Overview of Gaussian MIMO (Vector) BC

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Outline

Capacity Region of Gaussian MIMO BC

System Structure

Know Capacity Regions

- Aligned and Inconsistently Degraded MIMO BC → Superposition
- Aligned MIMO BC without Common Message

  → Writing on Dirty Paper

- Degraded Message Sets (A Common & One Private Message)

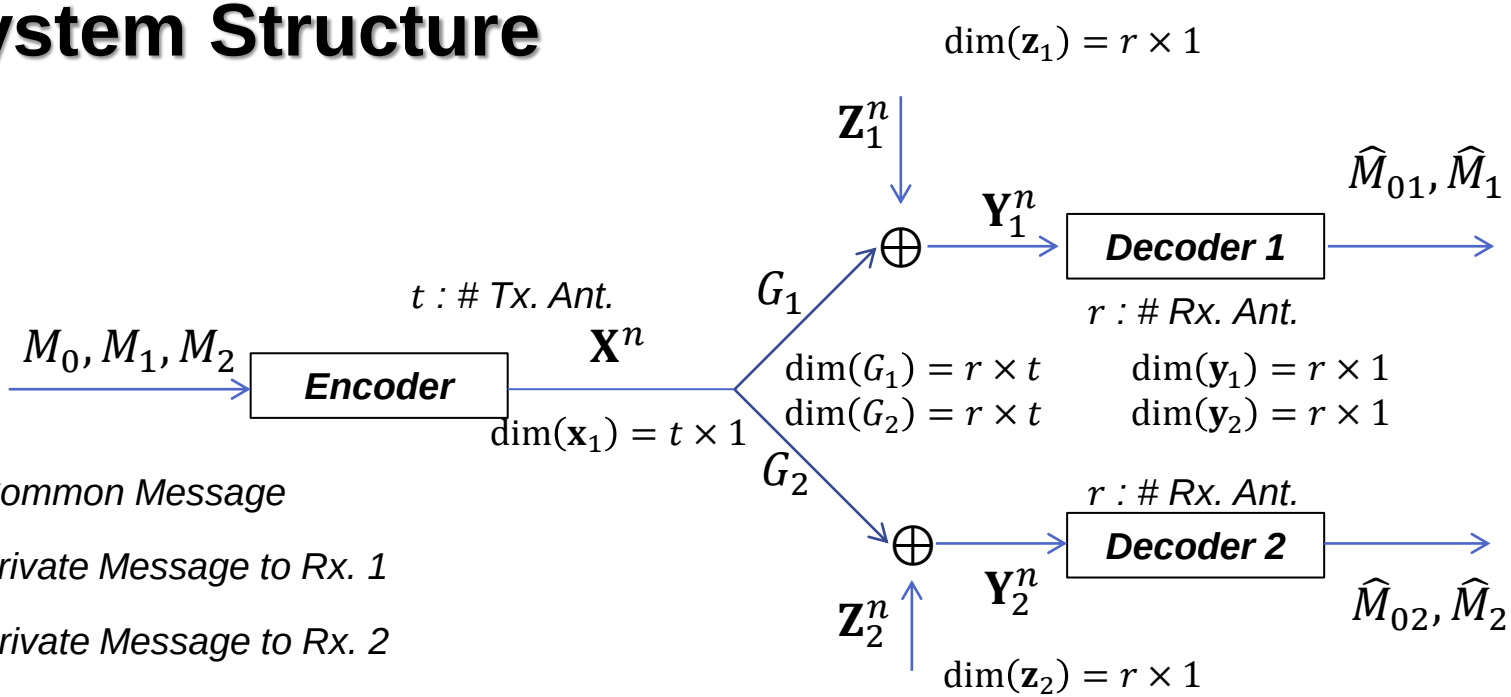
Duality of Gaussian MIMO BC & MAC

Gaussian MIMO MAC

Gaussian MIMO BC & MAC

Gaussian MIMO (Vector) BC

- **System Structure**



M_0 : *A Common Message*

M_1 : *A Private Message to Rx. 1*

M_2 : *A Private Message to Rx. 2*

Power Constraint

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}^T(m_0, m_1, m_2, i)\mathbf{x}(m_0, m_1, m_2, i) \le P$$

$$(m_0, m_1, m_2) \in [1: 2^{nR_0}] \times [1: 2^{nR_1}] \times [1: 2^{nR_2}]$$

channel

$$\mathbf{Y}_1 = G_1\mathbf{X} + \mathbf{Z}_1$$
$$\mathbf{Y}_2 = G_2\mathbf{X} + \mathbf{Z}_2$$

$$\mathbf{Z}_1 \sim \mathcal{N}(0, I_r)$$

$$\mathbf{Z}_2 \sim \mathcal{N}(0, I_r)$$

Capacity Region of Gaussian MIMO BC

- **Special Cases Known Capacity Region**

  ✓ **Aligned and *Inconsistently Degraded* MIMO BC**

  $\boldsymbol{t = r}$**, diagonal** $\boldsymbol{G_1}$**,** $\boldsymbol{G_2}$ ($G_1^T G_1$ and $G_2^T G_2$ have the same set of Eigenvalue)

