Dirty Paper Coding

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Outline

Introduction

Writing on Dirty Paper

- System Model & Channel Capacity
- Achievable Rate Region
- Encoding & Decoding
- Achievability Proof

Example of Dirty Paper Coding

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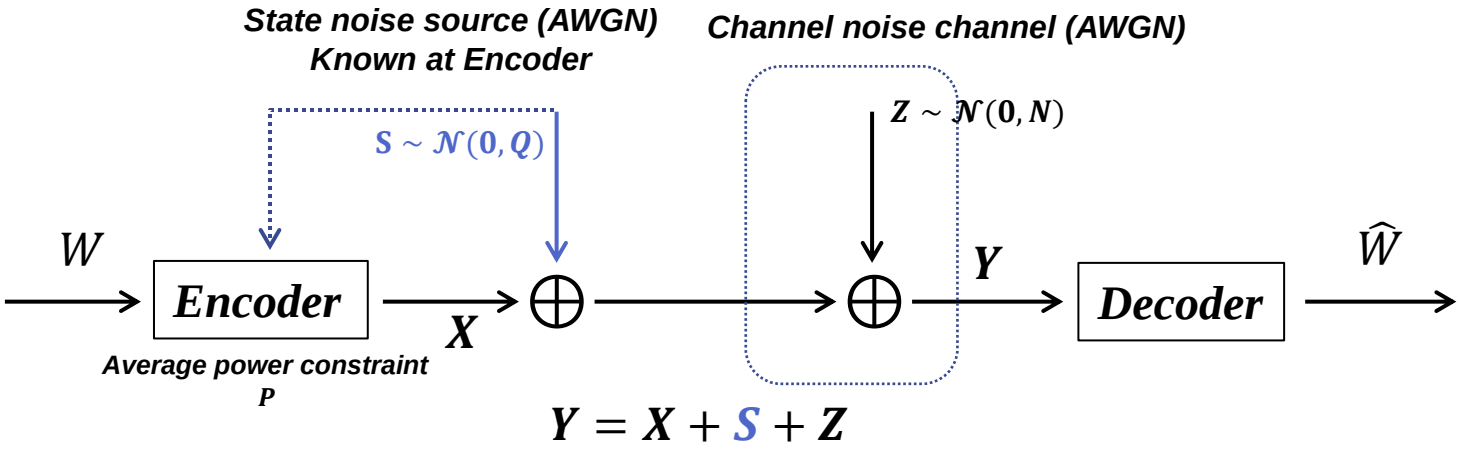
Example of Dirty Paper Coding

Writing on Dirty Paper (Costa, 83)

- **Costa's Idea**

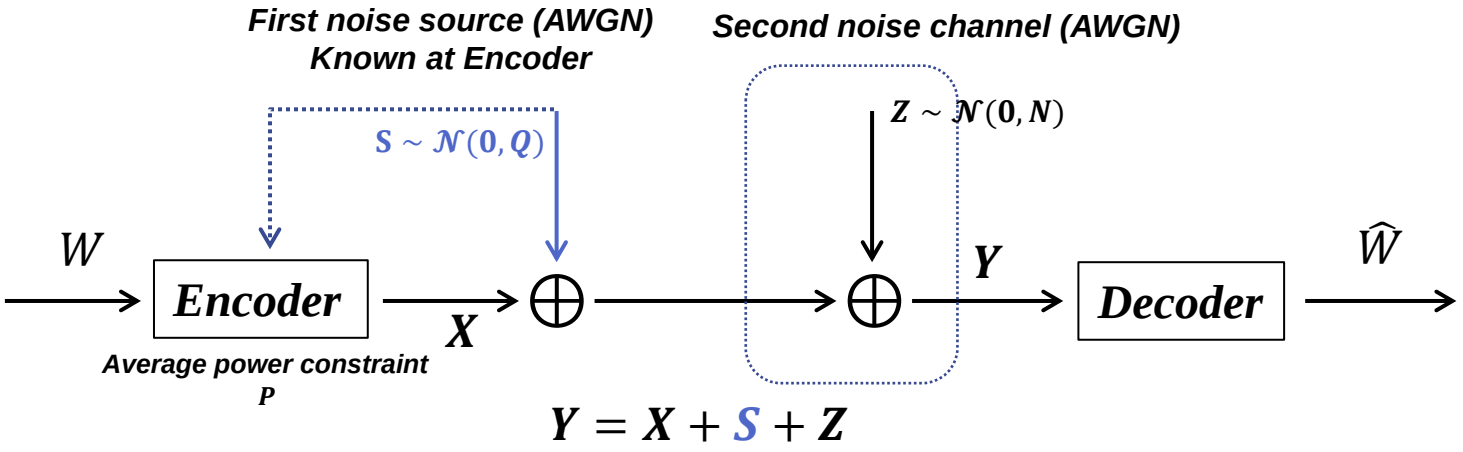


Capacity

$$\frac{1}{2}\ln\left(1+\frac{P}{N}\right) \quad \Leftarrow \quad \frac{1}{2}\ln\left(1+\frac{P}{N+Q}\right)$$

