cdot)}$ stands for closure
- Γ_n^* stands for the *unknown* entropy function region: set of all $2^n - 1$ dimensional vectors arising from entropy function

Explicit Characterization?

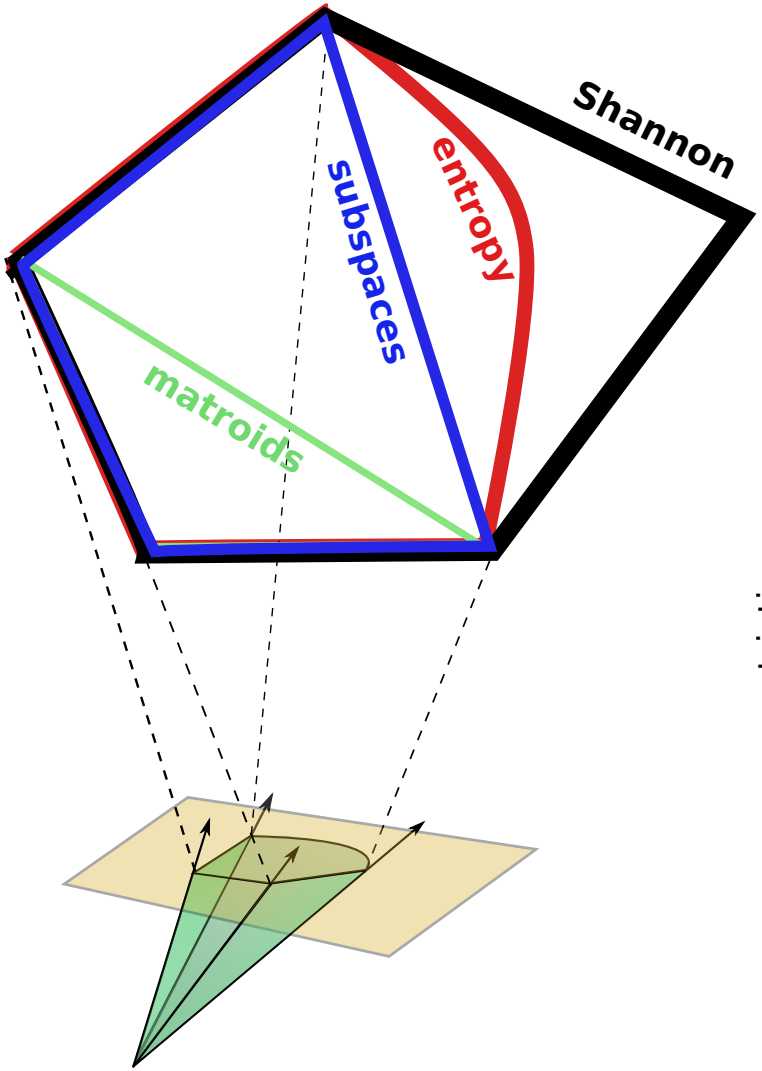


more restrictive

class of polymatroids	bound	comment
polymatroidal set functions	Γ_n	necessary but not sufficient
polymatroidal set functions arising from random variables	Γ_n^*	necessary and sufficient every possible code
Arising from a column partition of matrices over finite field of size q	$\Gamma_{n,k,q}^{\subseteq \text{space}}$	sufficient but not necessary, vector linear codes
Arising from matrices over finite field of size q	Γ_n^q	sufficient but not necessary, scalar linear codes

$$\mathcal{R}_k = \text{proj}_{\boldsymbol{\omega},\mathbf{r}}(\Gamma_n^k \cap \mathcal{L}_{123}^{\cap}), k \in \{\text{in},\text{out}\}$$

Explicit Characterization?



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$$\mathcal{R}_k = \text{proj}_{\boldsymbol{\omega},\mathbf{r}}(\Gamma_n^k \cap \mathcal{L}_{123}^{\cap}), k \in \{\text{in},\text{out}\} \leftarrow \text{Polyhedral Projection}$$

Leave no stone unturned

- We developed a C software library for polyhedral projection called `chm`
- Accepts arbitrary polyhedra as input, uses rational arithmetic, rational LP solver QSopt_ex, and gives exact answers
- The underlying algorithm Convex Hull Method was first used by Xu, Wang and Sun* for computing non-Shannon information inequalities
- We have used it for computing non-Shannon information inequalities and LP/Shannon outer bounds on rate regions
- Can also be used for solving interesting problems related to quantum information called marginal problems
- `chm` currently turns a blind eye to symmetries of input polyhedra

*W. Xu, J. Wang, J. Sun, A projection method for derivation of non-Shannon-type information inequalities, The proceedings of IEEE International Symposium on Information Theory (ISIT), pages 2116 – 2120, 2008

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Symmetry and Polyhedral Computation

- Symmetry can be exploited in polyhedral representation conversion (given extreme points compute inequalities or vice versa) [1]
- Symmetry can be exploited in linear programming [2]

[1] D. Bremner and M. Sikiric and A. Schürmann. Polyhedral representation conversion up to symmetries, Proceedings of the 2006 CRM workshop on polyhedral computation, AMS/CRM Lecture Notes 48 (2009), 45-71.

[2] R. Bödi and K. Herr, Symmetries in linear and integer programs, 2009. arXiv:0908.3329 [math.CO]

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Symmetry and Polyhedral Computation

- Symmetry can be exploited in polyhedral representation conversion (given extreme points compute inequalities or vice versa) [1]
- Symmetry can be exploited in linear programming [2]
- What about polyhedral projection (esp. CHM)?

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Network Symmetry Group*

- Those permutations of $\{1, ..., n\}$ that keep network constraints $\mathcal{L}_{123}$ invariant
- Form a group called Network Symmetry Group (NSG)
- Can be computed using algorithms for computing automorphism groups of graphs

*Jayant Apte, and John MacLaren Walsh, Symmetry in Network Coding, extended version, https://sites.google.com/site/jayantapteshomepage/

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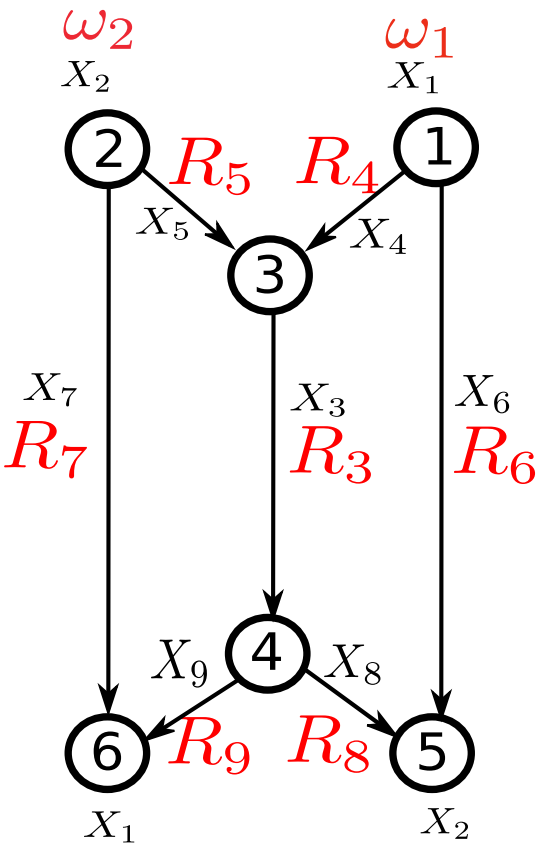
Example: Butterfly Network



