Isomorph-free exhaustive generation*

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B. D. McKay, Isomorph-free exhaustive generation, Journal of Algorithms, 26 (1998) 306-324.

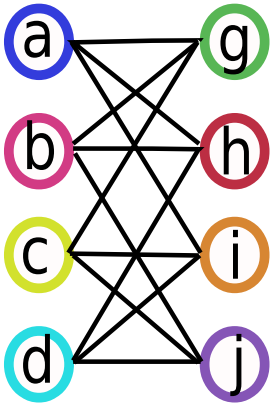
Some Background

Isomorphism

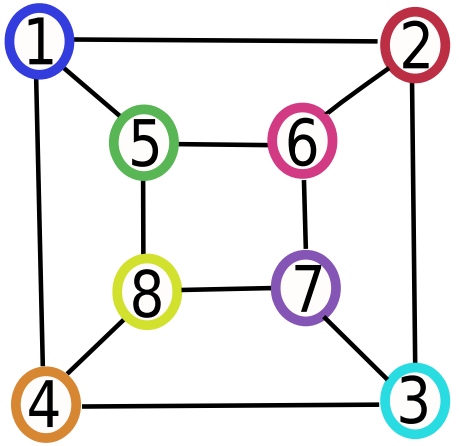
- An isomorphism between two combinatorial objects is a map that preserves sets and relations among elements
- eg. two undirected graphs are isomorphic if there exists a bijection between their vertex sets s.t. edge relations are preserved
- An isomorphism class is a collection of objects all isomorphic to one another

Isomorphism

G_1



G_2



$$G_1 \cong G_2$$

Labeled vs. Unlabeled

- Labeled object means an object with labels
- Unlabeled object means an entire isomorphism class

Problem

- Generate a list of all objects in a class up to a certain size
- Make sure there are no isomorphic objects in the list

Read-Faradzev type algorithm

- Define the notion of a canonical labeled object in an isomorphism class
- Make sure that there exists at least one maximal (in some sense) sub object contained in the canonical object which is also canonical
- Developed independently by Read and Faradzev
- No relabeling at intermediate steps

I. A. Faradzev, Generation of nonisomorphic graphs with a given degree sequence (Russian), in “Algorithmic Studies in Combinatorics” (Nauka, Moscow, 1978) 11–19

I. A. Faradzev, Constructive enumeration of combinatorial objects. Problemes Combinatoires et Theorie des Graphes Colloque Internat. CNRS 260. CNRS Paris (1978) 131–135.

R. C. Read, Every one a winner, Annals Discrete Math., 2 (1978) 107–120.

Examples of RF-type algorithms

- Cubic graph* generator:

G. Brinkmann, Fast generation of cubic graphs, J. Graph Theory, 23 (1996) 139–149.

- Regular graph* generator:

M. Meringer, Erzeugung regul˙arer Graphen, Diplomarbeit, Univ. Bayreuth, 1996.

- Generator for 1-factorizations* of complete graphs:

J. H. Dinitz, D. Garnick and B. D. McKay, There are 526,915,620 nonisomorphic one-factorizations of K 12 , J. Combinatorial Designs, 2 (1994) 273–285.

*Cubic graphs are undirected graphs with every vertex of degree 3

*Regular graphs are graphs with all vertices having same degree

*A 1-factor of a graph is collection of edges with every vertex appearing only once. A 1-factorization is partition of edges into 1- factors

Examples of RF-type algorithms

- Matroid generator:

  Yoshitake Matsumoto, Sonoko Moriyama, Hiroshi Imai, and David Bremner. Matroid enumeration for incidence geometry. Discrete Comput. Geom., 47(1):17–43, January 2012.

McKay-type algorithms

- Canonical Construction Path
- Only objects generated via a canonical augmentation are output
- Similar to RF-type algorithm in a sense that a tree/forest amongst unlabeled objects is traversed
- However, labels are not important. Objects may be relabeled freely at intermediate steps
- Canonical augmentation might lead to isomorphs, so some way of rejecting them is needed

Examples of McKay-type algorithms

- Cubic graph generator:

  B. D. McKay and G. F. Royle, Constructing the cubic graphs on up to 20 vertices, Ars Combinatoria, 21A (1986) 129–140.

- Matroid generator:

  Mayhew, D., Royle, G.F.: Matroids with nine elements. J. Comb. Theory, Ser. B 98, 415–431 (2008)

Other Methods

- Method of Homomorphisms

T. Grüner, R. Laue, M. Meringer: Algorithms for Group Actions: Homomorphism Principle and Orderly Generation Applied to Graphs. DIMACS Series in Discrete Mathematics and Theoretical Computer Science 28, 113-122, 1997.

McKay-type algorithms

Part 1: Theoretical Background

- Start with set of all labeled objects $\mathcal{L}$ and set of all unlabeled objects $\mathcal{U}$
- Create fancy sets called *lower objects* ($\check{\mathcal{L}}$) and *upper objects* ($\hat{\mathcal{L}}$) that obey a bunch of axioms
- Define *parent function* for each unlabeled object that maps given unlabeled object to an unlabeled object of smaller size
- Show that *parent function* creates forest of disjoint trees in $\mathcal{U}$
- Declare victory!

General framework

- G is a permutation group acting on (possibly infinite) set $\mathcal{L}$: the set of labeled objects
- Orbits of G in $\mathcal{L}$ are called unlabeled objects
- $\mathcal{U}$: set of all unlabeled objects
- Each labeled onbject has order $o(\cdot) \in \mathbb{N}$
- Order is constant inside an unlabeled object: every $S \in \mathcal{U}$ has order $o(S)$

Example: TFGs

- TFGs stands for triangle free graphs
- A triangle is K_3
- $\mathcal{L}$ is the set of all labeled TFGs. Labels $\{1, \ldots, n\}$ are used to label a TFG on n vertices
- For a TFG X with n vertices, *order* $o(X) = n$
- There is a group G permuting labels of labeled TFGs
- All relabelings of given TFG creates an unlabeled TFG
- $G = S_1 \times S_2 \times S_3 \times \ldots$, with S_n permuting labels of order n TFGs

General framework

- With each labeled object $X \in \mathcal{L}$, associate $L(X)$ and $U(X)$
- $L(X)$: finite set of *lower ojects*
- $U(X)$: finite set of *upper ojects*
- $\check{\mathcal{L}} = \cup_{X \in \mathcal{L}} L(X)$: set of all lower objects
- $\hat{\mathcal{L}} = \cup_{X \in \mathcal{L}} U(X)$: set of all upper objects
- Objects in $L(X)$ and $U(X)$ have same order as that of X
- G acts on $\mathcal{L} \cup \check{\mathcal{L}} \cup \hat{\mathcal{L}}$

General framework

- For distinct $X_1, X_2 \in \mathcal{L}$ following sets are mutually disjoint: $\{X_1\}, \{X_2\}, U(X_1), U(X_2), L(X_1), L(X_2)$
- A relation $R_f \subseteq \check{\mathcal{L}} \times \hat{\mathcal{L}}$
- R_f is accessed via two functions:

1) $f : \check{\mathcal{L}} \to 2^{\hat{\mathcal{L}}}$, $f(\check{Y}) = \{\hat{X} \in \hat{\mathcal{L}} \mid (\check{Y}, \hat{X}) \in R_f\}$

2) $f' : \hat{\mathcal{L}} \to 2^{\check{\mathcal{L}}}$, $f'(\hat{X}) = \{\check{Y} \in \check{\mathcal{L}} \mid (\check{Y}, \hat{X}) \in R_f\}$

- So far we defined $L(X), U(X), \check{\mathcal{L}}, \hat{\mathcal{L}}, R_f, f, f'$ and claimed existence of action of G on $\mathcal{L} \cup \check{\mathcal{L}} \cup \hat{\mathcal{L}}$

Example: TFGs

- We agree on a way of creating smaller TFGs from larger ones: deleting a vertex and all edges incident to it
- This information is contained in lower objects
  e.g. A pair $\langle X, v \rangle$, where X is a TFG and $v \in V(X)$
- $L(X)$ contains all such pairs
- We also agree on a way of creating larger TFGs from smaller ones: adding a new vertex and edges connecting it to an independent set of vertices
- This information is contained in upper objects
  e.g. a pair $\langle X', W \rangle$ where $W \subseteq V(X')$ is an independent set of X'
- $U(X')$ contains all such pairs

Example: TFGs

- Action of G on X can be easily extended to $U(X)$ and $L(X)$
- $\langle X, v\rangle^g = \langle X^g, v^g\rangle$
- $\langle X', W\rangle^g = \langle X'^g, W^g\rangle$
- R_f for TFGs encodes the complementarity of upper and lower objects
- $\langle X, v\rangle \in L(X)$ is related to $\langle X - v, W\rangle$ where W is teh set of vertices v is connected to in X
- These objects undo effects of each other
- We allow isomorphs of the complements in f and f'
- $f : \langle X, v\rangle \mapsto \{\langle X - v, W\rangle^g \mid g \in G\}$

Axioms C1-C7

C1. G fixes each of $\mathcal{L}, \check{\mathcal{L}}$ and $\hat{\mathcal{L}}$ setwise

C2. For each $X \in \mathcal{L}$ and $g \in G$ we have $L(X^g) = L(X)^g$ and $U(X^g) = U(X)^g$

C3. For each $\check{Y} \in \check{\mathcal{L}}$, $f(\check{Y}) \neq 0$

C4. For any $\check{Y} \in \check{\mathcal{L}}, g \in G, \hat{X}_1 \in f(\check{Y})$ and $\hat{X}_2 \in f(\check{Y}^g)$, there exists $h \in G$ s.t. $\hat{X}_1^h = \hat{X}_2$

C5. For any $\hat{X} \in \hat{\mathcal{L}}, g \in G, \check{Y}_1 \in f'(\hat{X})$ and $\check{Y}_2 \in f'(\hat{X}^g)$, there exists $h \in G$ s.t. $\check{Y}_1^h = \check{Y}_2$

C6. For each $X \in \mathcal{L}$ and $g \in G$, we have $o(X^g) = o(X)$

C7. For each $\check{Y} \in \check{\mathcal{L}}$ and $\hat{X} \in f(\check{Y})$. we have $o(\hat{X}) < o(\check{Y})$

C1. G fixes each of $\mathcal{L}, \check{\mathcal{L}}$ and $\hat{\mathcal{L}}$ setwise → **Since they are different types of objects**

C2. For each $X \in \mathcal{L}$ and $g \in G$ we have $L(X^g) = L(X)^g$ and $U(X^g) = U(X)^g$

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C2. For each $X \in \mathcal{L}$ and $g \in G$ we have $L(X^g) = L(X)^g$ and $U(X^g) = U(X)^g$ ⟶ **L and U commute with group action**

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g=Identity permutation?