State S does not effect the capacity

Writing on Dirty Paper (Costa, 83)



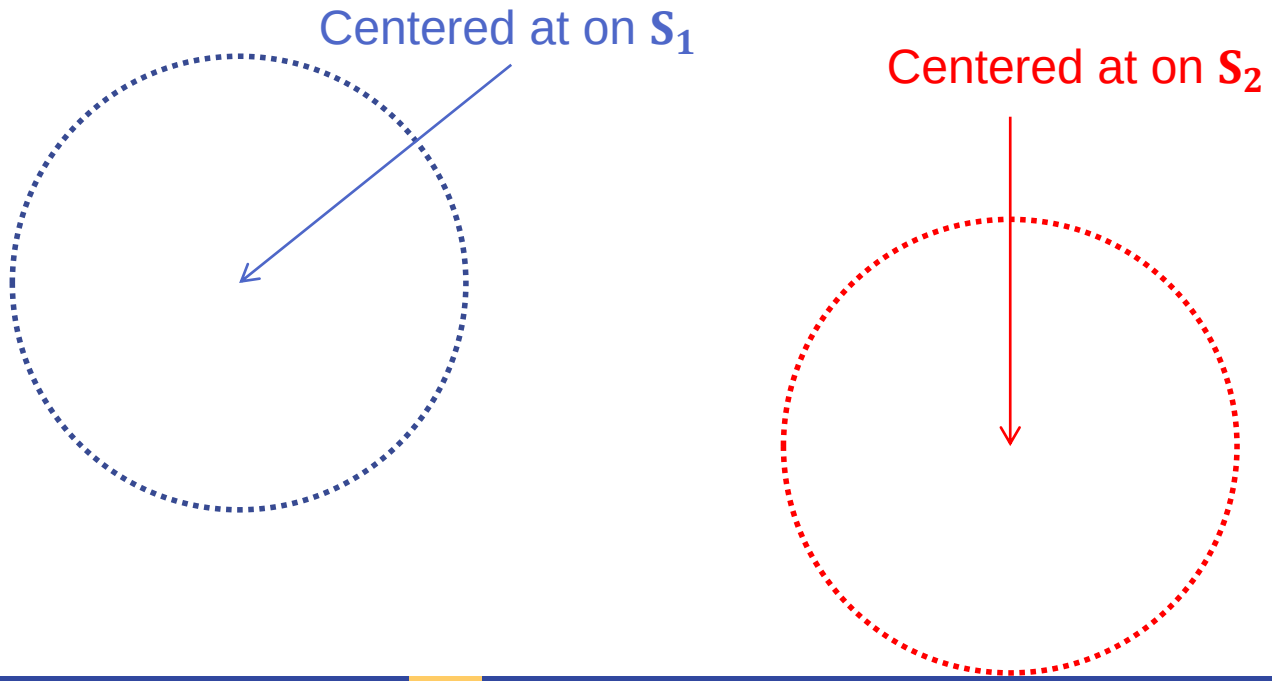
- ✓ **S** *: a piece of paper covered with normally distributed dots spot*
- ✓ **W** *: a message on it wit a limited amount of ink (**P**)*
- ✓ **How much information can we reliably send?**

$$\frac{1}{2}\ln\left(1+\frac{P}{N}\right) \quad \Leftarrow \quad \frac{1}{2}\ln\left(1+\frac{P}{N+Q}\right)$$

Dirt spots do not effect the capacity

Writing on Dirty Paper (Costa, 83)

- **Practical Importance**
  - ✓ **Diversity without capacity loss**
  - ✓ **Layering by dirt spots (Watermarking)**
  - ✓ **Extension to MIMO case for Diversity**