$\mathcal{L}_1$ source independence

$\mathcal{L}_2$ node constraints

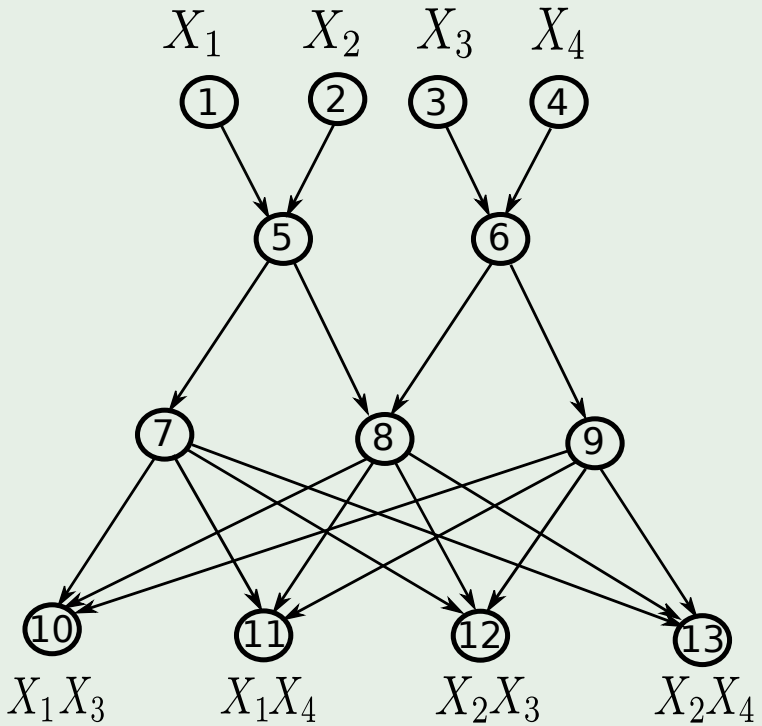
$\mathcal{L}_3$ rate constraints

	(1,2)(3)(4,5)(6,7)(8,9)
$h_1+h_2=h_{\{1,2\}}$	$h_1+h_2=h_{\{1,2\}}$
$h_1=h_{\{1,4,6\}}$	$h_2=h_{\{2,5,7\}}$
$h_2=h_{\{2,5,7\}}$	$h_1=h_{\{1,4,6\}}$
$h_{\{4,5\}}=h_{\{3,4,5\}}$	$h_{\{4,5\}}=h_{\{3,4,5\}}$
$h_3=h_{\{3,8,9\}}$	$h_3=h_{\{3,8,9\}}$
$h_{\{6,8\}}=h_{\{2,6,8\}}$	$h_{\{7,9\}}=h_{\{1,7,9\}}$
$h_{\{7,9\}}=h_{\{1,7,9\}}$	$h_{\{6,8\}}=h_{\{2,6,8\}}$
$h_1 \geq \omega_1$	$h_2 \geq \omega_2$
$h_2 \geq \omega_2$	$h_1 \geq \omega_1$
$h_3 \leq R_3$	$h_3 \leq R_3$
$h_4 \leq R_4$	$h_5 \leq R_5$
$h_5 \leq R_5$	$h_4 \leq R_4$
$h_6 \leq R_6$	$h_7 \leq R_7$
$h_7 \leq R_7$	$h_6 \leq R_6$
$h_8 \leq R_8$	$h_9 \leq R_9$
$h_9 \leq R_9$	$h_8 \leq R_8$

Example: M-Network

Example

An MSNC instance



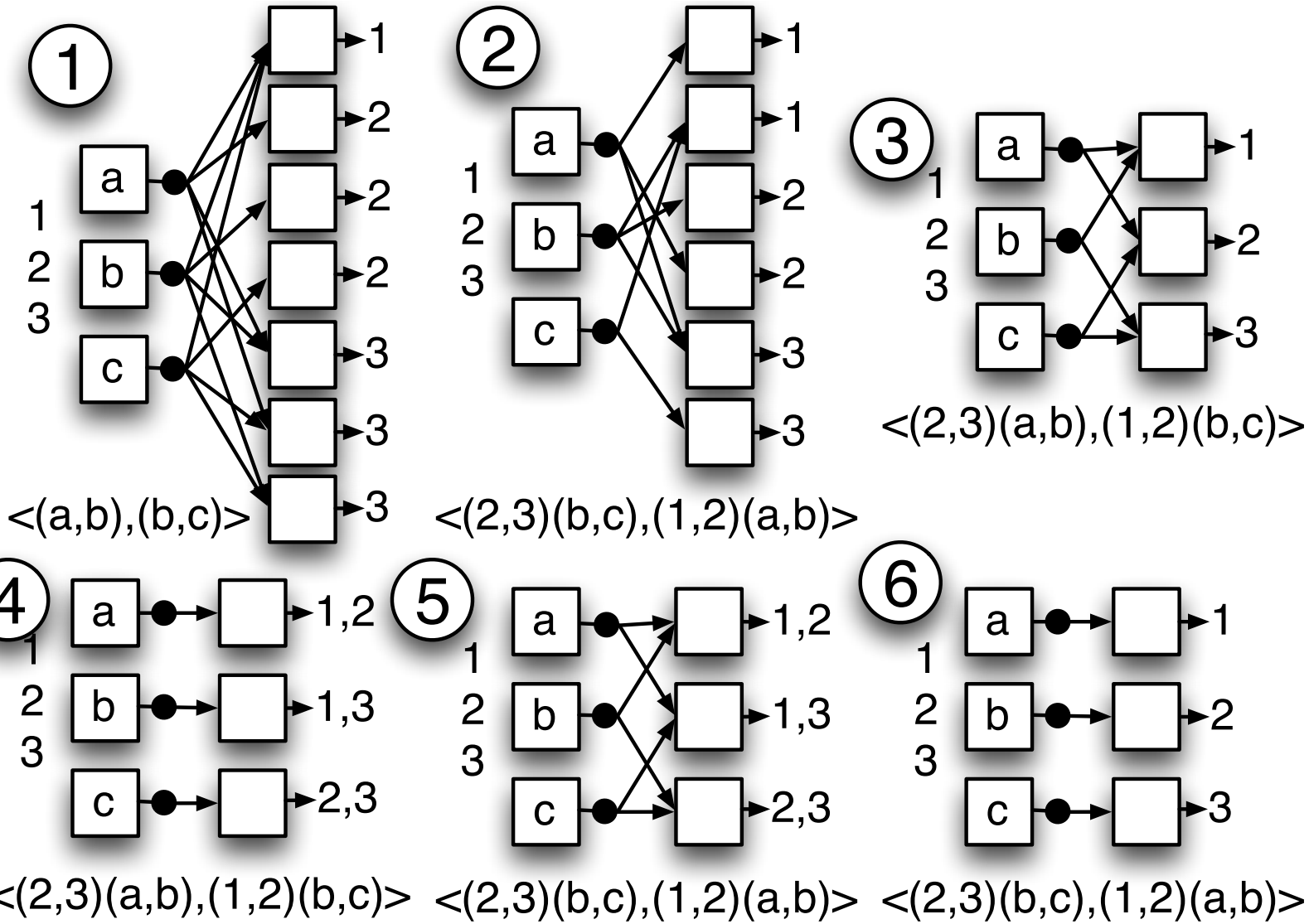
$\Rightarrow$

The Network Symmetry group is generated by:

$$
\begin{aligned}
g_1 =& (1,3)(2,4)(5,7)(6,8) \\
& (9,12)(10,11)(13,21) \\
& (14,23)(15,22)(16,24)(18,19) \\
g_2 =& (3,4)(7,8)(13,14)(15,16) \\
& (17,18)(19,20)(21,22)(23,24) \\
g_3 =& (1,2)(5,6)(13,15)(14,16) \\
& (17,19)(18,20)(21,23)(22,24)
\end{aligned}
$$