  : A Product of Gaussian BC → ***Superposition Coding***

  ✓ **Aligned MIMO BC** ($M_0 = 0$)

  Only Private Messages without a Common Message

  → Vector ***Writing on Dirty Paper***

  ✓ ***Degraded*** **a Private Message and a Common Message**

  Either $M_0 = 0$ or $M_0 = 0$

  → ***Degraded*** Message Set → ***Superposition Coding***

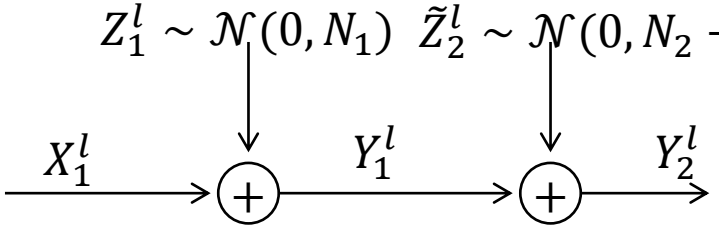
Case 1 : Gaussian Product BC

- **Parallel Gaussian BCs**

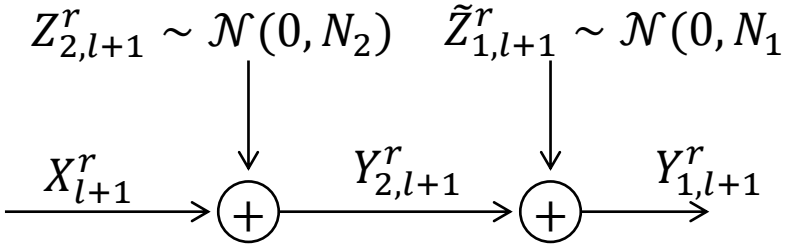
$$Y_{1k} = X_k + Z_{1k}$$

$$Y_{2k} = X_k + Z_{2k} \qquad k \in [1:r] \qquad Z_{jk} \sim \mathcal{N}(0, N_{jk}) \quad j = 1,2 \qquad M.I.$$

✓ ***Not Degraded, but Inconsistently Degraded BC***



$$N_{1k} \leq N_{2k} \qquad k \in [1:l]$$



$$N_{2k} > N_{1k} \qquad k \in [l+1:r]$$

Case 1 : Gaussian Product BC

- **Capacity Region**

$$R_0 + R_1 \leq \sum_{k=1}^{l} C\left(\frac{\beta_k P}{N_{1k}}\right) + \sum_{k=l+1}^{r} C\left(\frac{\alpha_k \beta_k P}{(1-\alpha_k)\beta_k P + N_{1k}}\right)$$

$$R_0 + R_2 \leq \sum_{k=1}^{l} C\left(\frac{\alpha_k \beta_k P}{(1-\alpha_k)\beta_k P + N_{2k}}\right) + \sum_{k=l+1}^{r} C\left(\frac{\beta_k P}{N_{2k}}\right)$$

$$R_0 + R_1 + R_2 \leq \sum_{k=1}^{l} C\left(\frac{\beta_k P}{N_{1k}}\right) + \sum_{k=l+1}^{r} \left[C\left(\frac{\alpha_k \beta_k P}{(1-\alpha_k)\beta_k P + N_{1k}}\right) + C\left(\frac{(1-\alpha_k)\beta_k P}{N_{2k}}\right)\right]$$

$$R_0 + R_1 + R_2 \leq \sum_{k=1}^{l} \left[C\left(\frac{\alpha_k \beta_k P}{(1-\alpha_k)\beta_k P + N_{2k}}\right) + C\left(\frac{\beta_k P}{N_{1k}}\right)\right] + \sum_{k=l+1}^{r} C\left(\frac{\beta_k P}{N_{2k}}\right)$$

For some $\alpha_k, \beta_k \in [0,1]$, $k \in [1:r]$, with $\sum_{k=1}^{r} \beta_k = 1$