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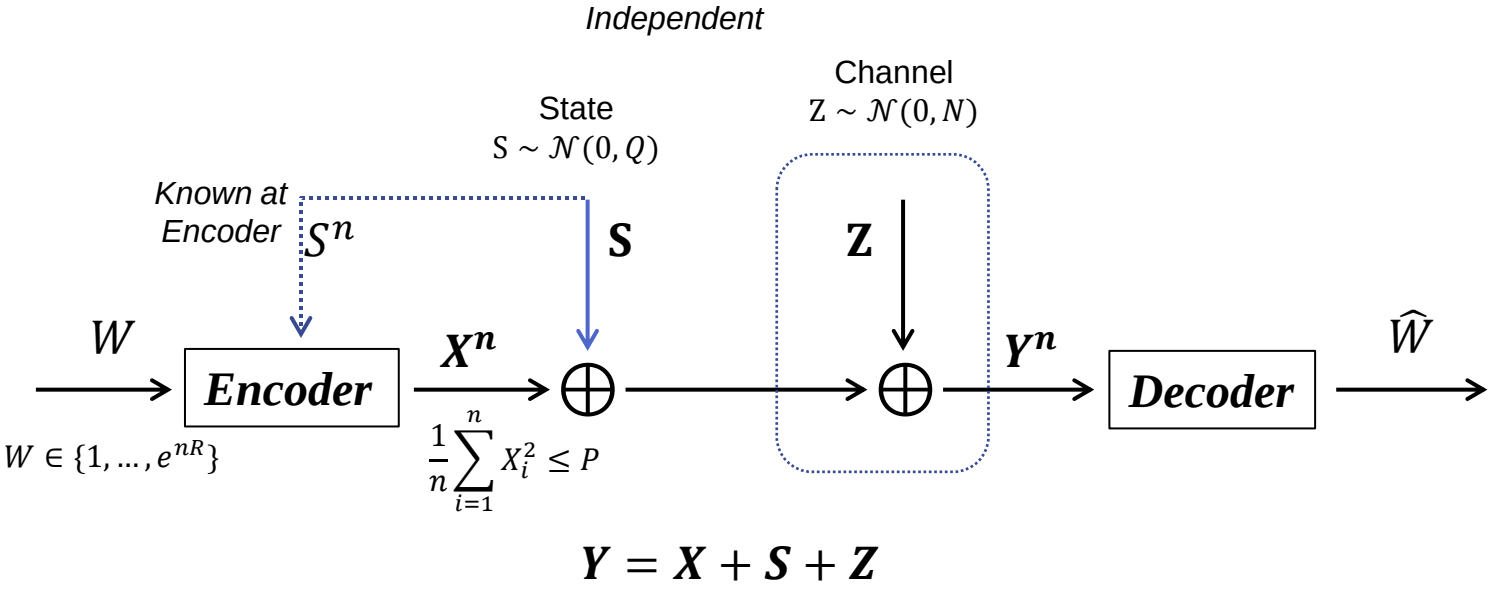
Writing on Dirty Paper

- **System Model**



$$Y = X + S + Z$$

✓ **Capacity**

$$\frac{1}{2}\ln\left(1+\frac{P}{N}\right) \quad \Leftarrow \quad \frac{1}{2}\ln\left(1+\frac{P}{N+Q}\right)$$

State S does not effect the capacity

Capacity Region

- **Refer to Gelfand Pinsker Theorem**
  - ✓ **Capacity of the with DM state (S) known only at Encoder**

$$C_{SI-E} = \max_{p(u,x|s)} [I(U;Y) - I(U;S)]$$

  - ✓ **Extension to DMC with continuous alphabet**

$$C_{SI-E} = \sup_{p(u,x|s)} [I(U_d;Y_p) - I(U_d;S_q)]$$

  - ✓**Proof :** Refer to Gelfand Pinsker Paper or El Gamal' book (Ch 7.6)

- **Capacity Region of DPC**
  - ✓ **Normally Distributed State S**
  - ✓ **Input power constraint P**
  - ✓ **AWGN channel with variance of Q**

$$\sup_{p(u,x|s)} [I(U;Y) - I(U;S)]$$

$$\frac{1}{2}\log\left(1+\frac{P}{N}\right)$$

Encoding

- **Encoding Scheme**
  - ✓ **Generate $e^{n[I(U;Y)-\epsilon]}$ iid sequences U**

    According to Uniform Distribution over the set of typical U
  - ✓ **Distribute these sequences uniformly over e^{nR} bins**

    Each sequence u with bin index $i(u)$
  - ✓ **Given state vector S and the message W**

    Look in bin W for a sequence U s.t. joint typical (U, S)
  - ✓ **Choose X s.t. jointly typical (X, U, S), send it**

Decoding

- **Decoding Scheme**
  - ✓ **Look for unique sequence U s.t. jointly typical (U, Y)**
  - ✓ **Declare Error**