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Order is unchanged by group action

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A result of complementarity

C6. For each $X \in \mathcal{L}$ and $g \in G$, we have $o(X^g) = o(X)$

C7. For each $\check{Y} \in \check{\mathcal{L}}$ and $\hat{X} \in f(\check{Y})$. we have $o(\hat{X}) < o(\check{Y})$

Verify axioms for TFGs

C1. G fixes each of $\mathcal{L}, \check{\mathcal{L}}$ and $\hat{\mathcal{L}}$ setwise

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Verify axioms for TFGs

C1. G fixes each of $\mathcal{L}, \check{\mathcal{L}}$ and $\hat{\mathcal{L}}$ setwise **(By construction)**

C2. For each $X \in \mathcal{L}$ and $g \in G$ we have $L(X^g) = L(X)^g$ and $U(X^g) = U(X)^g$ **(By construction)**

C3. For each $\check{Y} \in \check{\mathcal{L}}$, $f(\check{Y}) \neq 0$ **(Every vertex deletion can be undone with addition of a vertex, except for K_1)**

C4. For any $\check{Y} \in \check{\mathcal{L}}, g \in G, \hat{X}_1 \in f(\check{Y})$ and $\hat{X}_2 \in f(\check{Y}^g)$, there exists $h \in G$ s.t. $\hat{X}_1^h = \hat{X}_2$ **(By construction)**

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C6. For each $X \in \mathcal{L}$ and $g \in G$, we have $o(X^g) = o(X)$ **(By construction)**

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Isomorphism and automorphism

- Two objects $X_1, X_2 \in \mathcal{L}$ are *isomorphic* if $X_2 = X_1^g$ for some $g \in G$
- For any lableled object $X \in \mathcal{L}$, the *automorphism group* $\mathsf{Aut}(X)$ of X is the stabilizer $\{g \in G \mid X^g = X\}$
- C2(L, U commute with action of G)$\Rightarrow$ $\mathsf{Aut}(X)$ fixes $L(X)$ and $U(X)$ setwise

Part 1: Theoretical Background

- Start with set of all labeled objects $\mathcal{L}$ and set of all unlabeled objects $\mathcal{U}$ ✓
- Create fancy sets called *lower objects* ($\check{\mathcal{L}}$) and *upper objects* ($\hat{\mathcal{L}}$) that obey a bunch of axioms ✓
- Define *parent function* for each unlabeled object that maps given unlabeled object to an unlabeled object of smaller size
- Show that *parent function* creates forest of disjoint trees in $\mathcal{U}$
- Declare victory!

Towards a forest

Types of unlabeled objects

- An unlabeled object $S \in \mathcal{U}$ is called *irreducible* if $L(X) = \emptyset$ for each $X \in S$
- Other unlabeled objects are *reducible*
- $\mathcal{U}_1$: All reducible unlabeled objects
- $\mathcal{U}_0$: All irreducible unlabeled objects
- $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1$
- **TFGs**: K_1 is the only irreducible TFG

The m function

- The last ingredient for defining parent function is $m : \mathcal{L} \to 2^{\check{\mathcal{L}}}$ that satisfies:

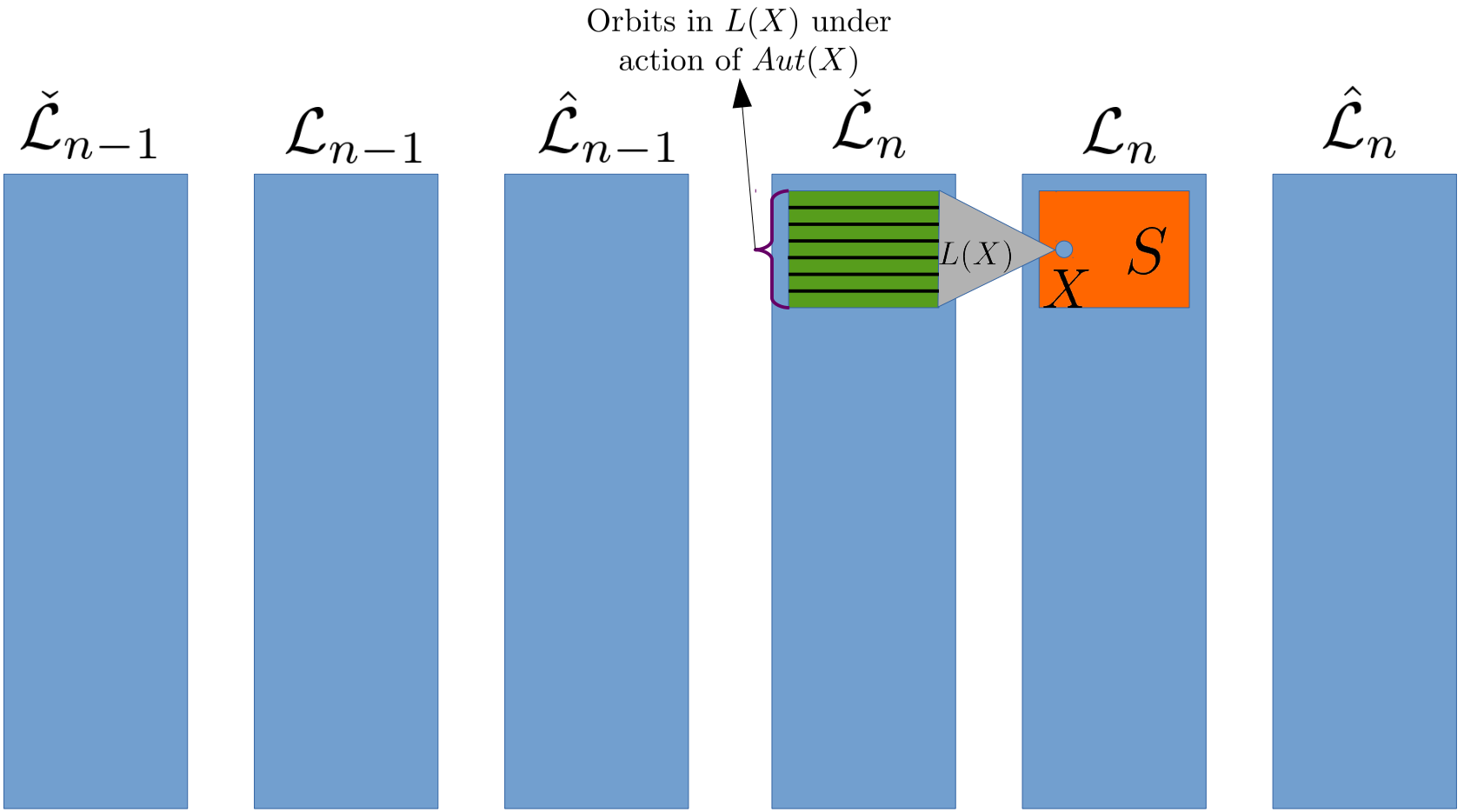
M1. If $L(X) = \emptyset$, then $m(X) = \emptyset$

M2. If $L(X) \neq \emptyset$, then $m(X)$ is an orbit of the action of $\mathsf{Aut}(X)$ on $L(X)$

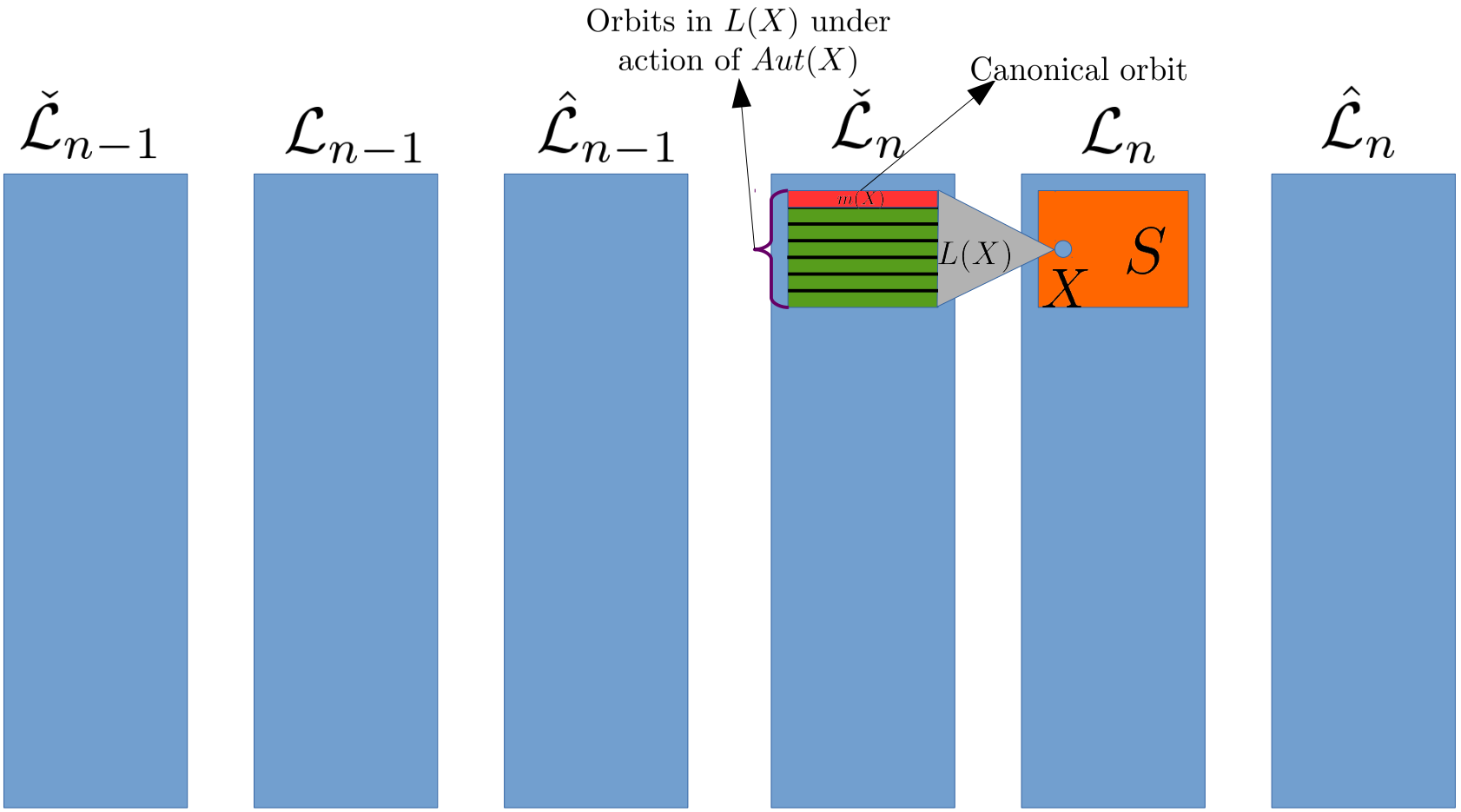
M3. For each $X \in \mathcal{L}$ and $g \in G$ we have $m(X^g) = m(X)^g$