Case 1 : Gaussian Product BC

- **Rate Region**

$$R_0 + R_1 \leq I(X_1; Y_{11}) + I(U_2; Y_{12})$$

$$R_0 + R_2 \leq I(X_2; Y_{22}) + I(U_1; Y_{21})$$

$$R_0 + R_1 + R_2 \leq I(X_1; Y_{11}) + I(U_2; Y_{12}) + I(X_2; Y_{22}|U_2)$$

$$R_0 + R_1 + R_2 \leq I(X_2; Y_{22}) + I(U_1; Y_{21}) + I(X_1; Y_{11}|U_1)$$

For some pmf $p(u_1, x_1)p(u_2, x_2)$

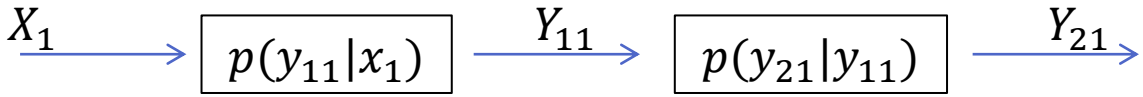
- **Achievability & Converse Proof**

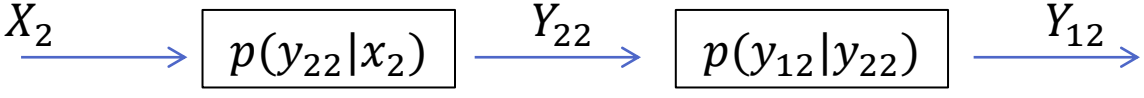
→ ***Superposition Coding (Degraded Gaussian BC)***

Case 1 : Gaussian Product BC

- **Achievability Proof**

$$(\boldsymbol{x}_1, p(y_{11}|x_1)p(y_{21}|y_{11}), \boldsymbol{y}_{11} \times \boldsymbol{y}_{21})$$





$$(\boldsymbol{x}_2, p(y_{22}|x_2)p(y_{12}|y_{22}), \boldsymbol{y}_{12} \times \boldsymbol{y}_{22})$$

Rate Splitting Divide M_j, $j = 1,2$ into two indep. Messages :

M_{j0} at rate R_{j0}, M_{jj} at rate R_{jj}

Case 1 : Gaussian Product BC

Codebook Generation

Fix a pmf $p(u_1, x_1)p(u_2, x_2)$.

Randomly and indep. Generate $2^{n(R_0+R_{10}+R_{20})}$ sequence pairs

$$(u_1^n, u_2^n)(m_0, m_{10}, m_{20}) \text{ according to } \prod_{i=1}^n p_{U_1}(u_{1i})p_{U_2}(u_{2i})$$

$$(m_0, m_{10}, m_{20}) \in [1: 2^{nR_0}] \times [1: 2^{nR_{10}}] \times [1: 2^{nR_{20}}]$$

For (m_0, m_{10}, m_{20}), randomly and conditionally indep. Generate $2^{nR_{jj}}$ sequences

$$x_j^n(m_0, m_{10}, m_{20}, m_{jj}) \text{ according to } \prod_{i=1}^n p_{X_j|U_j}(x_{ji}|u_{ji}(m_0, m_{10}, m_{20})$$

$$m_{jj} \in [1: 2^{nR_{jj}}], j = 1,2$$

Encoding

To send the message triple $(m_0, m_1, m_2) = (m_0, (m_{10}, m_{11}), (m_{20}, m_{22}))$

Transmit $(x_1^n(m_0, m_{10}, m_{20}, m_{11}), x_2^n(m_0, m_{10}, m_{20}, m_{22}))$

Case 1 : Gaussian Product BC

Decoding and analysis of the probability of error

Decoder 1 : find unique triple $(\hat{m}_{01}, \hat{m}_{10}, \hat{m}_{11})$

such that $((u_1^n, u_2^n)(\hat{m}_{01}, \hat{m}_{10}, \hat{m}_{11}), x_1^n(\hat{m}_{01}, \hat{m}_{10}, m_{20}, \hat{m}_{11}), y_1^n, y_2^n) \in T_\epsilon^{(n)}$

For some m_{10}.

Probability error for decoder 1

$$R_0 + R_1 + R_{20} < I(U_1, U_2, X_1; Y_{11}, Y_{12}) - \delta(\epsilon)$$
$$= I(X_1; Y_{11}) + I(U_2; Y_{12}) - \delta(\epsilon)$$
$$R_{11} < I(X_1; Y_{11}|U_1) - \delta(\epsilon)$$

Probability error for decoder 2

$$R_0 + R_{10} + R_2 < I(X_2; Y_{22}) + I(U_1; Y_{21}) - \delta(\epsilon)$$
$$R_{22} < I(X_2; Y_{22}|U_2) - \delta(\epsilon)$$

Case 2 : Private Messages

- **Capacity Region**

$$\mathbf{C = R_{WDP} = co(R_1 \cup R_2)}$$

✓ $\mathbf{R_1}$ **: DPC with Non-causal State** $\mathbf{X_2^n}$

$$R_1 < \frac{1}{2}\log|G_1K_1G_1^T + I_r|$$

$$R_2 < \frac{1}{2}\log\frac{|G_2K_2G_2^T + G_2K_1G_2^T + I_r|}{|G_2K_1G_2^T + I_r|}$$