    More than one or no such sequence exist
  - ✓ **Set the estimate $\widehat{W}$**

    Equal to the index of bin containing the obtained sequence U
  - ✓ **If $R < I(U;Y) - I(U;S) - \epsilon - \delta$**

    The probability of error averaged over all codes

    decrease exponentially to zero as $n \to \infty$

Achievability Proof

- $\boldsymbol{R \leq \frac{1}{2}\log\left(1+\frac{P}{N}\right)}$

Step 1 : Set $U = X + \alpha S$

$$
\begin{aligned}
I(U;Y) &= h(Y) - h(Y|U) \\
&= h(X+S+Z) - h(X+S+Z|X+\alpha S) \\
&= h(X+S+Z) + h(X+\alpha S) - h(X+\alpha S, X+S+Z) \\
&= \frac{1}{2}\ln[(2\pi e)^n (P+Q+N)(P+\alpha^2 Q) \\
&\quad - \frac{1}{2}\ln\{(2\pi e)^n [(P+Q+N)(P+\alpha^2 Q) - (P+\alpha Q)^2]\} \\
&= \frac{1}{2}\ln\frac{(P+Q+N)(P+\alpha^2 Q)}{PQ(1-\alpha)^2 + N(P+\alpha^2 Q)}
\end{aligned}
$$

Achievability Proof

$$
\begin{aligned}
I(U;S) &= h(U) - h(U|S) \\
&= h(X + \alpha S) - h(X + \alpha S|S) \\
&= h(X + \alpha S) + h(X) \\
&= \frac{1}{2}\ln[(2\,\pi e)^n(P + \alpha^2 Q)] - \frac{1}{2}\ln[(2\,\pi e)^n P] \\
&= \frac{1}{2}\ln\frac{(P + \alpha^2 Q)}{P}
\end{aligned}
$$

Let $R(\alpha) = I(U;Y) - I(U;S)$,

$$
R(\alpha) = \frac{1}{2}\ln\frac{P(P + Q + N)}{PQ(1-\alpha)^2 + N(P + \alpha^2 Q)}
$$

Achievability Proof

$$R(\alpha) = \frac{1}{2}\ln\frac{P(P+Q+N)}{PQ(1-\alpha)^2 + N(P+\alpha^2 Q)}$$

Find maximum $R(\alpha)$

Fixed $P(P+Q+N)$

minimize $PQ(1-\alpha)^2 + N(P+\alpha^2 Q)$

$$\alpha^* = \arg\min_{\alpha}[PQ(1-\alpha)^2 + N(P+\alpha^2 Q)] = \frac{-2PQ}{-2Q(N+P)} = \frac{P}{P+N}$$