A group of order 8

Example: Independent Distributed Source Coding



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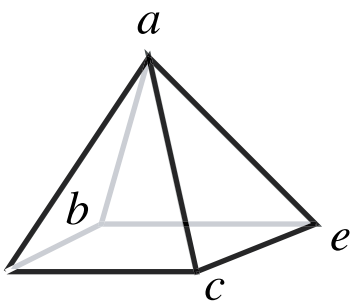
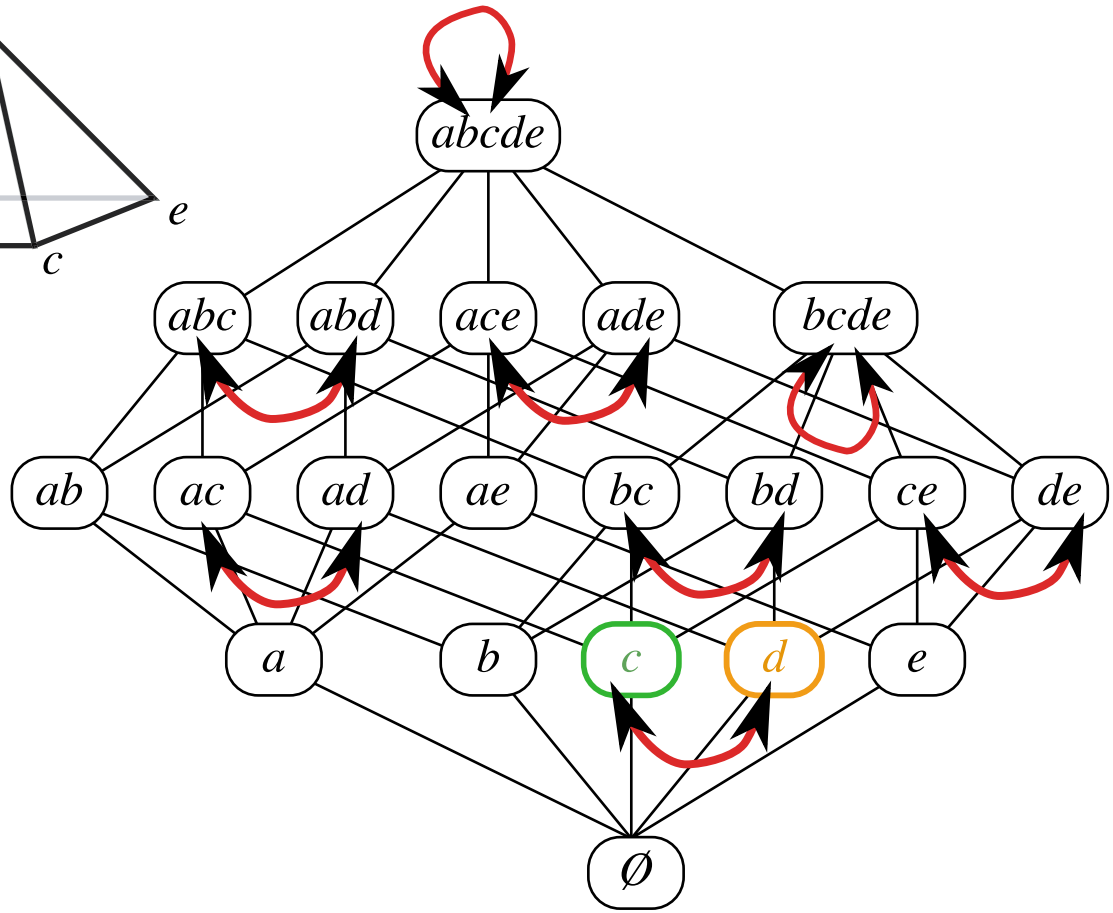
Combinatorial Symmetries

combinatorial symmetries





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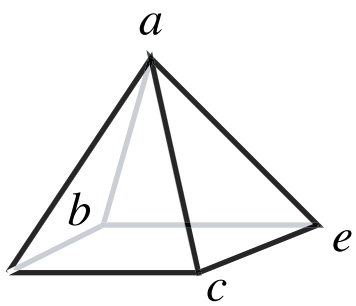
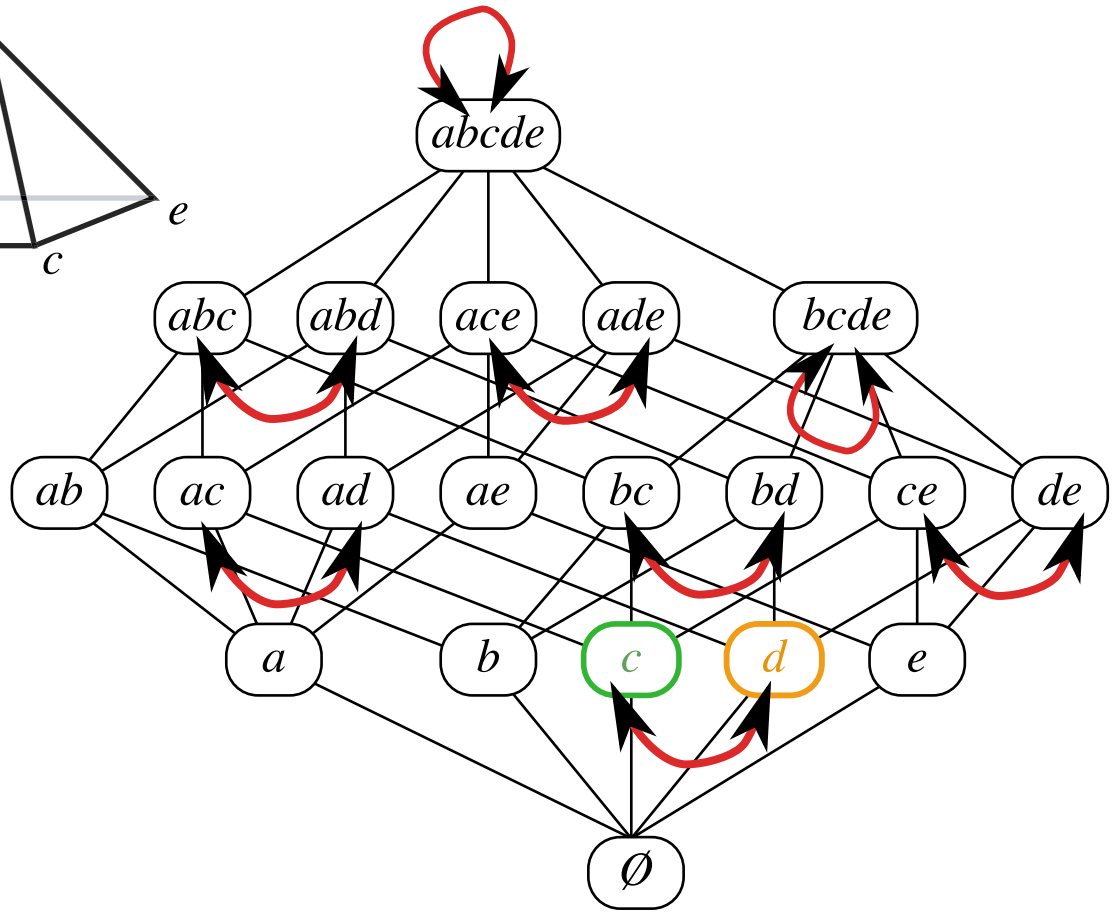


combinatorial symmetries

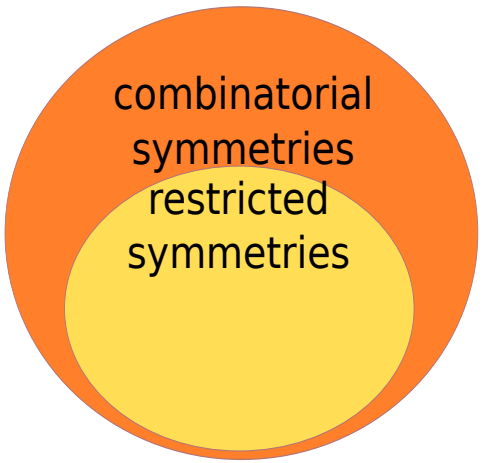


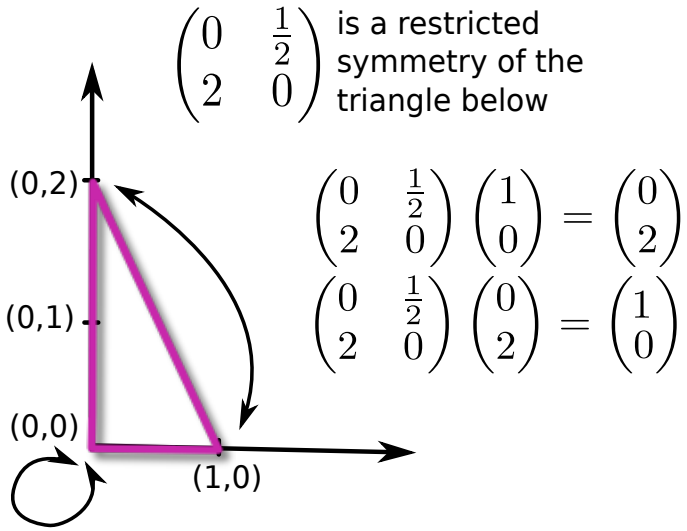
Very difficult to compute (require representation conversion*):

*Volker Kaibel and Alexander Schwartz. On the complexity of polytope isomorphism problems. Graphs and Combinatorics, 19(2):215–230, 2003

Restricted Symmetries

A subgroup of combinatorial symmetry group:





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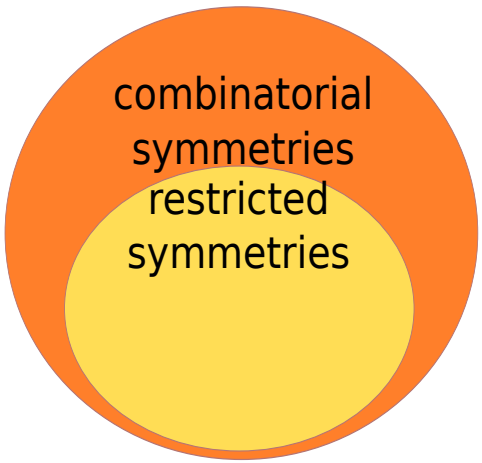
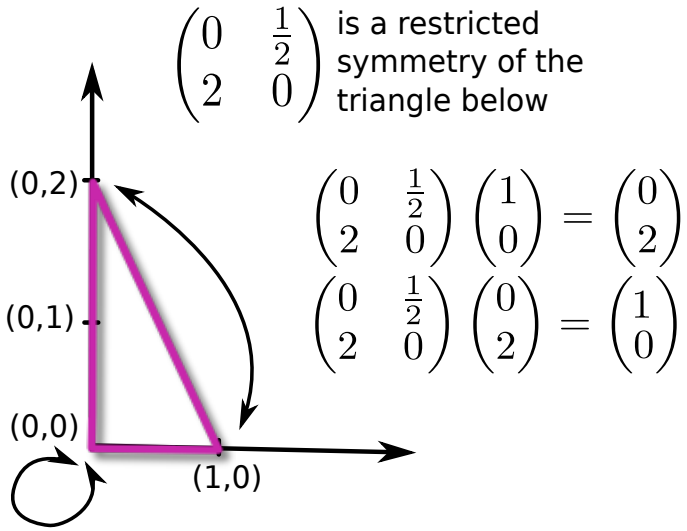


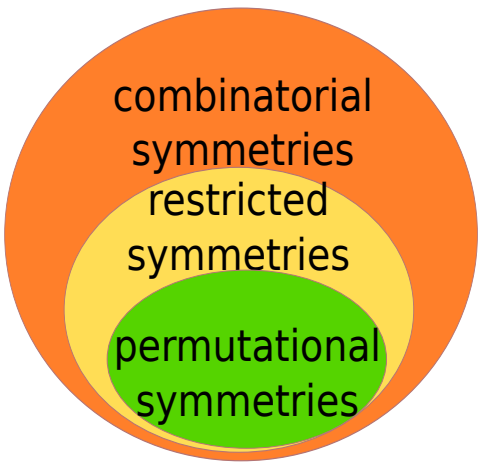


Can be obtained by computing automorphism group of an edge colored graph*:

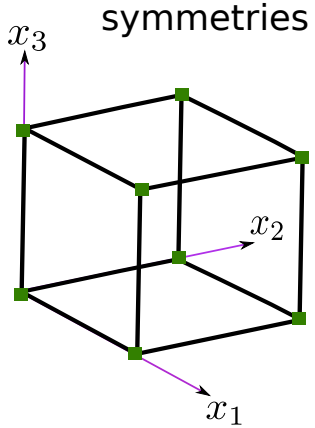
*Computing symmetry groups of polyhedra. David Bremner, Mathieu Dutour Sikiric, Dmitrii V. Pasechnik, Thomas Rehn, Achill Schrmann. arXiv:1210.0206v3 [math.CO], 2014

Permutational Symmetries

A subgroup of restricted symmetry group:



All 3! permutations
of standard basis vectors
of $\mathbb{R}^3$ are permutational
symmetries of the cube



Can be thought of as restricted symmetry arising from permutation matrices

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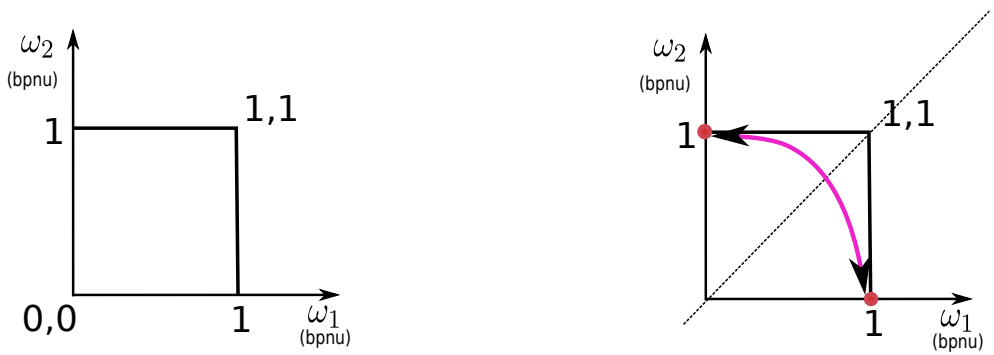
Symmetries of Rate Regions

- NSG can be interpreted as permutational symmetry group of rate region

(ω_1, ω_2) is achievable rate tuple iff
(ω_2, ω_1) is achievable rate tuple



(bpnu: bits per network use)

Symmetries of Rate Regions

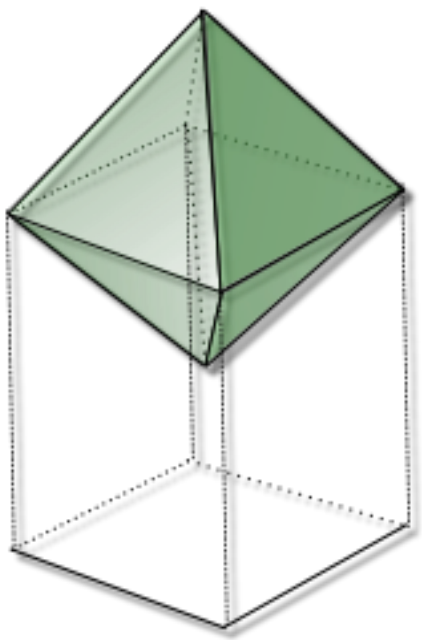
- Let $\mathcal{L}_{123}^{\cap}$ be the intersection of halfspaces and hyperplanes associated with network constraints
- NSG can be interpreted as permutational symmetry group of rate region $\mathcal{R}_*$
- NSG can be interpreted as permutational symmetry group of $\Gamma_n^{\text{in}} \cap \mathcal{L}_{123}^{\cap}$, $\Gamma_n^{\text{out}} \cap \mathcal{L}_{123}^{\cap}$[3]

[3] provided that Γ_n^{in} and Γ_n^{out} are setwise invariant under random variable permutations (which is indeed the case with most known bounds on entropy)

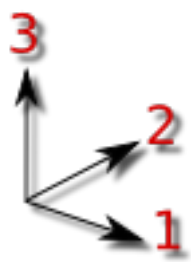
Symmetries of Rate Regions

- NSG can be interpreted as permutational symmetry group of $\text{proj}_{\boldsymbol{\omega},\mathbf{r}}(\Gamma_n^{\text{in}} \cap \mathcal{L}_{123}^{\cap})$, $\text{proj}_{\boldsymbol{\omega},\mathbf{r}}(\Gamma_n^{\text{out}} \cap \mathcal{L}_{123}^{\cap})$

$\mathcal{P}$



(1,2) is a permutational symmetry



$\text{proj}_2\mathcal{P}$

(1,2) is still a permutational symmetry after projection

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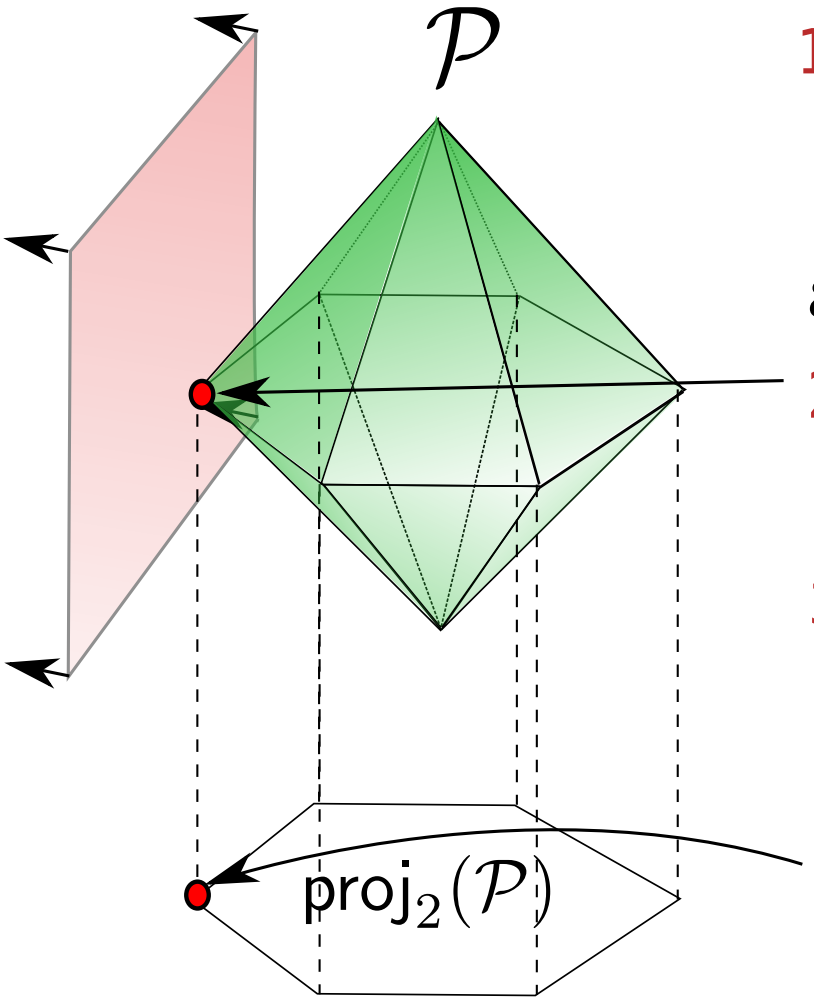
Key Idea 1