The m function

L(X) is fixed *setwise* under action of Aut(X)



The m function



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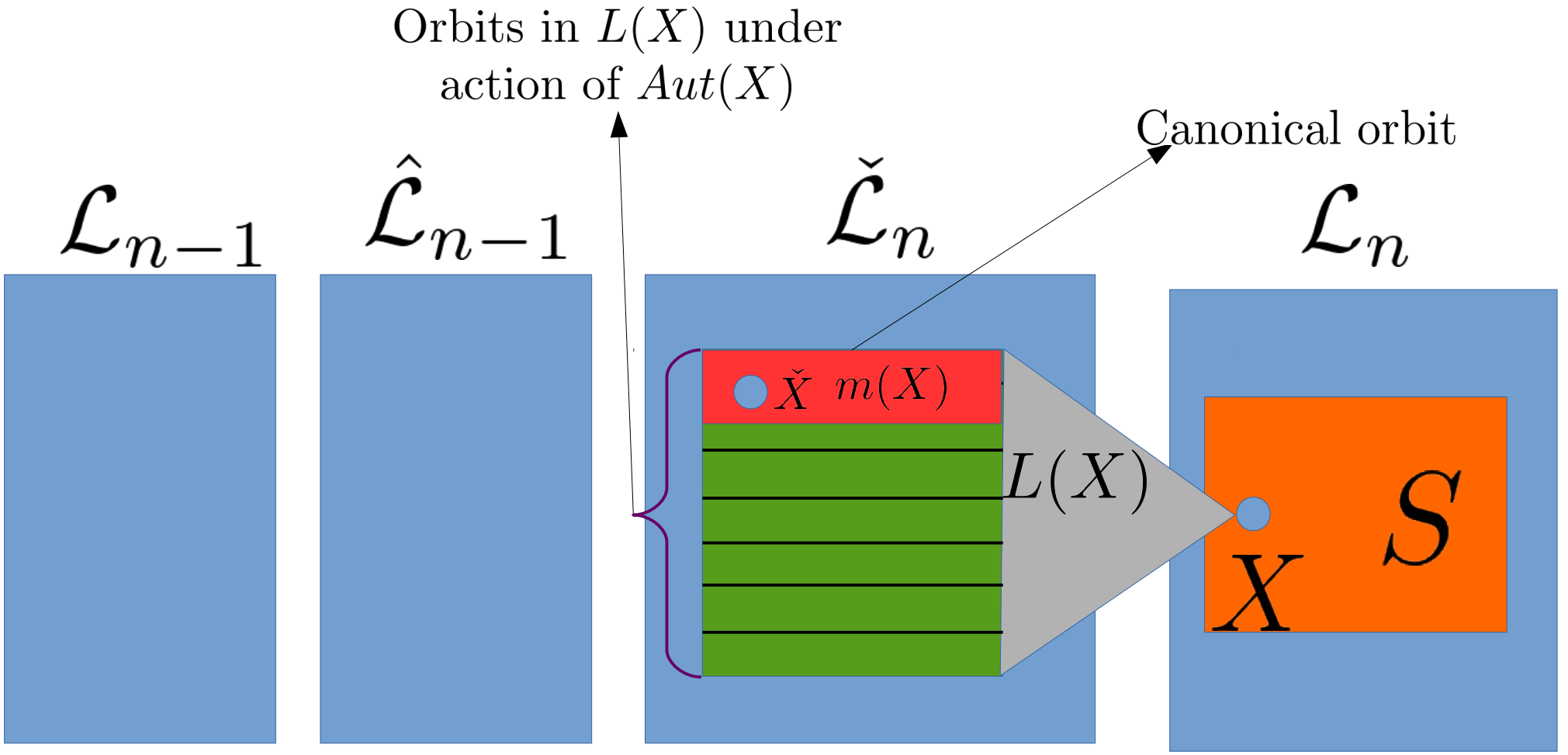
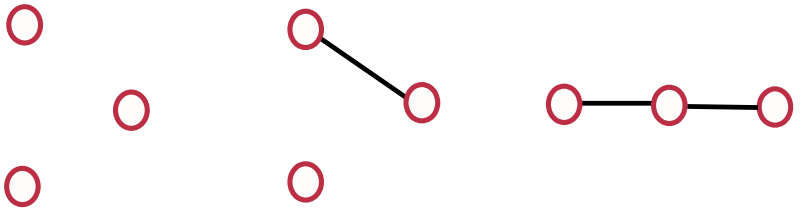
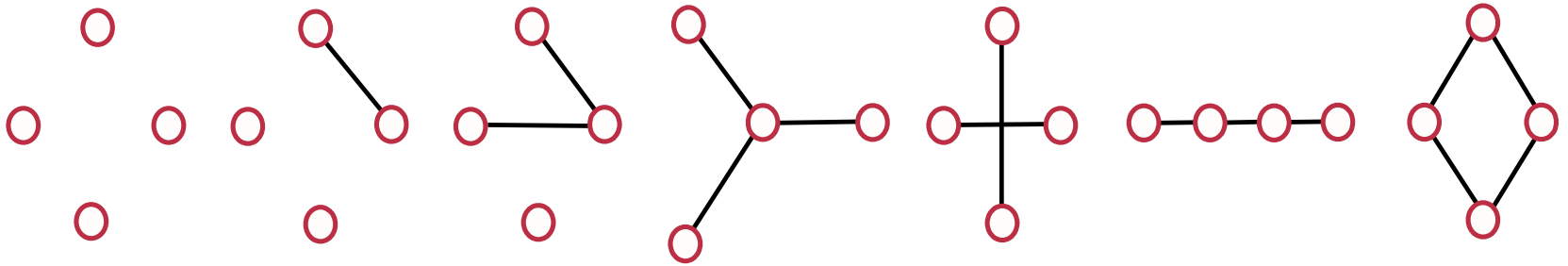


Example: TFGs

$n = 3$
$T_3 = 3$



$n = 4$
$T_4 = 7$



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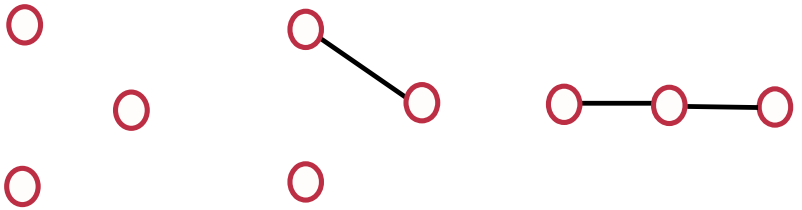

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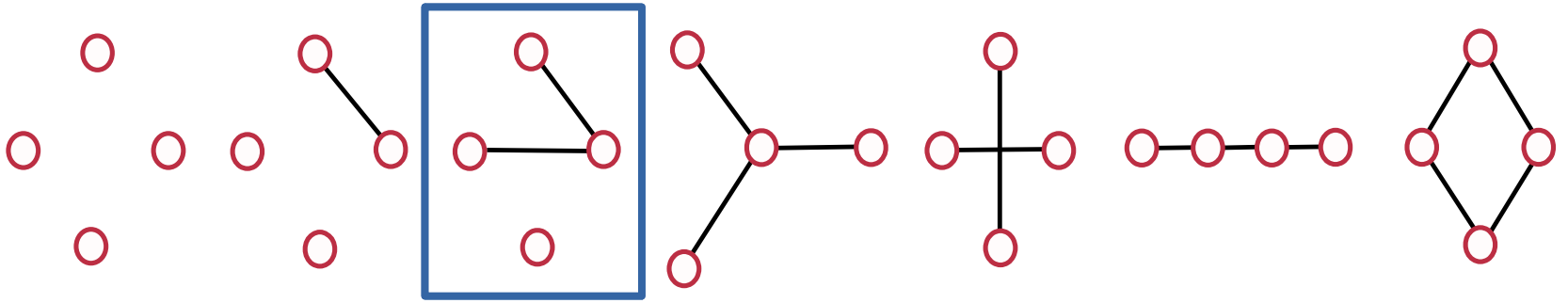
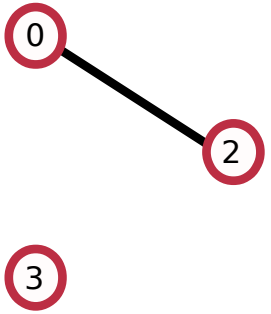
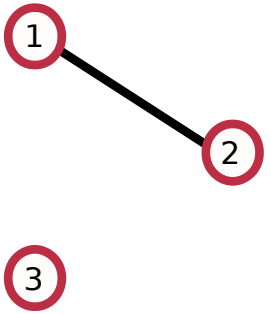


L(X)

$\langle X, 2 \rangle$

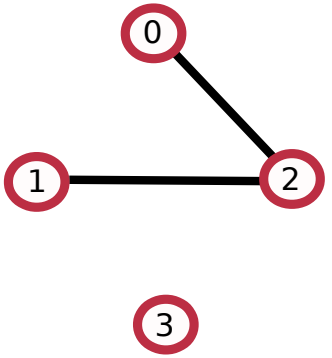


$\langle X, 1 \rangle$



$\langle X, 0 \rangle$

$\langle X, 3 \rangle$



X

Sage to the rescue

The Sage Notebook
Version 6.3

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TFG_generation

last edited Jan 13, 2015, 8:53:23 PM by admin

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File... | Action... | Data... | sage | Typeset

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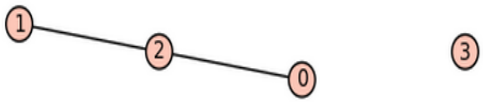
```
d={0:[2],1:[2],2:[0,1],3:[]}
```

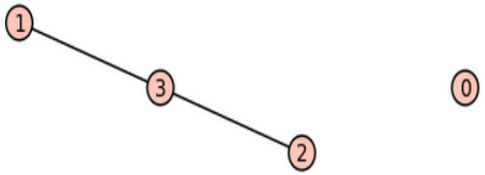
```
G=Graph(d)
G.show()
```




```
G.automorphism_group()
```

```
Permutation Group with generators [(0,1)]
```

```
H=G.canonical_label()
H.show()
```