$$\boldsymbol{R(\alpha) \le R(\alpha^*)|_{\alpha^*=\frac{P}{P+N}}}$$

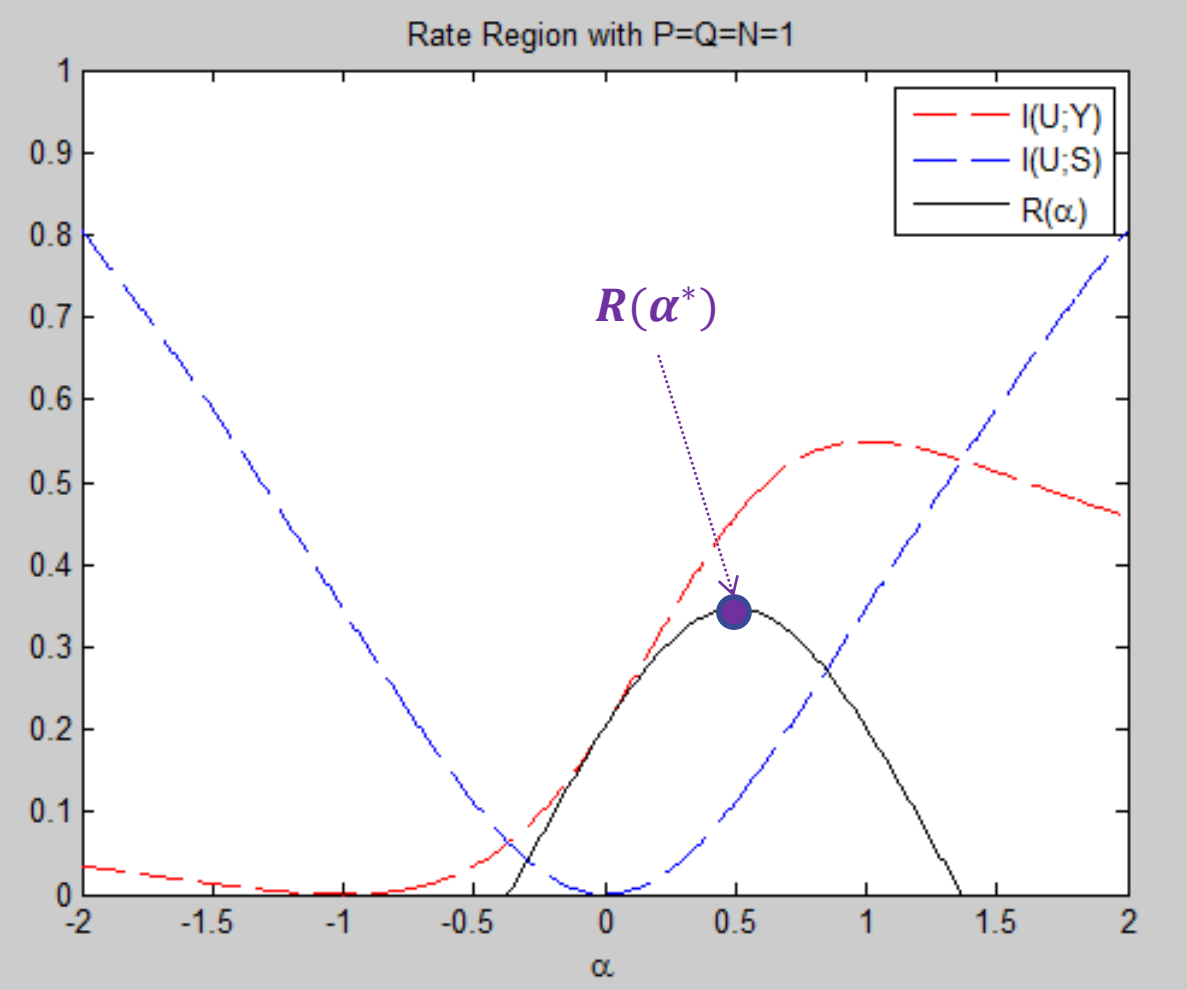
$$= \frac{1}{2}\ln[1+\frac{P}{N}]$$

Also,

$$I(U^*,Y) = \frac{1}{2}\ln[1+\frac{P(P+Q+N)}{N(P+N)}] \qquad I(U^*,S) = \frac{1}{2}\ln[1+\frac{PQ}{P+N}]$$

Achievability Proof

- **Rate Regions and Capacity**



Encoding & Decoding with α^*

- **Encoding Scheme**

✓ **Generate $e^{n[I(U^*;Y)-\epsilon]}$ sequences U**

With components independently drawn according to $\mathcal{N}(0, P + \alpha^{*2} Q)$

✓ **Place these sequences into $e^{nR} = e^{n(C^*-2\,\epsilon)}$ bins**

Each bin contain same number of sequences

✓ **Share Codebook between Encoder and Decoder**

✓ **Given a State Vector $S = S_0$ and a message $W = k$**

Look for jointly typical pair (U, S_0) among the U in bin k

$$|(U - \alpha^* S_0)' S_0| \leq \delta$$

Encoder calculate $\mathbf{X_0 = U_0 - \alpha^* S_0}$ **, send** $\mathbf{X_0}$

Encoding & Decoding

- **Decoding Scheme**

✓ **Receive $Y = Y_0$**

Look for a sequence U s.t jointly typical (U, Y_0)