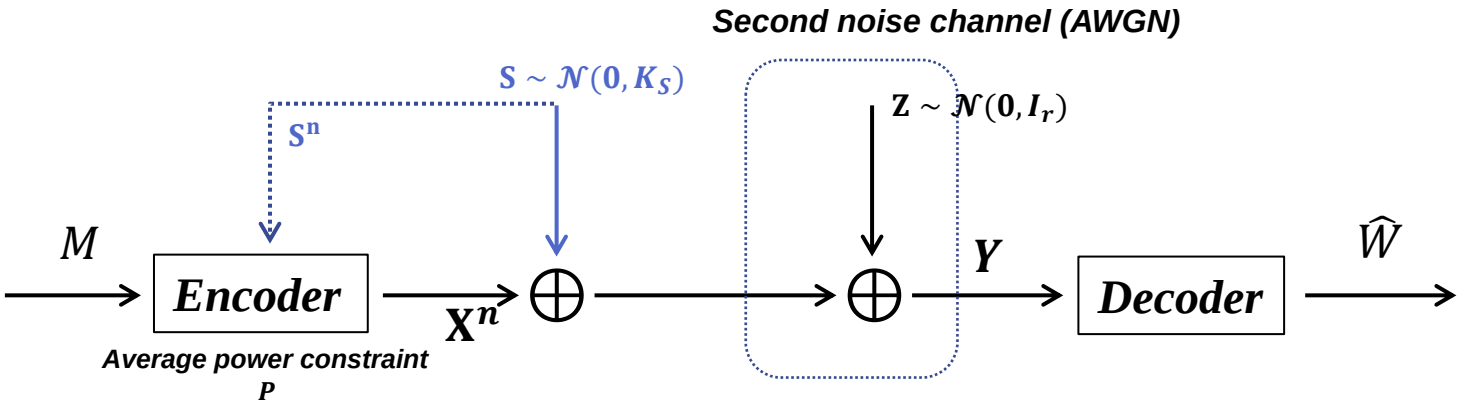
✓ $\mathbf{R_2}$ **: DPC with Non-causal State** $\mathbf{X_1^n}$

$$R_1 < \frac{1}{2}\log\frac{|G_1K_1G_1^T + G_1K_2G_1^T + I_r|}{|G_1K_2G_1^T + I_r|}$$

$$R_2 < \frac{1}{2}\log|G_2K_2G_2^T + I_r|$$

Vector Writing on Dirty Paper (1)

- **Vector Writing on Dirty Paper**



$$\mathbf{Y} = G\mathbf{X} + \mathbf{S} + \mathbf{Z}$$

$$\mathbf{C} = \max_{tr(K_X) \le \boldsymbol{P}} \frac{\mathbf{1}}{\mathbf{2}} \log |\boldsymbol{G}\, \boldsymbol{K_X} \boldsymbol{G^T} + \boldsymbol{I_r}|$$

Vector Writing on Dirty Paper (2)

- **Proof of Capacity** $\mathbf{C} = \max_{tr(K_X) \le P} \frac{1}{2} \log |G\, K_X G^T + I_r|$

$$C = \sup_{p(\mathbf{u}|\mathbf{s}),\mathbf{x}(\mathbf{u}|\mathbf{s}):\mathrm{E}[\mathbf{X}^T\mathbf{X}) \le P} [\, I(\mathbf{U};\mathbf{Y}) - I(\mathbf{U};\mathbf{S})]$$

Let $\mathbf{U} = \mathbf{X} + A\mathbf{S}$, where $X \sim \mathcal{N}(0, K_X)$ is independent of $\mathbf{S}$

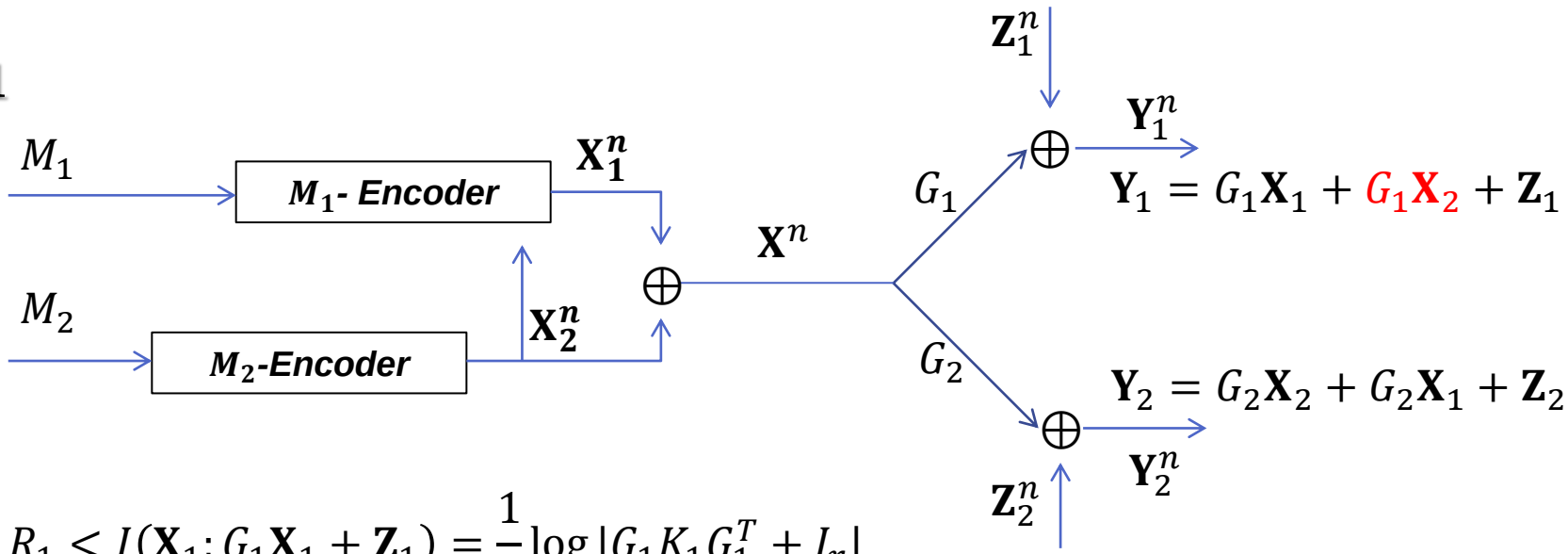
$$A = K_X G^T (G\, K_X G^T + I_r)^{-1}$$