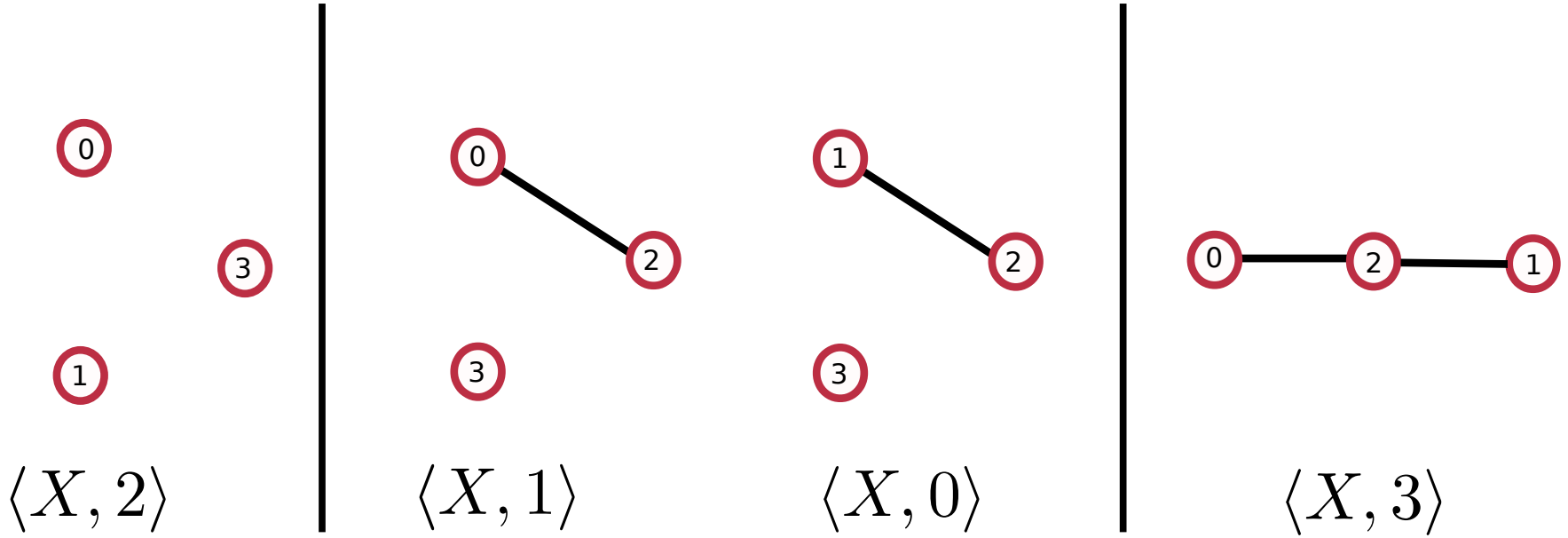


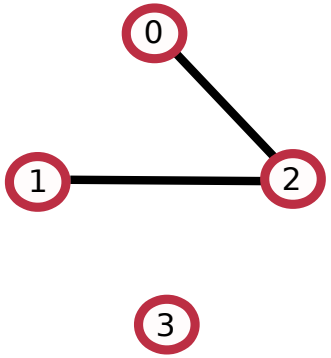

```

```

evaluate

Orbits in L(G) under Aut(G)



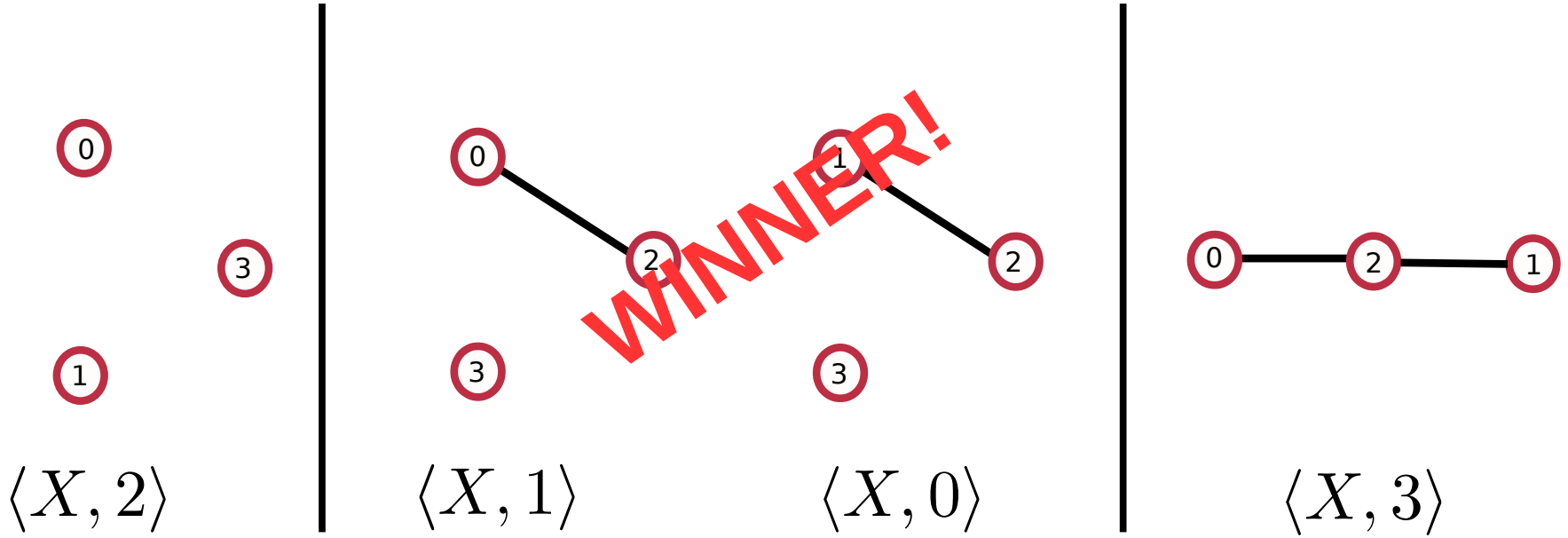


X

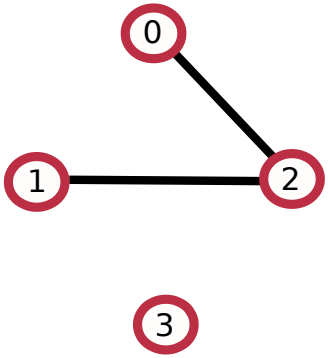
m function for TFGs

- Let v^* be the vertex labeled 0 under canonical labeling
- Define $m(X)$ to be set of lower objects $\langle X, v \rangle$ s.t. v is equivalent to v^* under $\mathsf{Aut}(X)$
- In our example, m(X) is the orbit containing $\langle G, 0 \rangle$ and $\langle G, 1 \rangle$

Orbits in L(G) under Aut(G)



$\langle X, 2\rangle$ $\langle X, 1\rangle$ $\langle X, 0\rangle$ $\langle X, 3\rangle$



X

Lemma: There exists a unique mapping $p : \mathcal{U}_1 \to \mathcal{U}$ with following property:

P1. For each $S \in \mathcal{U}_1$, $X \in S$ and $\check{X} \in m(X)$, we have
$\hat{Y} \in f(\check{X}) \Rightarrow \hat{Y} \in U(Y)$ for some $Y \in p(S)$