✓ **Declare Error**

More than one or no such sequence

Find U_0

✓ **Set the estimate $\widehat{W}$**

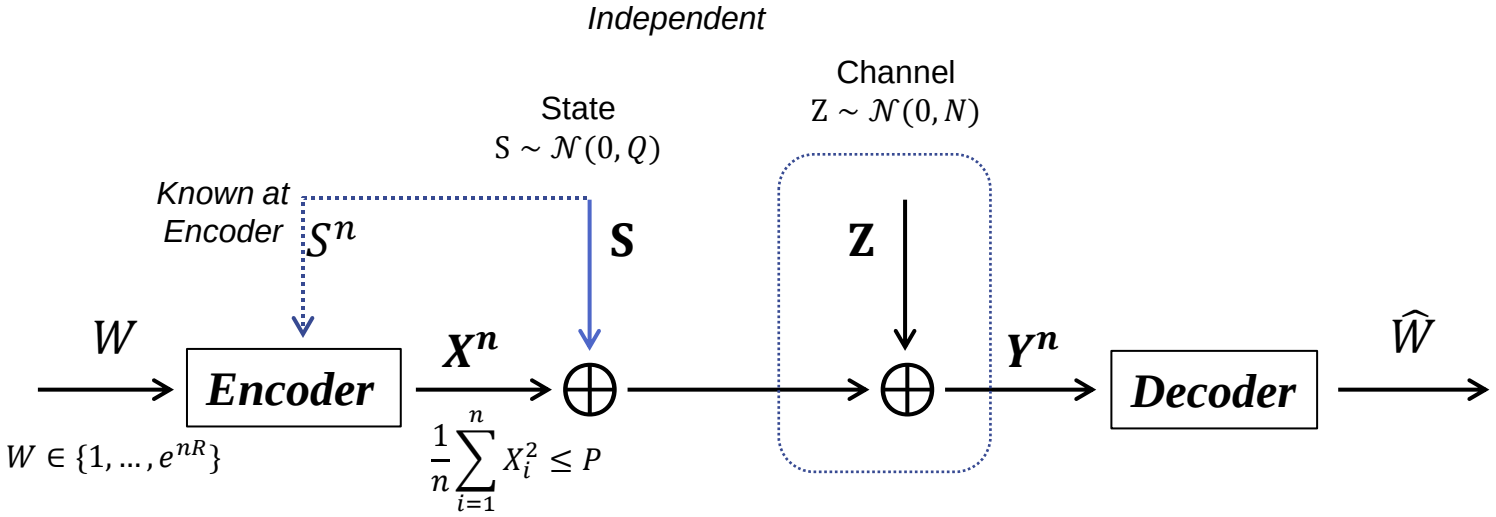
Equal to the index of bin containing this sequence

The probability of error averaged over random choice of code

decrease exponentially to zero as $n \to \infty$

Conclusion of DPC

- **System Model**



$$Y = X + S + Z$$

✓ **Capacity**

$$\frac{1}{2}\ln\left(1+\frac{P}{N}\right) \quad \Leftarrow \quad \frac{1}{2}\ln\left(1+\frac{P}{N+Q}\right)$$

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Example of DPC

- **DPC for Gaussian BC**

$$Y_1 = X + S_1 + Z_1$$
$$Y_2 = X + S_2 + Z_2$$

$$R_1 \le C\left(\frac{\alpha P}{N_1}\right) \qquad \alpha \in [0, 1]$$
$$R_2 \le C\left(\frac{(1-\alpha)P}{\alpha P + N_2}\right)$$

✓ **Assumptions**

Two States S_1 and S_2 are only known at Encoder

$S_1 \sim \mathcal{N}(0, Q_1)$ $\quad$ $S_2 \sim \mathcal{N}(0, Q_2)$

$Z_1 \sim \mathcal{N}(0, N_1)$ $\quad$ $Z_2 \sim \mathcal{N}(0, N_2)$ $\quad$ $N_2 \ge N_1$

Average Power Constraint P on X

Example of DPC

- **Encoding & Decoding**

Split X into two independent parts X_1 and X_2

$$X_1 \sim \mathcal{N}(0, \alpha P) \text{ and } X_2 \sim \mathcal{N}(0, (1-\alpha)P)$$

For weaker receiver Y_2,

$$Y_2 = X_2 + S_2 + (X_1 + Z_2) \text{ with known state } S_2$$

Taking $U_2 = X_2 + \beta_2 S_2$ with $\beta_2 = \frac{(1-\alpha)P}{P+N_2}$

Then, achieve $\boldsymbol{R_2 \le I(U_2; Y_2) - I(U_2; S_2) \le C\left(\frac{(1-\alpha)P}{\alpha P + N_2}\right)}$

Example of DPC

For stronger receiver Y_1,

$Y_1 = X_1 + (X_2 + S_1) + Z_1$ with known state S_1

Taking $U_1 = X_1 + \beta_1(X_2 + S_1)$ with $\beta_1 = \frac{\alpha P}{\alpha P + N_1}$