$$\begin{aligned} I(\mathbf{U};\mathbf{Y}) - I(\mathbf{U};\mathbf{S}) &= h(\mathbf{U}|\mathbf{S}) - h(\mathbf{U}|\mathbf{Y}) \\ &= h(\mathbf{X} + A\mathbf{S}|\mathbf{S}) - h(\mathbf{X} + A\mathbf{S}|\mathbf{Y}) \\ &= h(\mathbf{X}) - h(\mathbf{X}|G\mathbf{X} + \mathbf{Z}) \\ &= I(\mathbf{X};\ G\mathbf{X} + \mathbf{Z}) \\ &= \frac{1}{2} \log |I_r + G\, K_X G^T| \end{aligned}$$

$$\begin{aligned} h(\mathbf{X} + A\mathbf{S}|\mathbf{Y}) &= h(\mathbf{X} + A\mathbf{S} - A\mathbf{Y}|\mathbf{Y}) \\ &= h(\mathbf{X} + A(\mathbf{S} - \mathbf{Y})|\mathbf{Y}) \\ &= h(\mathbf{X} + A(G\boldsymbol{X} + \boldsymbol{Z})|\mathbf{Y}) \\ &= h(\mathbf{X} + A(G\boldsymbol{X} + \boldsymbol{Z})) \\ &= h(\mathbf{X} + A(G\boldsymbol{X} + \boldsymbol{Z})|G\mathbf{X} + \mathbf{Z}) \\ &= h(\mathbf{X}|G\mathbf{X} + \mathbf{Z}) \end{aligned}$$

Vector Writing on Dirty Paper (3)