1) Solve LP over $\mathcal{P}$ with 0 in non-projection cost vector entries

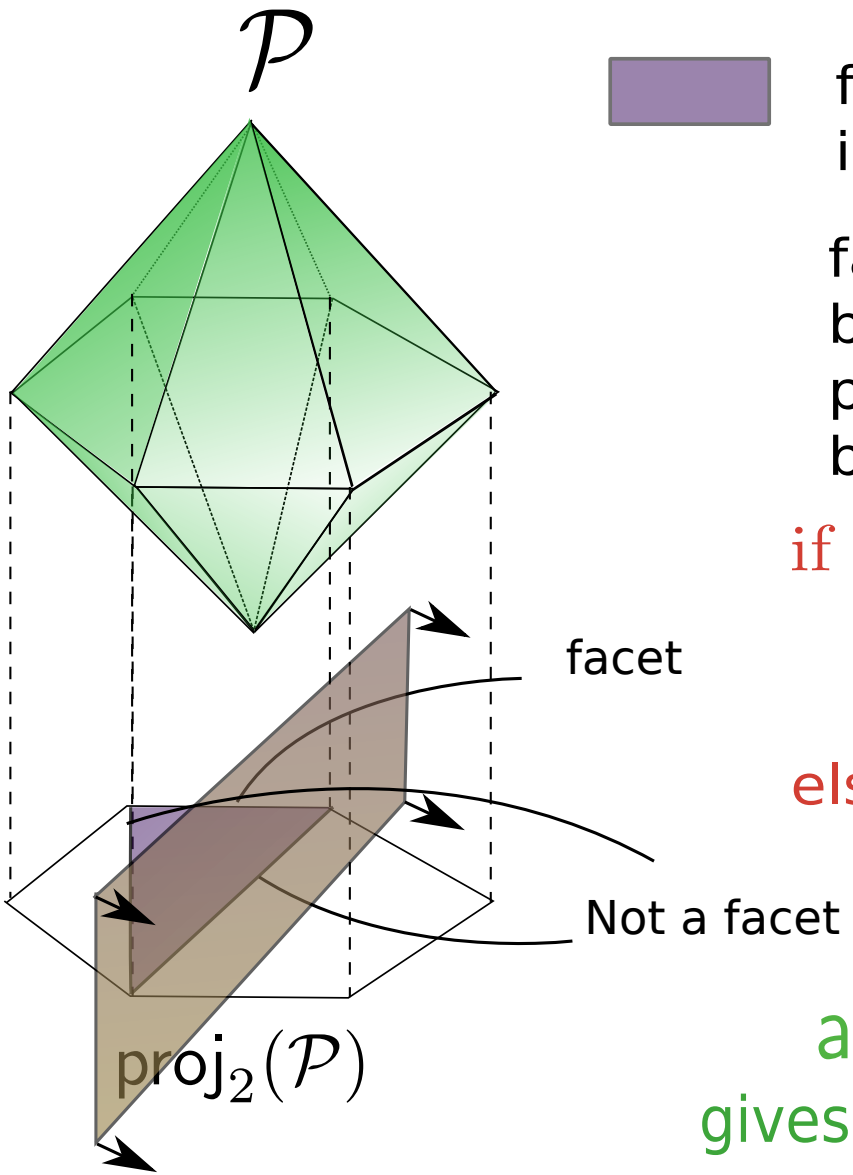
$$\mathbf{a} = \text{argmax}_{\mathcal{P}} c_1 x_1 + c_2 x_2 + 0 x_3$$

2) Project extreme point of $\mathcal{P}$ attaining the optimum

3) Obtain an extreme point of projection

$$\mathbf{a}' = \text{proj}_2(\mathbf{a})$$

Key Idea 2



full-dimensional inner bound

facet $\mathbf{a}^t\mathbf{x}_{1:k} \leq 0$ of inner bound that is not a facet of projection can be identified by solving an LP over $\mathcal{P}$

if $\max_{\mathcal{P}} \mathbf{a}^t\mathbf{x}_{1:k} > b$

$\mathbf{a}^t\mathbf{x}_{1:k} \leq 0$ is not a facet of projection

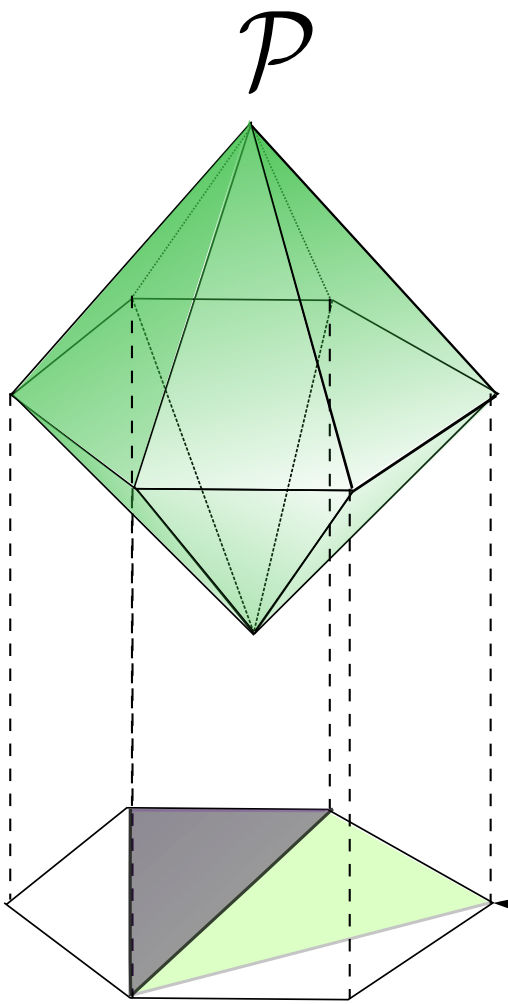
else

$\mathbf{a}^t\mathbf{x}_{1:k} \leq 0$ is a facet of projection

$\text{argmax}_{\mathcal{P}} \mathbf{a}^t\mathbf{x}_{1:k} > b$

gives previously unknown vertex

Key Idea 3



previously known full-dimensional inner bound

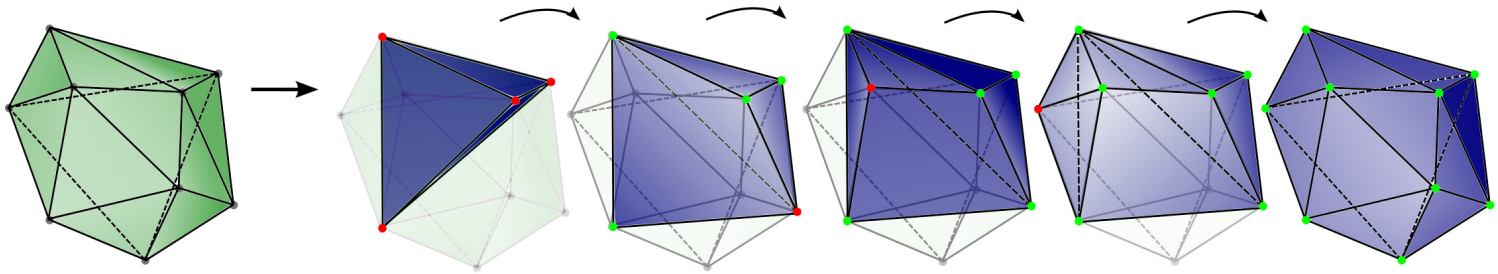
new and improved improved full-dimensional inner bound