Then, achieve

$$\boldsymbol{R_1 \leq C(\frac{\alpha P}{N_1})}$$

Converse Proof of Capacity Region of DPC

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Converse Proof

System Structure and Gel'fand Pinsker Theorem

Csiszar Sum Equality

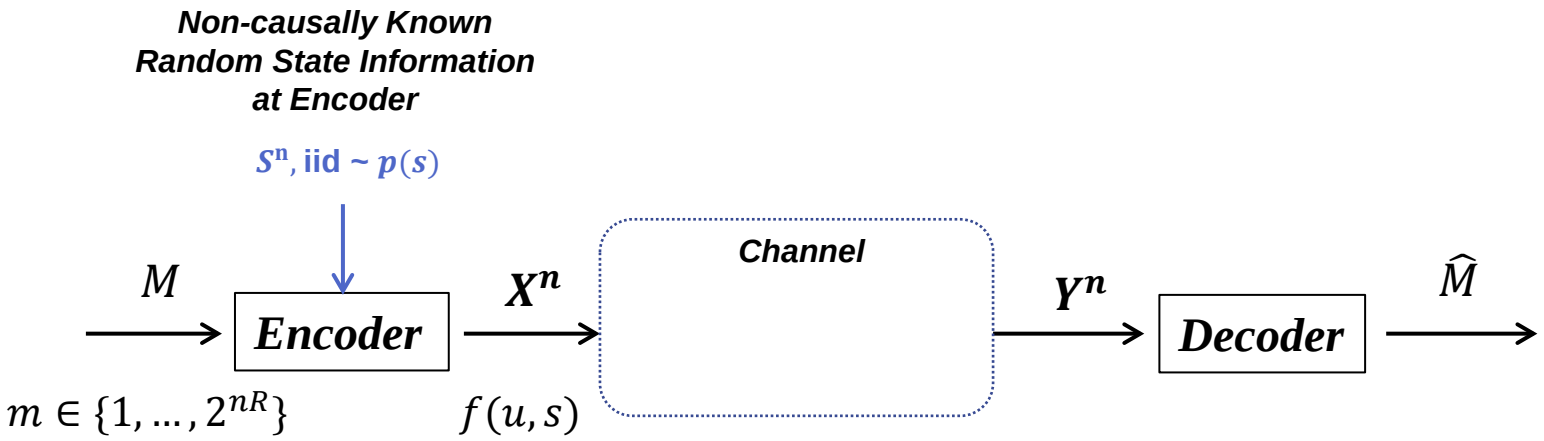
Applying Non-causality in Capacity

References

1. S. I. Gelf'and and M. S. Pinsker, "Coding for channel with random parameters", Problems of Control and Information Theory, Vol 9, No 1, pp 19-31, 1980
2. Abbas El Gamal and Young-han Kim, "Network Information Theory"
3. Thomas Cover, "Elements of Information Theory"

Non-causal State Information at Encoder

- **DMC with DM State**



✓ **Applying Gel'fand Pinsker Theorem**

$$C_{SI-E} = \max_{p(u|s),x=f(u,s)} [I(U;Y) - I(U;S)]$$

$$C_{Gel'fand} = \max_{p(u,s)} [I(U;Y) - I(U;S)]$$

Converse Proof

· Let R be an achievable rate, $\exists\ (2^{nR}, n)$ code with $P_e \to 0$ as $n \to \infty$.

· Find the auxiliary random variable U_i that forms

Markov Chain $\boldsymbol{U_i \to (X_i, S_i) \to Y_i}$,

$$nR = H(M) \quad \leftarrow +0$$

$$= H(M) - H(M|Y^n) + H(M|Y^n)$$

$$= I(M;Y^n) + H(M|Y^n)$$

← By Fano's lemma $H(M|Y^n) \le n\,\epsilon_n$

$$\le I(M;Y^n) + n\epsilon_n$$

$$= H(Y^n) - H(Y^n|M) + n\,\epsilon_n$$

Converse Proof

$$= H(Y^n) - H(Y^n|M) + n\,\epsilon_n$$

← By Chain Rule

$$= \sum_{i=1}^{n} H(Y_i|Y^{i-1}) - \sum_{i=1}^{n} H(Y_i|Y^{i-1}, M) + n\,\epsilon_n$$

← conditioning reduces entropy

$$\leq \sum_{i=1}^{n} H(Y_i) - \sum_{i=1}^{n} H(Y_i|Y^{i-1}, M) + n\,\epsilon_n$$

$$= \sum_{i=1}^{n} I(Y_i; Y^{i-1}, M) + n\,\epsilon_n$$

Converse Proof

$$= \sum_{i=1}^{n} I(Y_i; Y^{i-1}, M) \; + n\,\epsilon_n$$

Define n random variables $U_1, \ldots, U_n$ as follow

$$U_i = (M, Y_1, \ldots, Y_{i-1}, S_{i+1}, \ldots, S_n) = (M, Y^{i-1}, S_{i+1}^{n}) \cong (Q, U_Q)$$