- $\mathbf{R_1}$

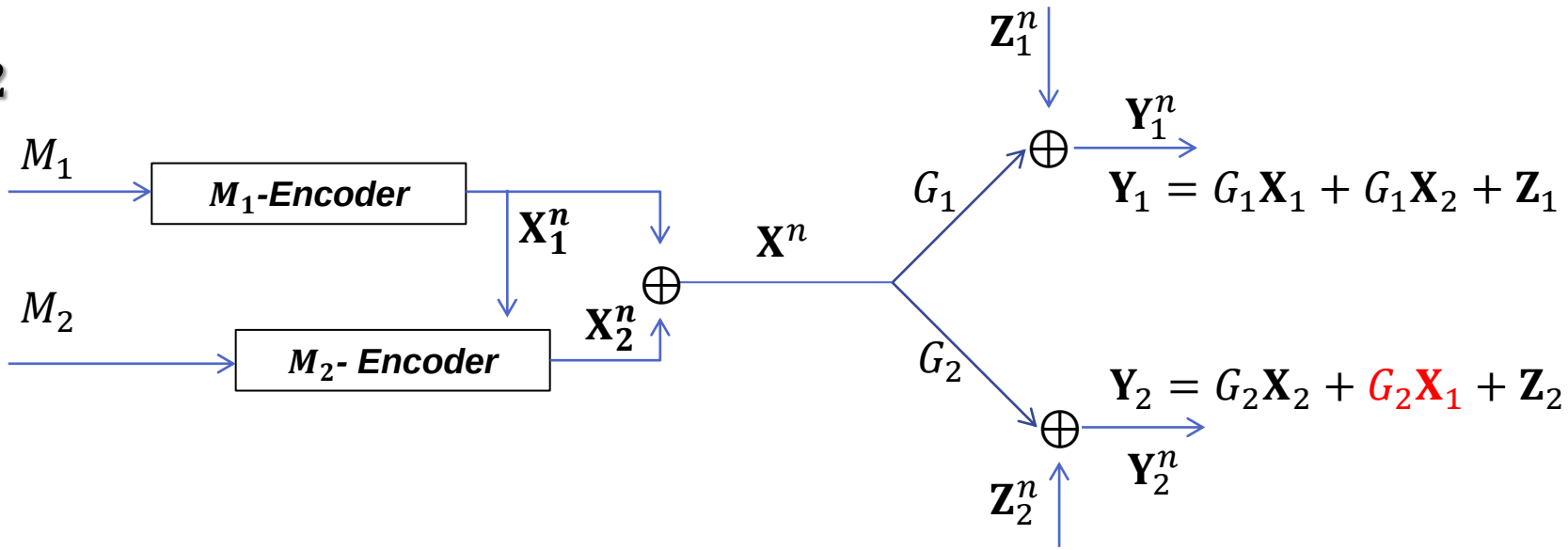


$$R_1 < I(\mathbf{X}_1; G_1\mathbf{X}_1 + \mathbf{Z}_1) = \frac{1}{2}\log|G_1K_1G_1^T + I_r|$$

$$R_2 < I(\mathbf{X}_2; G_2\mathbf{X}_1 + G_2\mathbf{X}_2 + \mathbf{Z}_2) = \frac{1}{2}\log\frac{|G_2K_1G_2^T + G_2K_2G_2^T + I_r|}{|G_2K_1G_2^T + I_r|}$$

Vector Writing on Dirty Paper

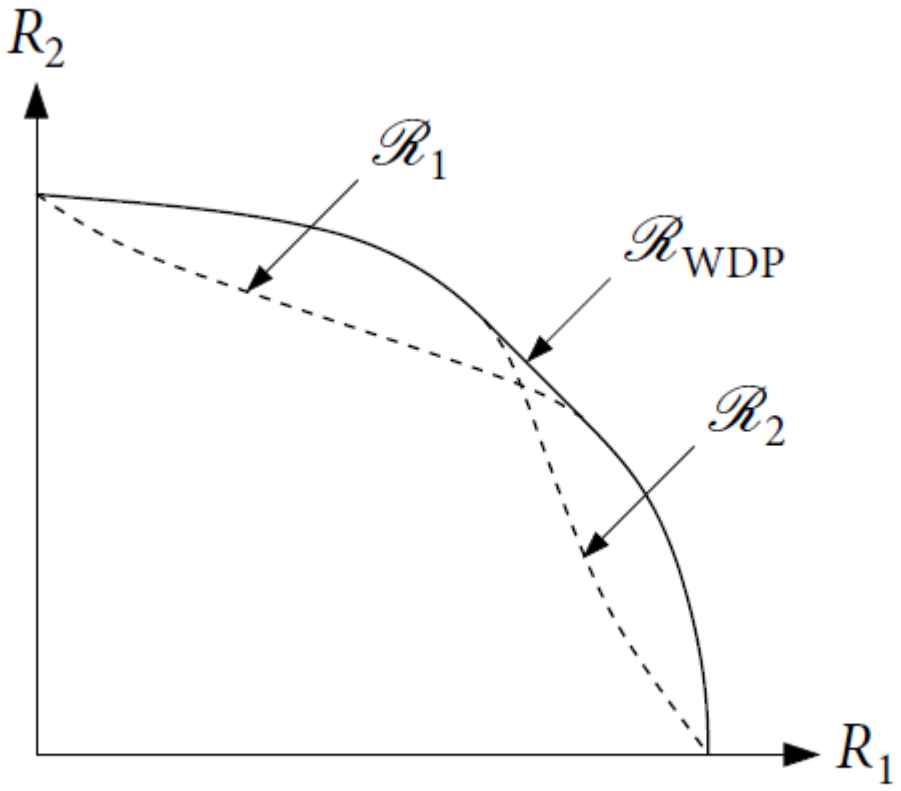
- $\mathbf{R_2}$



$$R_1 < I(\mathbf{X}_1; G_1\mathbf{X}_1 + G_1\mathbf{X}_2 + \mathbf{Z}_1) = \frac{1}{2}\log\frac{|G_1K_1G_1^T + G_1K_2G_1^T + I_r|}{|G_1K_2G_1^T + I_r|}$$

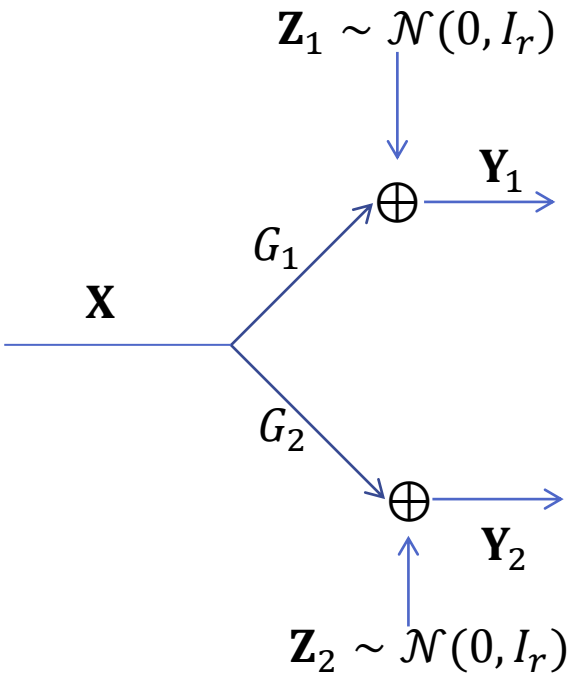
$$R_2 < I(\mathbf{X}_2; G_2\mathbf{X}_2 + \mathbf{Z}_2) = \frac{1}{2}\log|G_2K_2G_2^T + I_r|$$