Update the inner bound using an iteration of double description method*

$\text{argmax}_{\mathcal{P}} \mathbf{a}^t \mathbf{x}_{1:k} > b$

previously unknown vertex

Iterations of CHM



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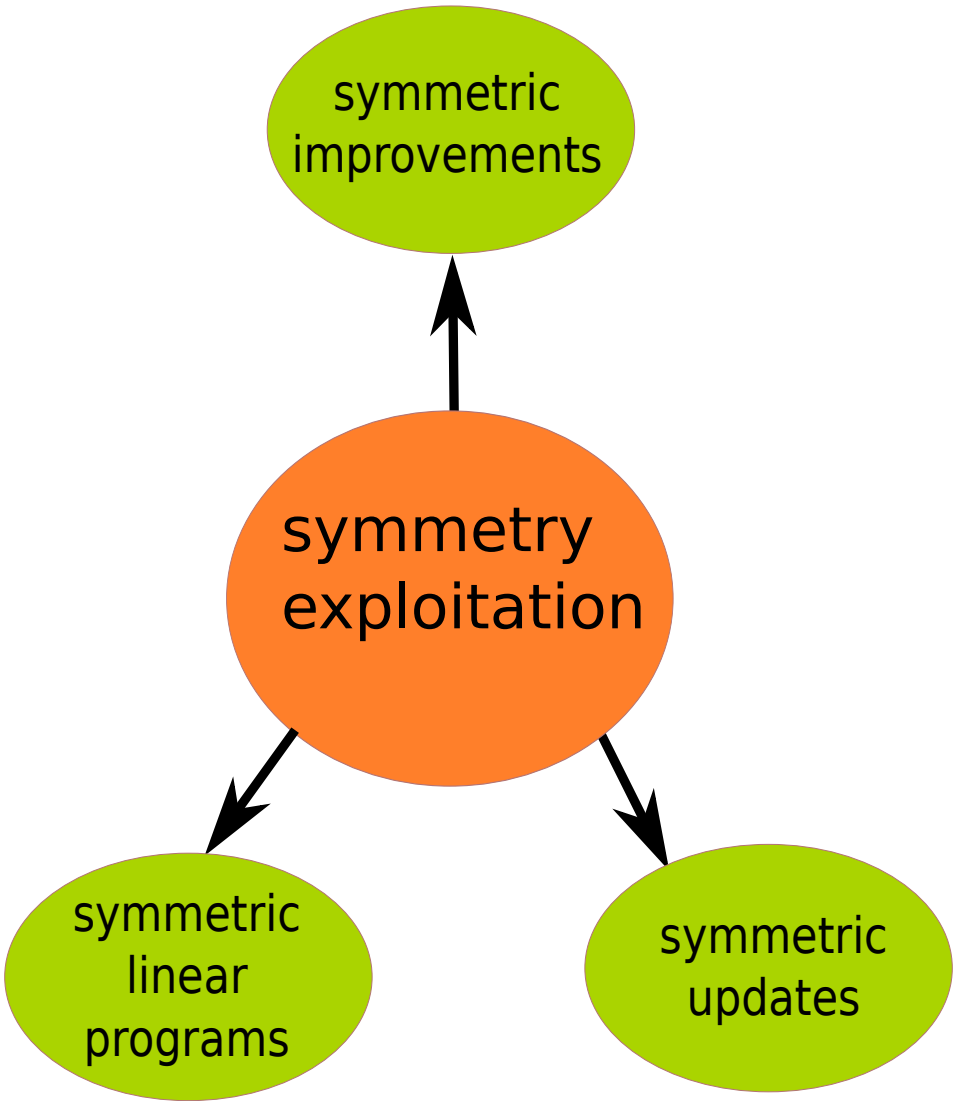
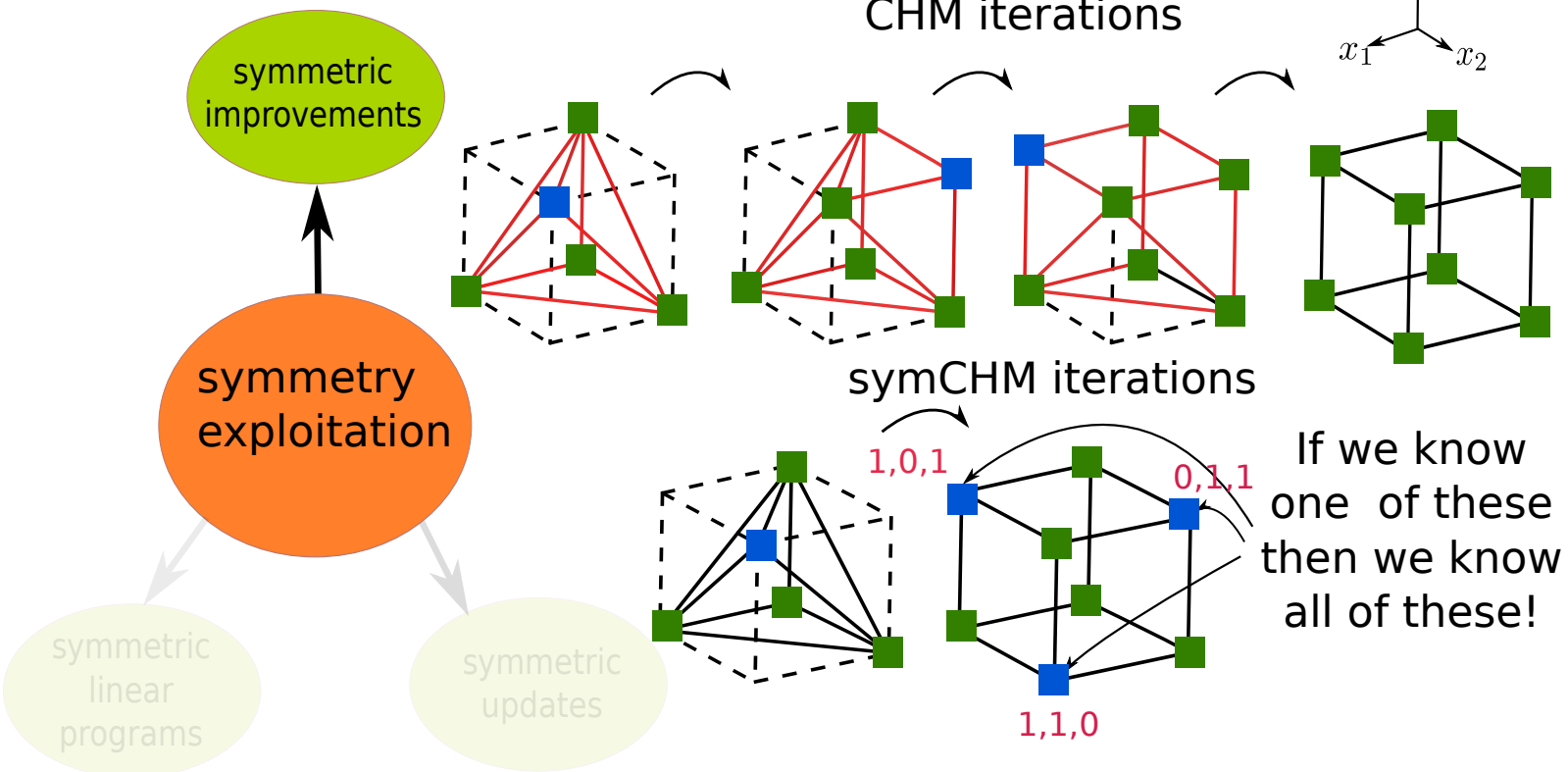
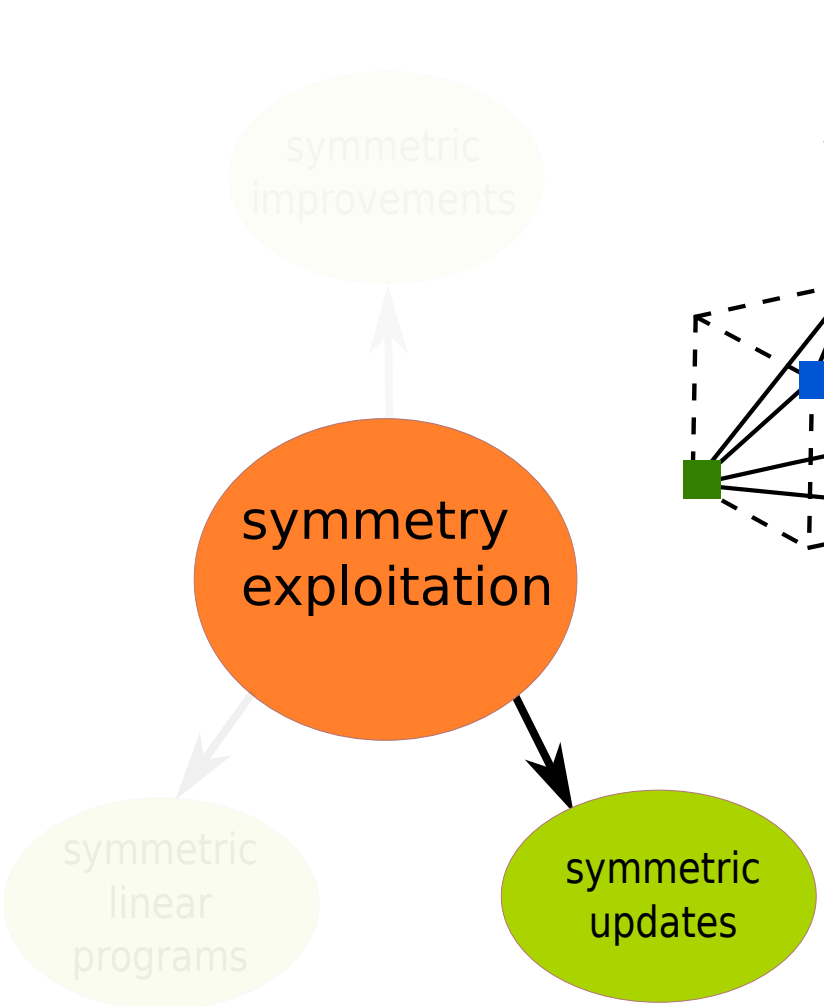
Three uses of symmetry