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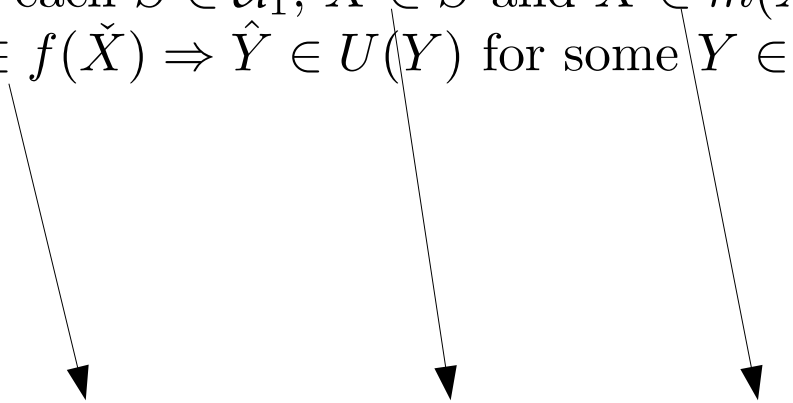
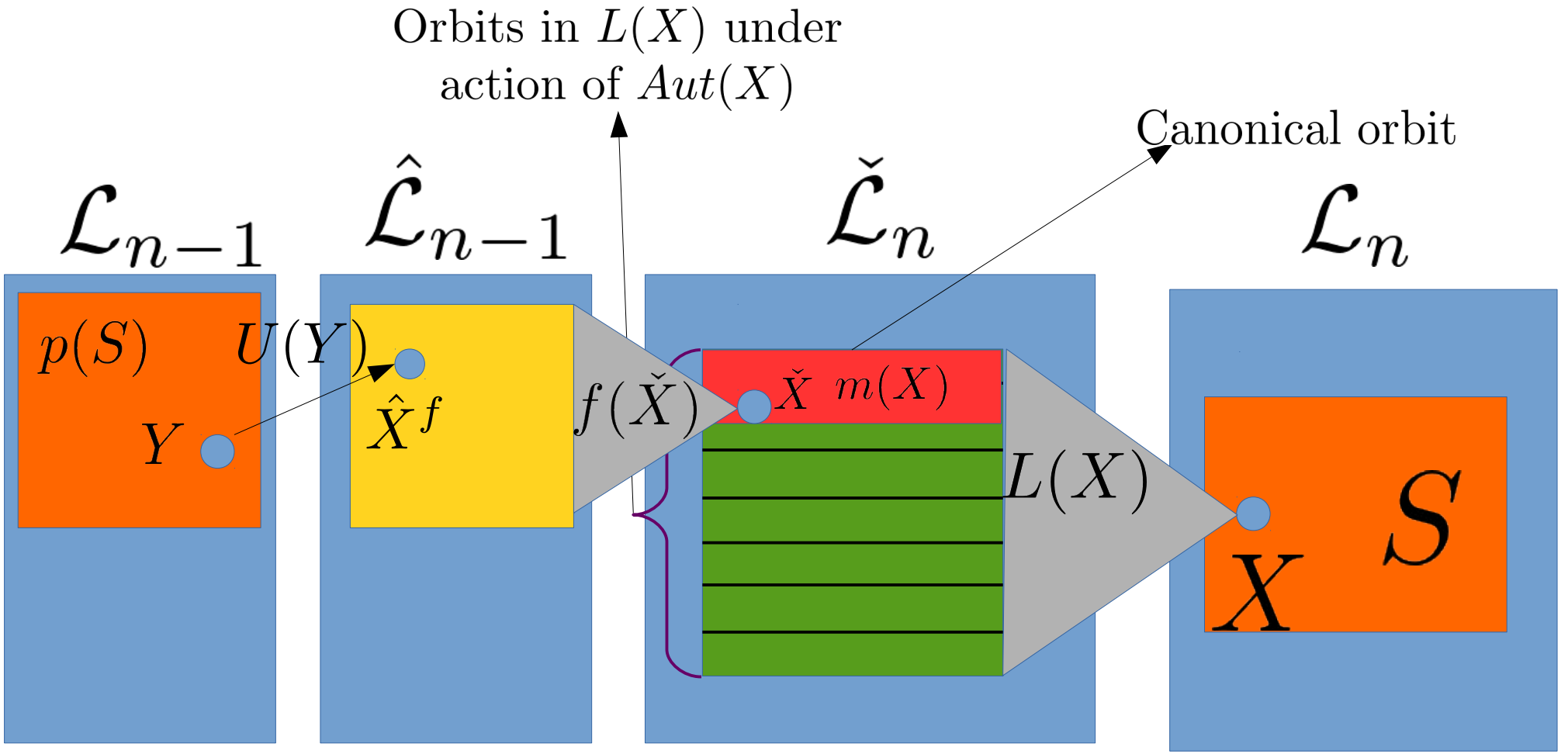
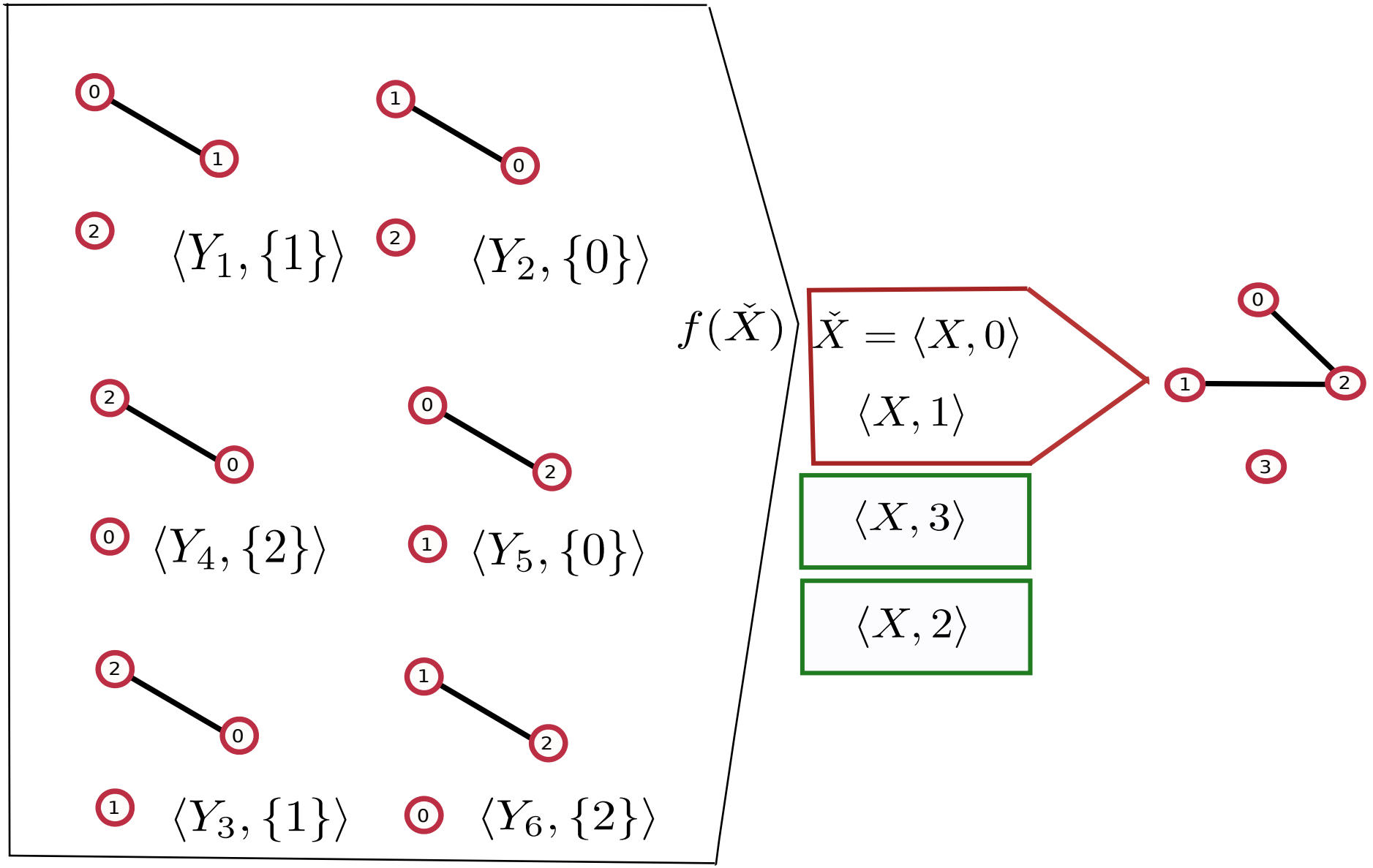


Completely arbitrary choices

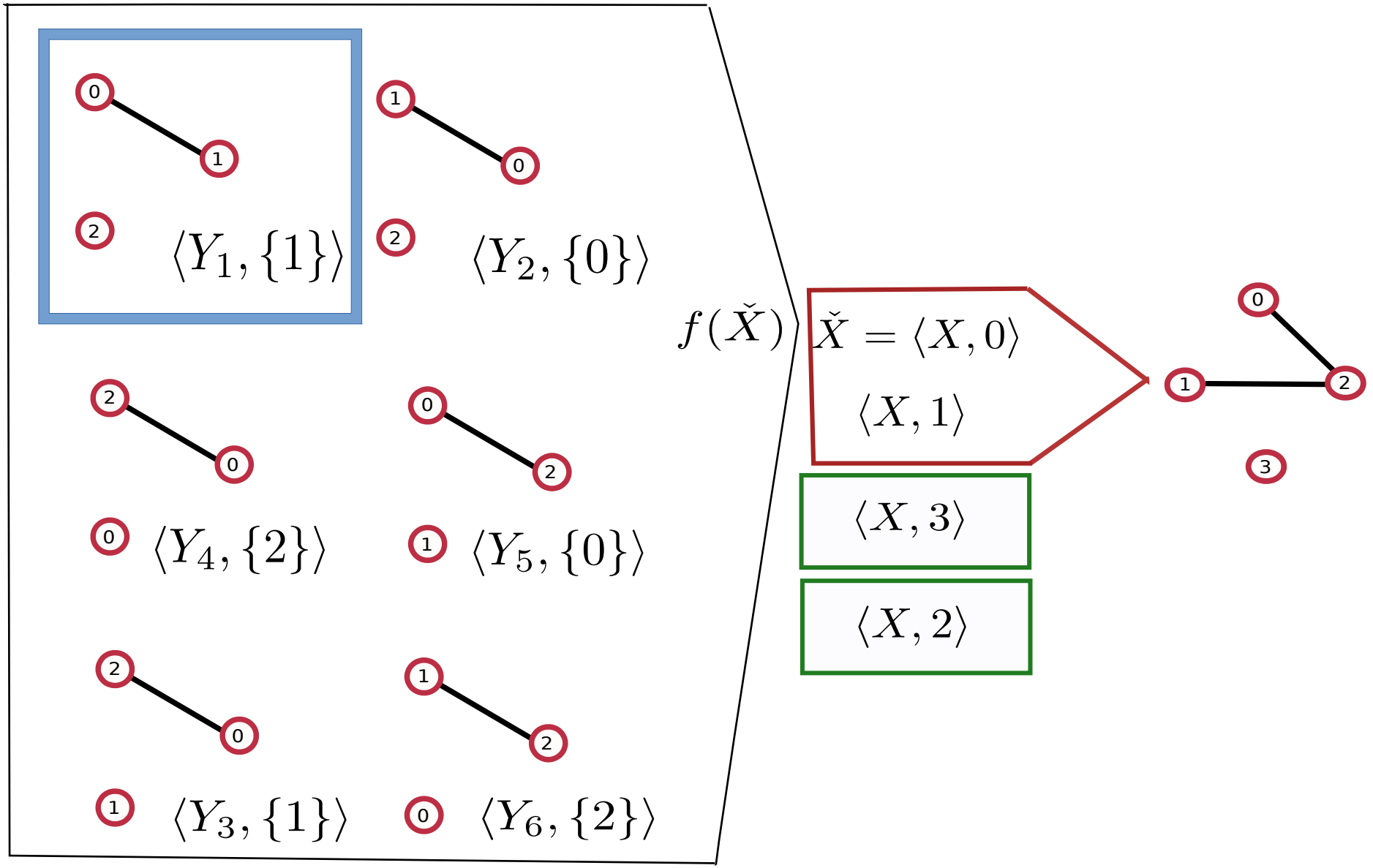
Existence


Orbits in $L(X)$ under
action of $Aut(X)$
Canonical orbit
$\mathcal{L}_{n-1}$
$\hat{\mathcal{L}}_{n-1}$
$\check{\mathcal{L}}_n$
$\mathcal{L}_n$
$p(S)$
$U(Y)$
Y
$\hat{X}^f$
$f(\check{X})$
$\check{X}$ $m(X)$
$L(X)$
X
S