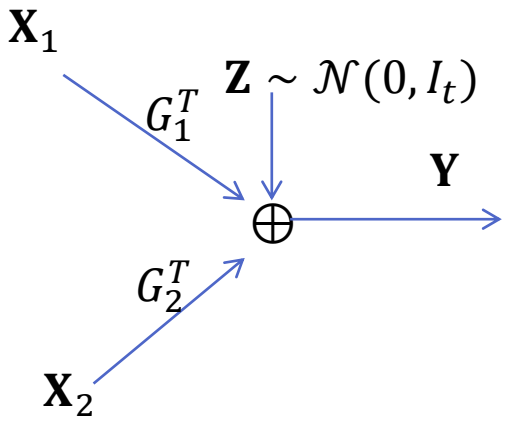
Capacity Region of Gaussian MIMO BC



The capacity region of a GV-BC: $\mathscr{C} = \mathscr{R}_{\text{WDP}} = \text{co}(\mathscr{R}_1 \cup \mathscr{R}_2)$.

BC-MAC Duality

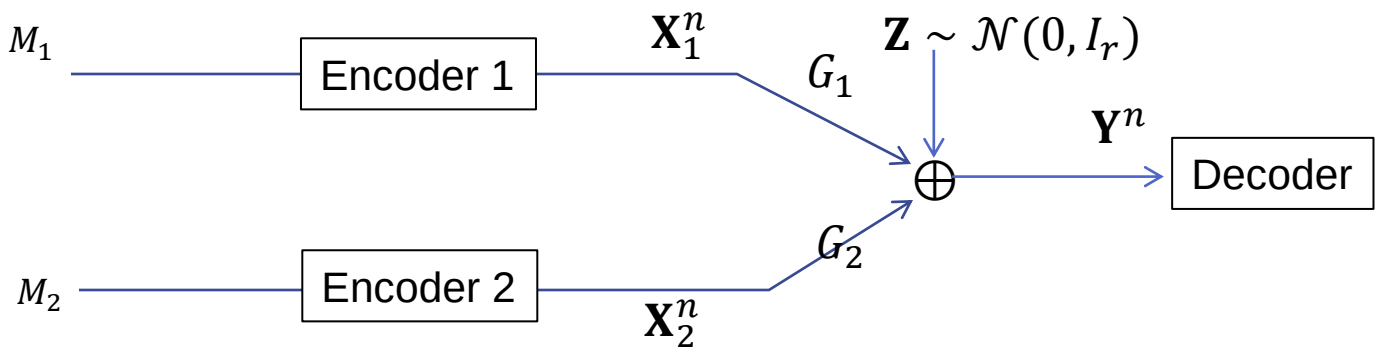




$$C_{BC}^{DP}(P; G_1, G_2) = \bigcup_{\sum_{i=1}^{2} tr(P_i) \leq P} C_{MAC}(P_1, P_2; G_1^T, G_2^T)$$

MIMO Multiple Access Channel

- **System Structure**



Power Constraint

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_j^T(m_j, i)\mathbf{x}_j(m_j, i) \le P$$

$$m_j \in \left[1 : 2^{nR_j}\right],\ j = 1,2$$

channel

$$\mathbf{Y} = G_1\mathbf{X}_1 + G_2\mathbf{X}_2 + \mathbf{Z}$$

$$\mathbf{Z} \sim \mathcal{N}(0, I_r)$$

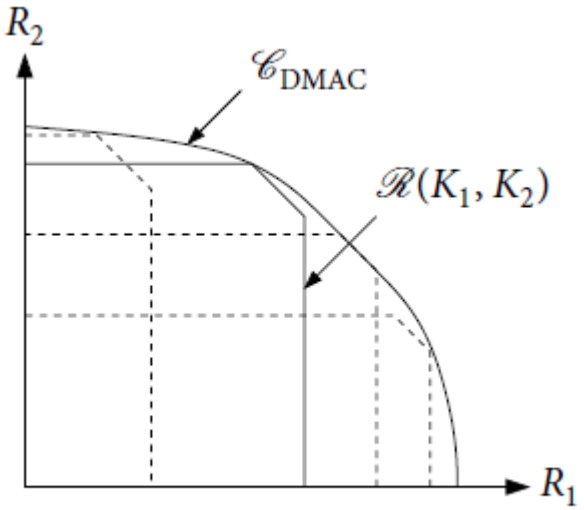
MIMO MAC

- **Capacity Region**

$$R_1 \le \frac{1}{2}\log|G_1 K_1 G_1^T + I_r|$$

$$R_2 \le \frac{1}{2}\log|G_2 K_2 G_2^T + I_r|$$

$$R_1 + R_2 \le \frac{1}{2}\log|G_1 K_1 G_1^T + G_2 K_2 G_2^T + I_r|$$



Dual GV-MAC capacity region $\mathscr{C}_{\text{DMAC}}$.