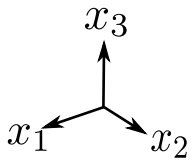


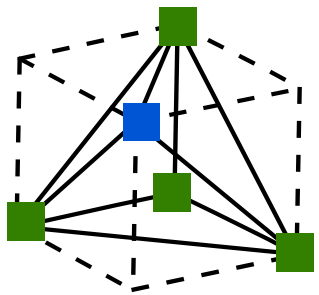
Exploitation 1: Symmetric Improvements

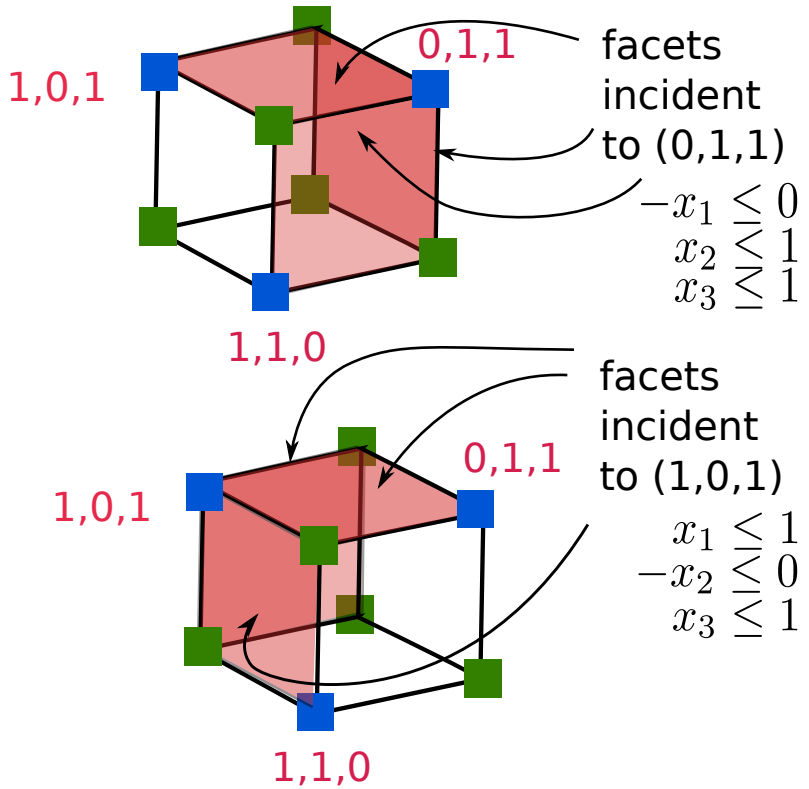


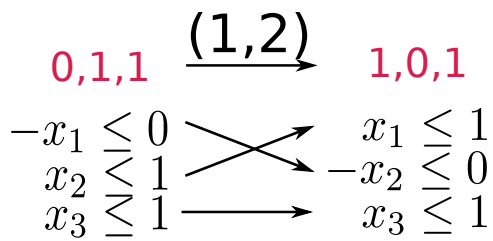
Exploitation 2: Symmetric Updates



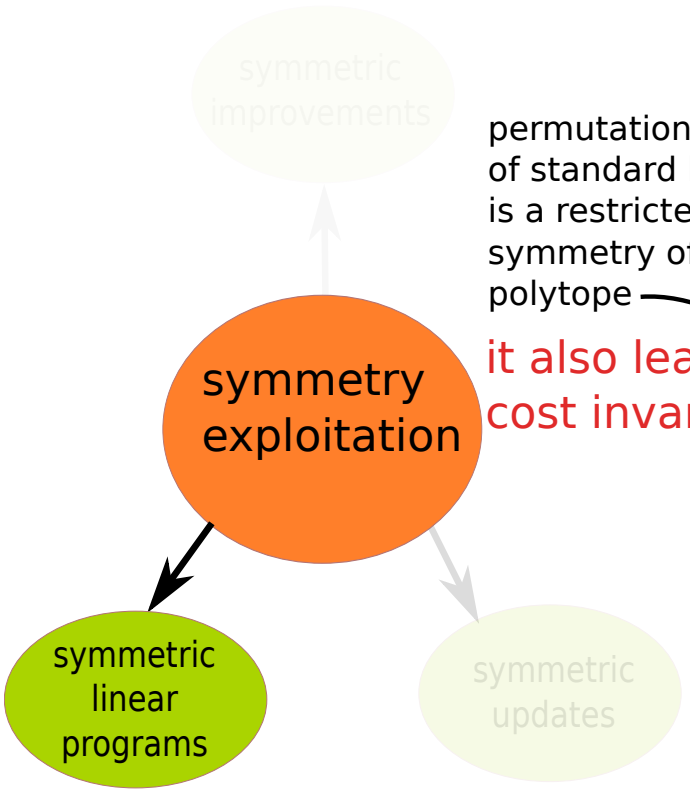






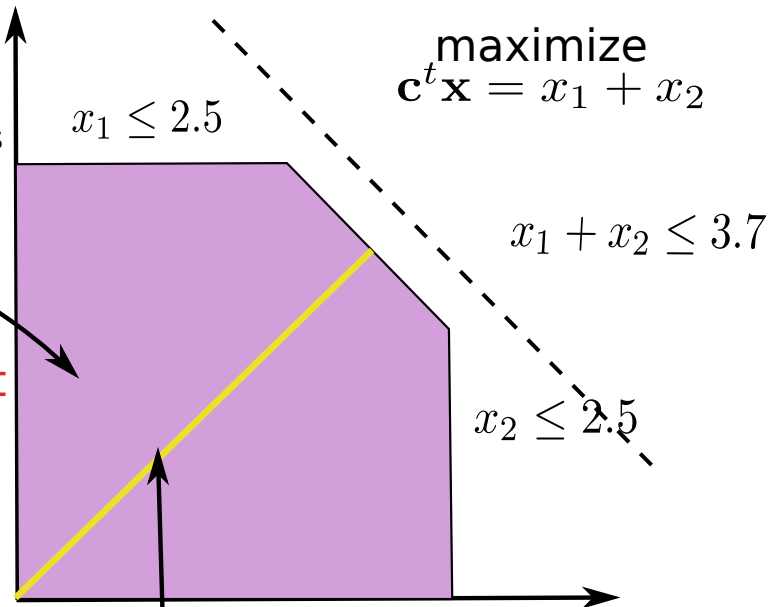


Exploitation 3: Symmetric LPs



permutation of standard bases is a restricted symmetry of this polytope

it also leaves cost invariant



there exists a solution of this LP in *fixed space*: the subspace of points mapping to themselves under the basis vector permutation