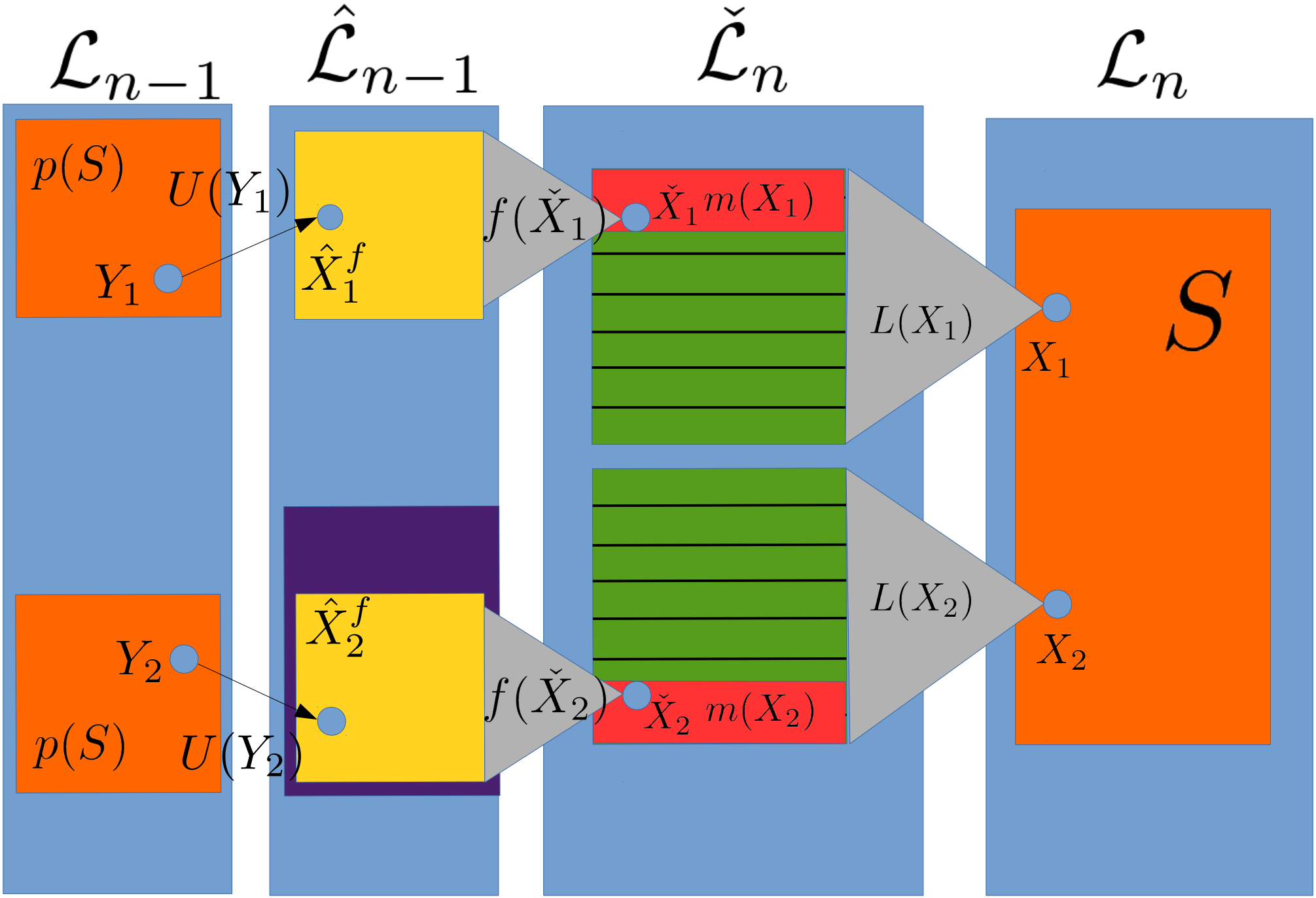
Example: TFGs

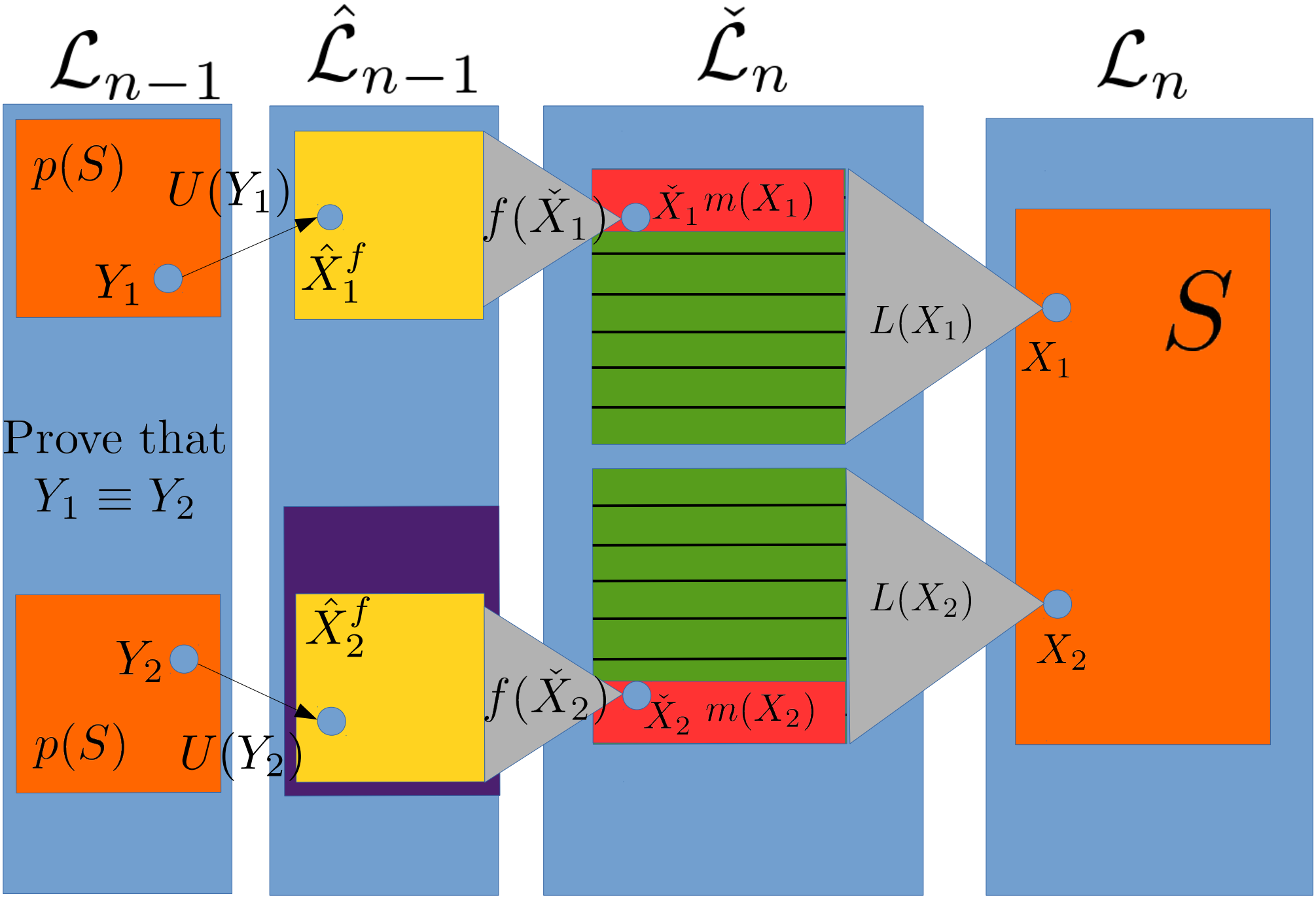


Example: TFGs



Uniqueness


$\mathcal{L}_{n-1}$
$\hat{\mathcal{L}}_{n-1}$
$\check{\mathcal{L}}_n$
$\mathcal{L}_n$
$p(S)$
$U(Y_1)$
Y_1
$\hat{X}_1^f$
$f(\check{X}_1)$
$\check{X}_1\ m(X_1)$
$L(X_1)$
X_1
S
Y_2
$p(S)$
$U(Y_2)$
$\hat{X}_2^f$
$f(\check{X}_2)$
$\check{X}_2\ m(X_2)$
$L(X_2)$
X_2


$\mathcal{L}_{n-1}$
$\hat{\mathcal{L}}_{n-1}$
$\check{\mathcal{L}}_n$
$\mathcal{L}_n$
$p(S)$
$U(Y_1)$
Y_1
$\hat{X}_1^f$
$f(\check{X}_1)$
$\check{X}_1\ m(X_1)$
$L(X_1)$
X_1
S
Prove that
$Y_1 \equiv Y_2$
Y_2
$p(S)$
$U(Y_2)$
$\hat{X}_2^f$
$f(\check{X}_2)$
$\check{X}_2\ m(X_2)$
$L(X_2)$
X_2

Lemma: There exists a unique mapping $p : \mathcal{U}_1 \to \mathcal{U}$ with following property:

P1. For each $S \in \mathcal{U}_1$, $X \in S$ and $\check{X} \in m(X)$, we have
$\hat{X}^f \in f(\check{X}) \Rightarrow \hat{X}^f \in U(Y)$ for some $Y \in p(S)$

- Assume the contrary
- $\exists X_1, X_2 \in S$ s.t. $p(S)$ differs when computed from X_1 and X_2
- Let $m(X_i) = \check{X}_i$ and choose $\hat{Y}_i \in f(\check{X}_i)$ for $i = 1, 2$
- Define Y_i s.t. $\hat{Y}_i \in U(Y_i)$ for $i = 1, 2$
- **Claim**: $Y_1 \equiv Y_2$

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- Since $X_1, X_2 \in S$, $\exists g \in G$ s.t. $X_1^g = X_2$

Recall:

M2. If $L(X) \neq \emptyset$, then $m(X)$ is an orbit of the action of $\mathsf{Aut}(X)$ on $L(X)$

M3. For each $X \in \mathcal{L}$ and $g \in G$ we have $m(X^g) = m(X)^g$

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- $\exists h \in G$ s.t. $\check{X}_1^h = \check{X}_2$

Recall:

C4. For any $\check{Y} \in \check{\mathcal{L}}, g \in G, \hat{X}_1 \in f(\check{Y})$ and $\hat{X}_2 \in f(\check{Y}^g)$, there exists $h \in G$ s.t. $\hat{X}_1^h = \hat{X}_2$

Lemma: There exists a unique mapping $p : \mathcal{U}_1 \to \mathcal{U}$ with following property:

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Recall:

C2. For each $X \in \mathcal{L}$ and $g \in G$ we have $L(X^g) = L(X)^g$ and $U(X^g) = U(X)^g$

Lemma: There exists a unique mapping $p : \mathcal{U}_1 \to \mathcal{U}$ with following property:

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- $\Rightarrow\Leftarrow$

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- $Y_2 = Y_1^k$
- $\Rightarrow\Leftarrow$

Ancestors and Descendants

- The unlabeled object $p(S)$ is called *parent* of unlabeled object S
- The set $\{S, p(S), p(p(S)), \ldots\}$ will be called *ancestors* if S
- Inverse concepts of *child* and *descendent* are defined in the same way