- **Boundary Point R^***

$$R_1^* = \frac{1}{2}\log|G_1 K_1^* G_1^T + I_r|$$

$$R_2^* = \frac{1}{2}\log|G_1 K_1 G_1^T + G_2 K_2 G_2^T + I_r| - \frac{1}{2}\log|G_1 K_1 G_1^T + I_r|$$

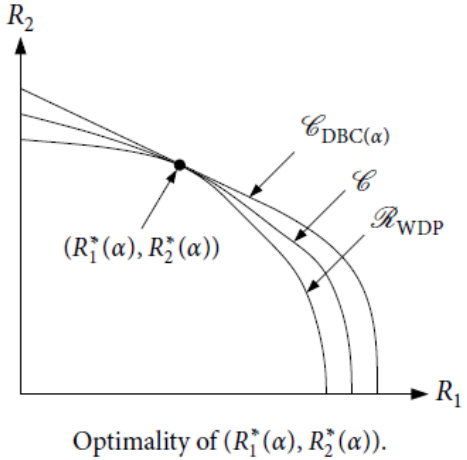
$$= \frac{1}{2}\log\frac{|G_1 K_1^* G_1^T + G_2 K_2^* G_2^T + I_r|}{|G_1 K_1^* G_1^T + I_r|}$$

Achievability Proof : DPC Capacity Region

- **Using Dual MAC**

$$\mathbf{R}_{WDP} = \mathbf{C}_{DMAC} = \bigcup_{K_1,K_2 \succcurlyeq 0: tr(K_1)+tr(K_2) \le P} \mathbf{R}(K_1, K_2)$$

(R_1^*, R_2^*) of $\mathbf{C}_{DMAC}$ lies on the boundary of (K_1, K_2)



Optimality of $(R_1^*(\alpha), R_2^*(\alpha))$.

$$\max_{\alpha \in [0,1], (R_1,R_2) \in \mathbf{C}_{DMAC}} [\alpha R_1 + \bar{\alpha} R_2]$$

$$\max_{\alpha \in [0,1], tr(K_1)+tr(K_2) \le P, K_1, K_2 \succcurlyeq 0\}} [\frac{\alpha}{2} \log \left| G_1 K_1 G_1^T + G_2 K_2 G_2^T + I_r \right| + \frac{\bar{\alpha} - \alpha}{2} \log |G_2 K_2 G_2^T + I_r|]$$

Introducing Dual Variables

$$tr(K_1) + tr(K_2) \le P \qquad \lambda \ge 0$$

$$K_1, K_2 \succcurlyeq 0 \qquad \gamma_1, \gamma_2 \succcurlyeq 0$$

Achievability Proof : DPC Capacity Region

$$L(K_1, K_2, \gamma_1, \gamma_2, \lambda) = \frac{\alpha}{2}\log\left|G_1 K_1 G_1^T + G_2 K_2 G_2^T + I_r\right| + \frac{\bar{\alpha} - \alpha}{2}\log\left|G_2 K_2 G_2^T + I_r\right|$$

$$+tr(\gamma_1 K_1) + tr(\gamma_2 K_2) - \lambda[tr(K_1) + tr(K_2) - P)$$

Applying KKT

$$\lambda^* G_1 S_1 G_1^T + \gamma_1^* - \lambda^* I_r = 0$$

$$\lambda^* G_2 S_2 G_2^T + \gamma_2^* - \lambda^* I_r = 0$$

$$\lambda^*[tr(K_1^*) + tr(K_2^*) - P] = 0$$

$$tr(\gamma_1 K_1) = tr(\gamma_2 K_2) = 0$$

$$S_1 = \frac{\alpha}{2\lambda^*}(G_1^T K_1^* G_1 + G_2^T K_2{}^* G_2 + I_r)^{-1}$$

$$S_2 = \frac{\alpha}{2\lambda^*}(G_1^T K_1^* G_1 + G_2^T K_2{}^* G_2 + I_r)^{-1} + \frac{\bar{\alpha} - \alpha}{2\lambda^*}(G_2^T K_2^* G_2 + I_r)^{-1}$$

$$K_1^{**} = \frac{\alpha}{2\lambda^*}(G_2^T K_2^* G_2 + I_r)^{-1} - S_1$$

$$K_2^{**} = \frac{\alpha}{2\lambda^*} I_r - K_1^{**} - S_2$$