Outline

1. Motivation and Background
   - Explicit Expressions for MSNC Rate Regions
   - Symmetry Can Help in Polyhedral Computation
   - Symmetry in Network Coding
   - Classes of Polyhedral Symmetries
2. Symmetries of polyhedral Bounds and NSG
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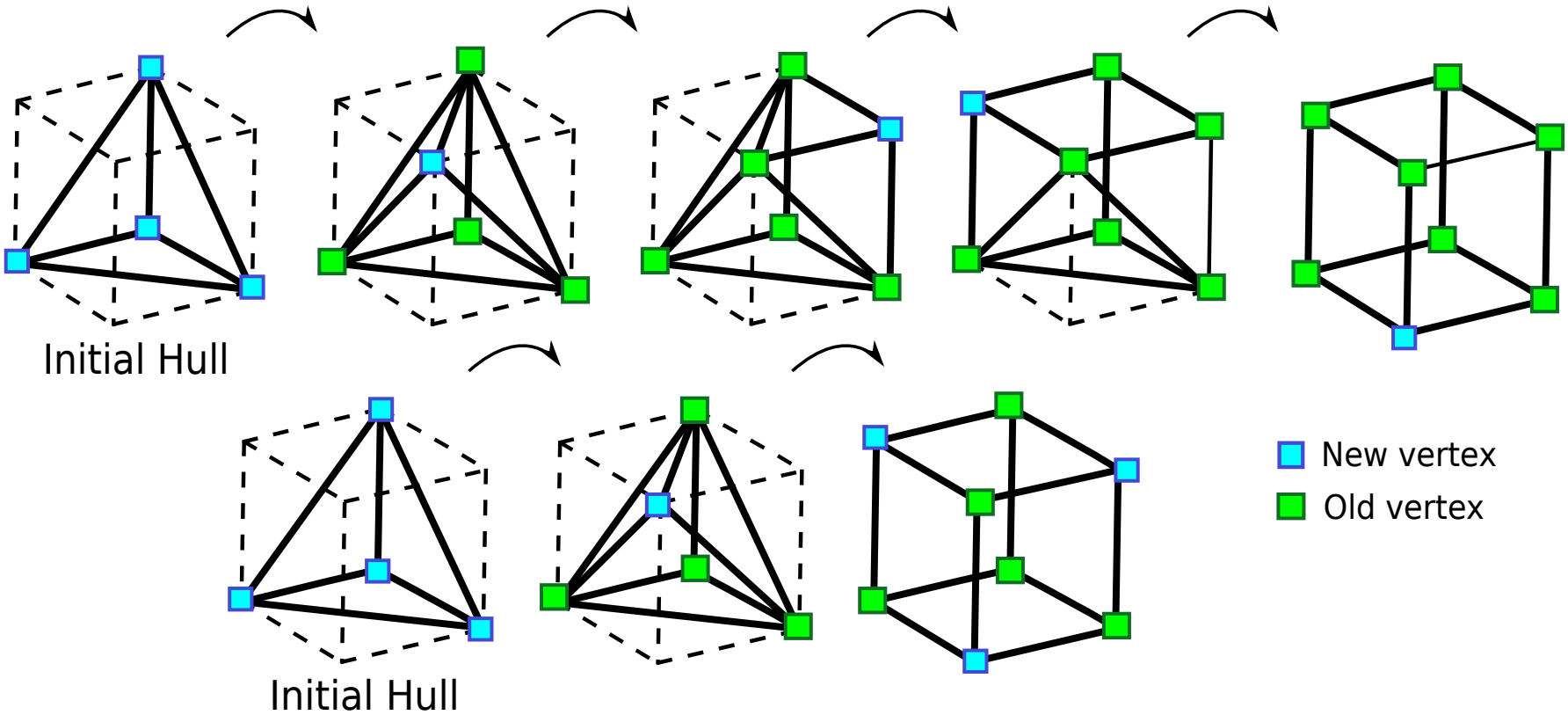
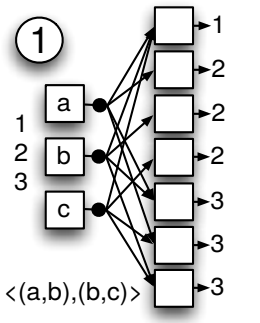
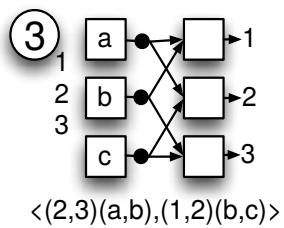
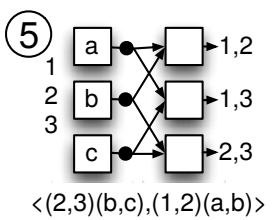
Comparison of CHM (top) and symCHM (bottom)

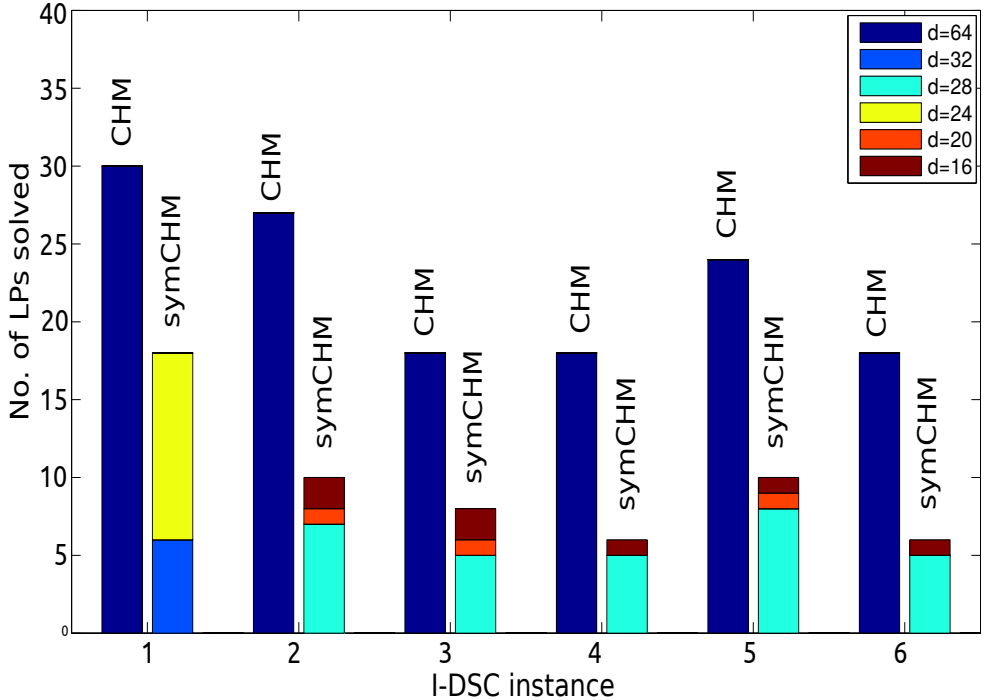


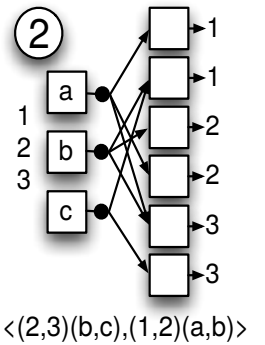
Complexity Comparison (No. and dimension of linear programs)

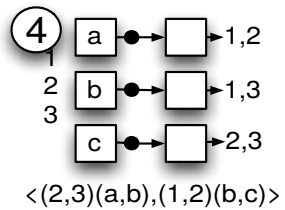


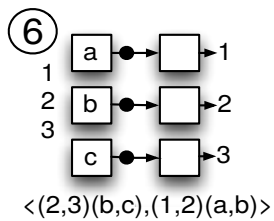












Conclusion

In this paper,

- We related network symmetries to symmetries of polyhedral bounds
- Developed a symmetry exploiting variant of Convex Hull Method

Future work:

- A Python implementation of symCHM in SageMath
- A C implementation of symCHM to go with the Entropic Vectors Computation Package
- Extensive computational experimemnts with arbitrary symmetric polyhedra

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Selected References I

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R. Bödi and K. Herr,
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Questions