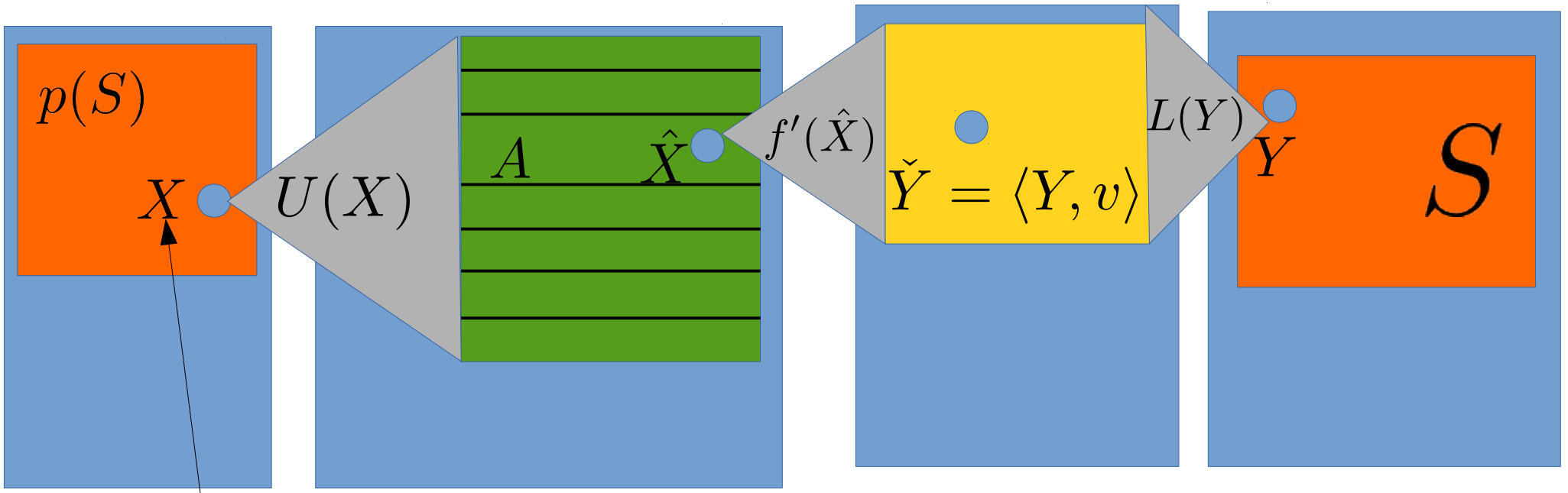
Recall:

C7. For each $\check{Y} \in \check{\mathcal{L}}$ and $\hat{X} \in f(\check{Y})$. we have $o(\hat{X}) < o(\check{Y})$

- C7 $\Rightarrow$ p is nilpotent $\Rightarrow$ Every $S \in \mathcal{U}_1$ has only finitely many ancestors
- The relation $\{(S, p(S) \mid S \in \mathcal{U}_1)\}$ on $\mathcal{U}$ is a set of disjoint directed rooted trees
- Roots of these trees are irreducible objects in $\mathcal{U}_0$

Part II: Traversing the tree

procedure `scan`(X: labeled object, n: integer)
  output X
  for each orbit A of the action of $\mathsf{Aut}(X)$ on $U(X)$ **do**
    select any $\hat{X} \in A$
    if $f'(\hat{X}) \neq \emptyset$ **then**
      select any $\check{Y} \in f'(\hat{X})$, and suppose $\check{Y} \in L(Y)$
      if $o(Y) \leq n$ **and** $\check{Y} \in m(Y)$ **then** `scan`(Y,n) **endif**
    endif
  endif



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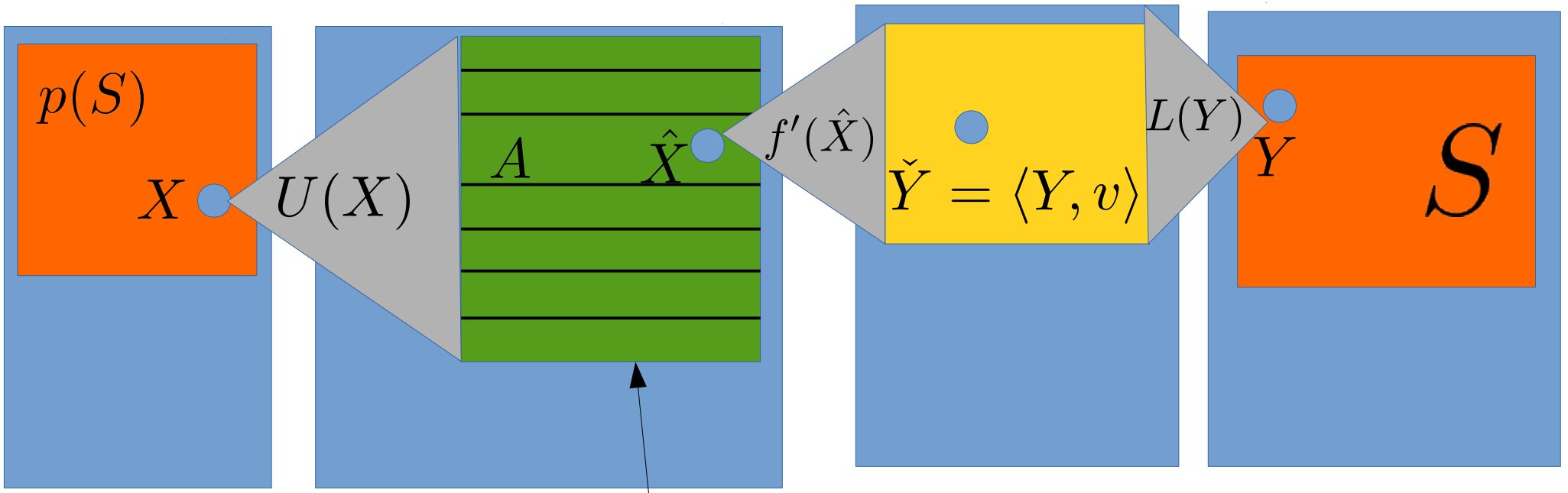
  endif



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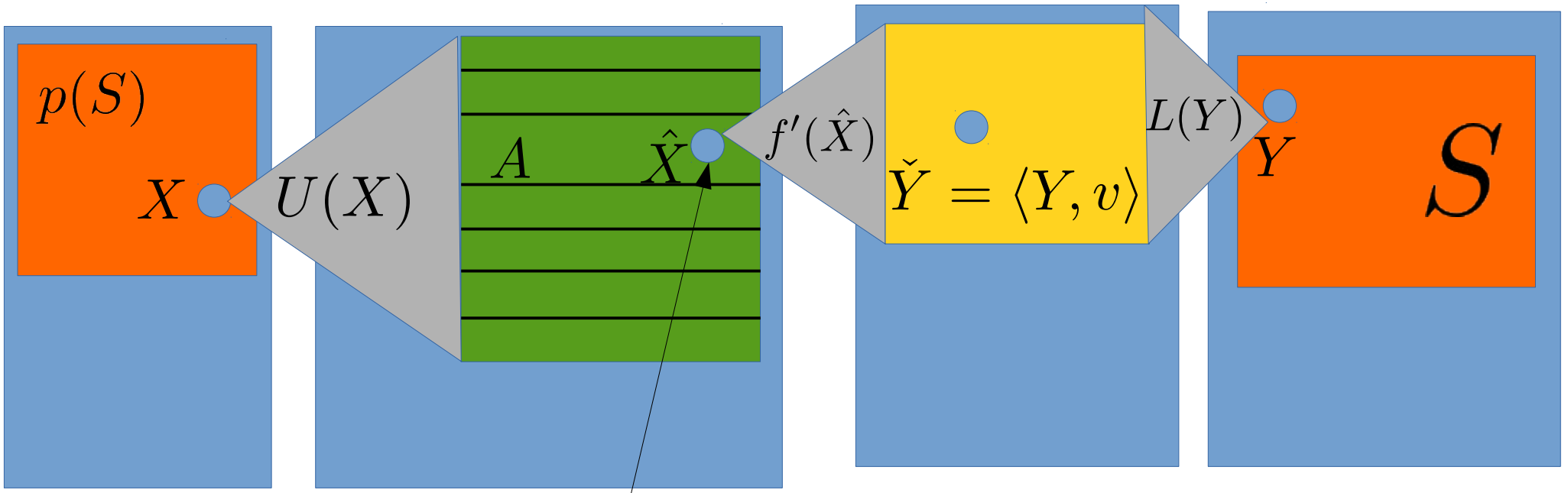
  endif



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  output X
  for each orbit A of the action of Aut(X) on $U(X)$ **do**
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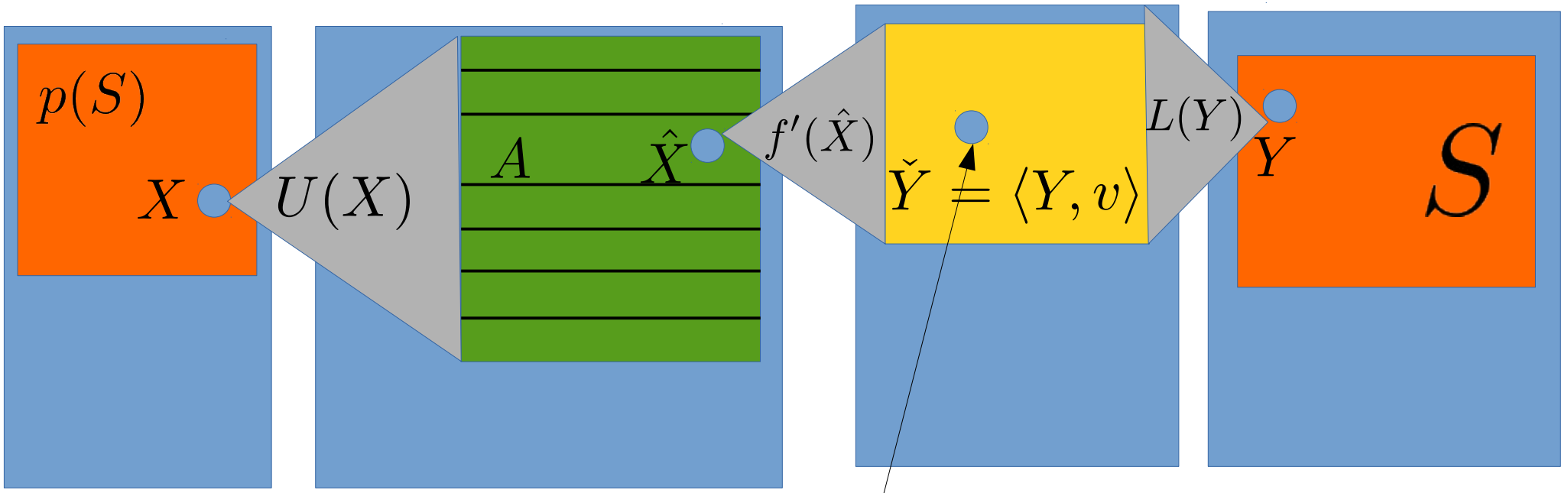
    if $f'(X) \neq \emptyset$ **then**
      select any $\check{Y} \in f'(\hat{X})$, and suppose $\check{Y} \in L(Y)$
      if $o(Y) \leq n$ **and** $\check{Y} \in m(Y)$ **then** scan(Y,n) **endif**
    endif
  endif