Where $Q = (M, Y^{i-1})$ and $U_Q = S_{i+1}^{n}$

When Q is independent $Y_Q = Y_i$, then

$$I(Y_Q; Q) = I(Y_Q; U_Q, Q) - I(Y_Q; U_Q|Q)$$

$$I(Y_Q; U_Q|Q) = I(Y_Q; U_Q, Q)$$

By Eq. 15.281 in Elements of Information Theory ,Thomas Cover

$$\sum_{i=1}^{n} I(Y_i; Y^{i-1}, M) \; = \sum_{i=1}^{n} [I(M, Y^{i-1}, S_{i+1}^{n}; Y_i) - I(Y_i; S_{i+1}^{n}|M, Y^{i-1})]$$

Converse Proof

$$\leq \sum_{i=1}^{n} [I(M, Y^{i-1}, S_{i+1}^{n}; Y_i) - \boldsymbol{I(Y_i; S_{i+1}^{n} | M, Y^{i-1})}] + n\,\epsilon_n$$

← By Csiszar Sum Equality

$$= \sum_{i=1}^{n} [I(M, Y^{i-1}, S_{i+1}^{n}; Y_i) - \boldsymbol{I(Y^{i-1}; S_i | S_{i+1}^{n}, M)}] + n\,\epsilon_n$$

Csiszar Sum Equality

For Two random vectors X^n *and* Y^n *with joint probability distribution, then*

$$\sum_{i=1}^{n} \boldsymbol{I(Y_i; X_{i+1}^{n} | Y^{i-1})} = \sum_{i=1}^{n} \boldsymbol{I(Y^{i-1}; X_i | X_{i+1}^{n})}$$

Csiszar Sum Equality Proof

$$\sum_{i=1}^{n} I\left(Y_i; X_{i+1}^{n} \middle| Y^{i-1}\right) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} I(X_j; Y_i | Y^{i-1}, X_{j+1}^{n})$$

Chain rule

Switching of summation

$$= \sum_{j=2}^{n} \sum_{i=1}^{j-1} I(X_j; Y_i | Y^{i-1}, X_{j+1}^{n})$$

Chain rule

$$= \sum_{j=2}^{n} I(X_j; Y^{j-1} | X_{j+1}^{n})$$

$$= \sum_{j=1}^{n} I(X_j; Y^{j-1} | X_{j+1}^{n})$$

Also define $I(X_1; Y^0 | X_2^n) = 0$

Switching of Index

$$= \sum_{i=1}^{n} I(X_i; Y^{i-1} | X_{i+1}^{n})$$

Converse Proof

$$= \sum_{i=1}^{n} [I(M, Y^{i-1}, S_{i+1}^{n}; Y_i) - I(Y^{i-1}; S_i | S_{i+1}^{n}, M)] + n\,\epsilon_n$$

Independency between (M, S_{i+1}^{n}) *and* S_i

$$= \sum_{i=1}^{n} [I(M, Y^{i-1}, S_{i+1}^{n}; Y_i) - I(S_{i+1}^{n}, M, Y^{i-1}; S_i)] + n\,\epsilon_n$$

Use $U_i = (M, Y^{i-1}, S_{i+1}^{n})$

$$nR \leq \sum_{i=1}^{n} [I(U_i; Y_i) - I(U_i; S_i)] + n\,\epsilon_n$$

$$\leq n \max_{p(u,x|s)} [I(U; Y) - I(U; S)] + n\,\epsilon_n$$

Converse Proof

$$R \leq \max_{p(u,x|s)} [I(U;Y) - I(U;S)] + \epsilon_n$$

Also show that it suffices to maximize over $p(u|s)$ and function $x = f(u,s)$

$$p(u,x|s) = p(u|s)p(x|u,s)$$

Since $x = f(u,s)$ is deterministic function of u and s,

$$p(x|u,s) = 0,1$$

For fixed $p(u|s)$,

Fixed

$$R \leq \max_{p(u|s),x=f(u,s)} [I(U;Y) - I(U;S)] + \epsilon_n$$

Only over $I(U;Y)$ Convex of $p(y|u)$ fixing $p(u|s)$

Converse Proof

Since mutual information $I(U;Y)$ is a convex function of $p(y|u)$ for a fixed $p(u|s)$

$$p(y|u) = \sum_{x,s} p(s|u)p(x|s,u)p(y|x,s)$$ linear in $p(x|u,s)$

$I(U;Y)$ is also convex of $p(x|u,s)$

Maximum is attained at an extreme points of the set of pmf $p(x|u,s)$