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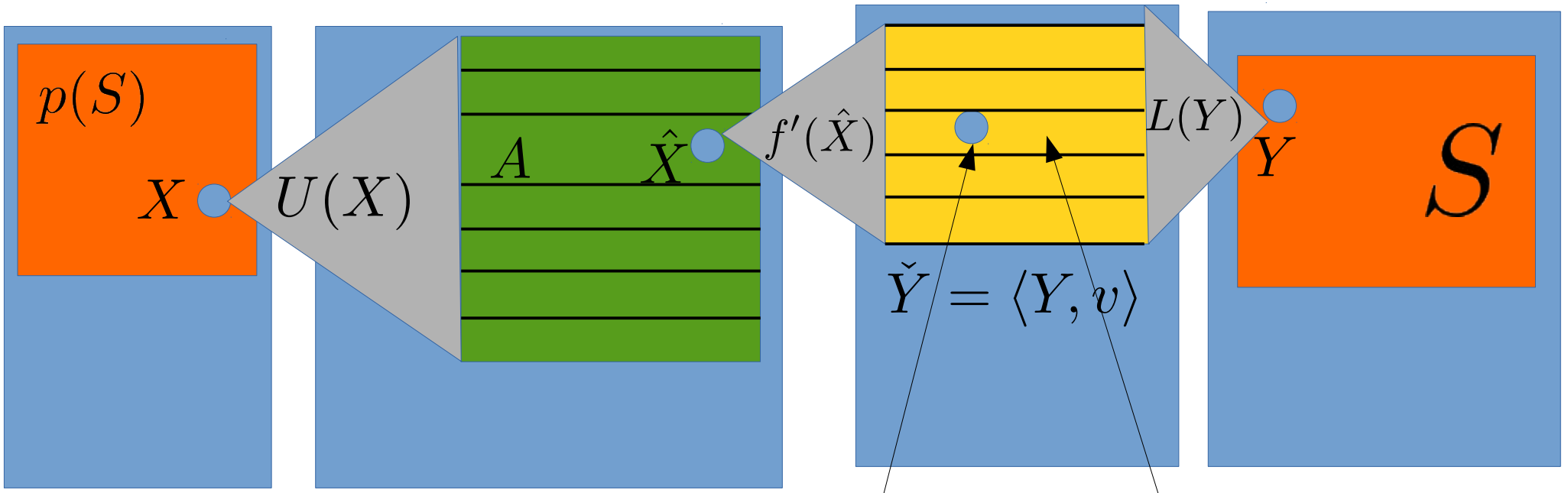
    endif
  endif



Is this orbit in $L(Y)$ under action of $\mathsf{Aut}(Y)$ canonical?

procedure `scan`(X: labeled object, n: integer)
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    endif
  endif

Theorem: Suppose $X_0 \in S_0 \in \mathcal{U}$ with $o(X_0) \leq n$. Then the call $\texttt{scan}(X_0, n)$ will output exactly one labelled object belonging to each unlabelled object of order at most n which is descended from S_0.

Theorem: Suppose $X_0 \in S_0 \in \mathcal{U}$ with $o(X_0) \leq n$. Then the call $\texttt{scan}(X_0, n)$ will output exactly one labelled object belonging to each unlabelled object of order at most n which is descended from S_0.

- A descendent S of S_0 belongs to *generation* i if S has $i+1$ ancestors.
- Generation 0 is the set S_0(every object is ancestor of itself)
- We prove by induction
- Assume true for all objects of of order $\leq n$ and generation at most i.
- Let S be an arbitrary descendent of S_0 with order $\leq n$ and generation $i+1$

Claim: Every unlabeled object has *at least* one labeled object output

- Let $T = p(S)$
- Take arbitrary $Y' \in S$, choose $\check{Y}' \in m(Y')$ and $\hat{X}' \in f(\check{Y}')$.
- By induction hypothesis, there is a pair X and $\hat{X} \in U(X)$ s.t. inside $\texttt{scan}(X, n)$, inside **for**, $\hat{X}$ was selected.
- $\hat{X}^g = \hat{X}'$ for some $g \in G$

Recall:

C5. For any $\hat{X} \in \hat{\mathcal{L}}, g \in G, \check{Y}_1 \in f'(\hat{X})$ and $\hat{Y}_2 \in f'(\hat{X}^g)$, there exists $h \in G$ s.t. $\check{Y}_1^h = \check{Y}_2$

Claim: Every unlabeled object has *at least* one labeled object output

- Let $T = p(S)$
- Take arbitrary $Y' \in S$, choose $\check{Y}' \in m(Y')$ and $\hat{X}' \in f(\check{Y}')$.
- By induction hypothesis, there is a pair X and $\hat{X} \in U(X)$ s.t. inside $\texttt{scan}(X, n)$, inside **for**, $\hat{X}$ was selected.
- $\hat{X}^g = \hat{X}'$ for some $g \in G$
- C5 $\Rightarrow \check{Y}^h = \check{Y}'$ for some $h \in G$

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Recall:

C2. For each $X \in \mathcal{L}$ and $g \in G$ we have $L(X^g) = L(X)^g$ and $U(X^g) = U(X)^g$

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Recall:

M3. For each $X \in \mathcal{L}$ and $g \in G$ we have $m(X^g) = m(X)^g$

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- $\hat{X}^g = \hat{X}'$ for some $g \in G$
- C5 $\Rightarrow \check{Y}^h = \check{Y}'$ for some $h \in G$
- C2$\Rightarrow Y^h = Y' \Rightarrow Y \in S$
- M3$\Rightarrow \check{Y} \in m(Y)$
- $\Rightarrow \texttt{scan}(Y, n)$ is called and Y is output

Claim: Every unlabeled object has *at most* one labeled object output

- Let distinct Y_1, Y_2 be output s.t. $Y_2 = Y_1^g$ for some $g \in G$
- Let $\check{Y}_1 \in m(Y_1)$ and $\check{Y}_2 \in m(Y_2)$ presented to the **if** preceding $\texttt{scan}(Y_1, n)$ and $\texttt{scan}(Y_2, n)$ resp.

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procedure $\texttt{scan}(X$: labeled object, n: integer)
  output X
  for each orbit A of the action of $\mathsf{Aut}(X)$ on $U(X)$ **do**
    select any $\hat{X} \in A$
    if $f'(\hat{X}) \neq \emptyset$ **then**
      select any $\check{Y} \in f'(\hat{X})$, and suppose $\check{Y} \in L(Y)$
      if $o(Y) \leq n$ **and** $\check{Y} \in m(Y)$ **then** $\texttt{scan}$(Y,n) **endif**
    endif
  endif

Claim: Every unlabeled object has *at most* one labeled object output

- Let distinct Y_1, Y_2 be output s.t. $Y_2 = Y_1^g$ for some $g \in G$
- Let $\check{Y}_1 \in m(Y_1)$ and $\check{Y}_2 \in m(Y_2)$ presented to the **if** preceding $\texttt{scan}(Y_1, n)$ and $\texttt{scan}(Y_2, n)$ resp.

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- M3 $\Rightarrow \check{Y}_2 \in m(Y_2) = m(Y_1^g) = m(Y_1)^g$

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Recall:

M2. If $L(X) \neq \emptyset$, then $m(X)$ is an orbit of the action of $\mathsf{Aut}(X)$ on $L(X)$

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- M3 $\Rightarrow \check{Y}_2 \in m(Y_2) = m(Y_1^g) = m(Y_1)^g$
- M2 $\Rightarrow \check{Y}_2 = \check{Y}_1$

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- M3 $\Rightarrow \check{Y}_2 \in m(Y_2) = m(Y_1^g) = m(Y_1)^g$
- M2 $\Rightarrow \check{Y}_2 = \check{Y}_1$
- Let $\hat{X}_1 \in f(\check{Y}_1)$ and $\hat{X}_2 \in f(\check{Y}_2)$

Recall:

C4. For any $\check{Y} \in \check{\mathcal{L}}, g \in G, \hat{X}_1 \in f(\check{Y})$ and $\hat{X}_2 \in f(\check{Y}^g)$, there exists $h \in G$ s.t. $\hat{X}_1^h = \hat{X}_2$

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- M3 $\Rightarrow \check{Y}_2 \in m(Y_2) = m(Y_1^g) = m(Y_1)^g$
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- C4$\Rightarrow \hat{X}_2 = \hat{X}_1^k$ for some $k \in G$
- $\hat{X}_1 \in U(X_1)$ and $\hat{X}_2 \in U(X_2)$ and C2 $\Rightarrow X_2 = X_1^k$
- Induction hypothesis $\Rightarrow X_1, X_2$ were both output $\Rightarrow X_1 = X_2$
- Hence $\hat{X}_1$ and $\hat{X}_2$ are upper objects of same object (X_1) and $k \in G$ maps $\hat{X}_1$ to $\hat{X}_2 \Rightarrow \hat{X}_1, \hat{X}_2$ are in same orbit of $\mathsf{Aut}(X_1)$
- **for** loop selects only one element per orbit$\Rightarrow \hat{X}_1 = \hat{X}_2$
- $\Rightarrow Y_1 = Y_2$ (choices made in scan after $\hat{X} \in A$ are unique/canonical)

What if we can't compute automorphism groups?

procedure scan2(X : labelled object, n : integer)

  output X

  $C := \emptyset$

  for each $\hat{X} \in U(X)$ such that $f'(\hat{X}) \neq \emptyset$ **do**

    select any $\check{Y} \in f'(\hat{X})$, and suppose $\check{Y} \in L(Y)$

    if $o(Y) \leq n$ **and** $\check{Y} \in m(Y)$ **then** $C := C \cup \{Y\}$ **endif**

  endfor

  remove isomorphs from the set C

  for each $Y \in C$ **do** scan2(Y, n) **endif**

endprocedure