A new computational approach for determining rate regions and optimal codes for coded networks

Congduan Li, Jayant Apte, John Walsh, Steven Weber

Drexel University, Dept. of ECE
Philadelphia, PA 19104
congduan.li@drexel.edu

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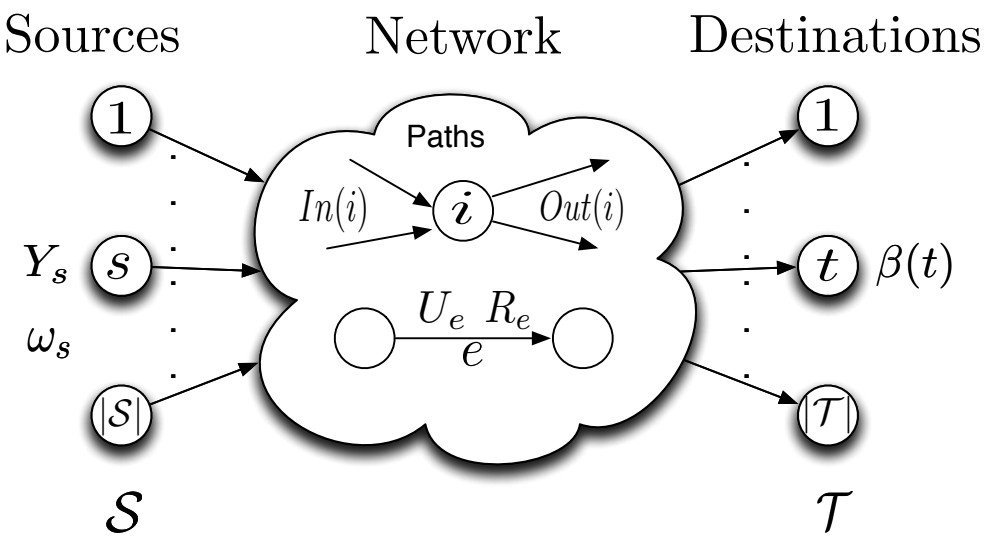
Motivation

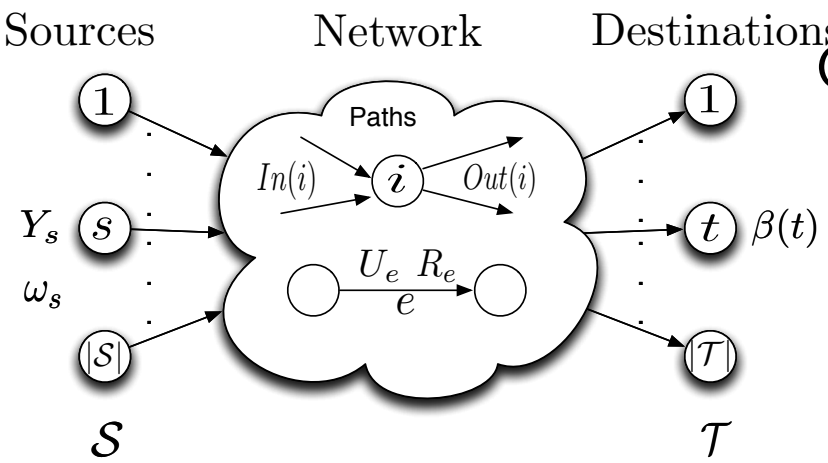
A multisource multicast network:



- Effective algorithm to calculate entire rate regions (not the all-equal capacity line)
- If simple linear codes acheivable

Rate Region

Collect $N = \{Y_s, U_e\}$. Suppose we know the region Γ_N^* of entropic vectors. Here Γ_N^* is a $2^N - 1$ dimensional cone and any point in it is **entropic** (has a valid joint distribution associated with the N variables) [1].



Constraints: (Hyperplanes)

$$\mathcal{L}_1 = \{\mathbf{h} \in \Gamma_N^* : h_{Y_s} \geq \omega_s\}$$

$$\mathcal{L}_2 = \{\mathbf{h} \in \Gamma_N^* : h_{Y_\mathcal{S}} = \Sigma_{s \in \mathcal{S}} h_{Y_s}\}$$

$$\mathcal{L}_3 = \{\mathbf{h} \in \Gamma_N^* : h_{X_{\mathrm{Out}(k)}|Y_s} = 0\}$$

$$\mathcal{L}_4 = \{\mathbf{h} \in \Gamma_N^* : h_{X_{\mathrm{Out}(i)}|X_{\mathrm{In}(i)}} = 0\}$$

$$\mathcal{L}_5 = \{\mathbf{h} \in \Gamma_N^* : h_{Y_{\beta(t)}|U_{\mathrm{In}(t)}} = 0\}$$

$$\mathcal{R} = \mathrm{Ex}(\mathrm{proj}_{U_e}(\overline{\mathrm{con}(\Gamma_N^* \cap \mathcal{L}_{234})} \cap \mathcal{L}_{15}))$$

Bounds on Rate Region

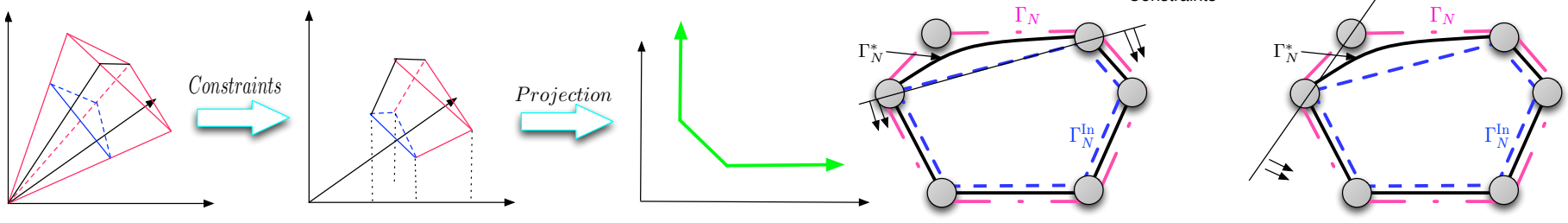
- Replace Γ_N^* with Shannon outer bound Γ_N:

$$\mathcal{R}_{out} = \text{Ex}(\text{proj}_{U_e}(\Gamma_N \cap \mathcal{L}_{12345}))$$

- Replace Γ_N^* with a closed inner bound Γ_N^{In}:

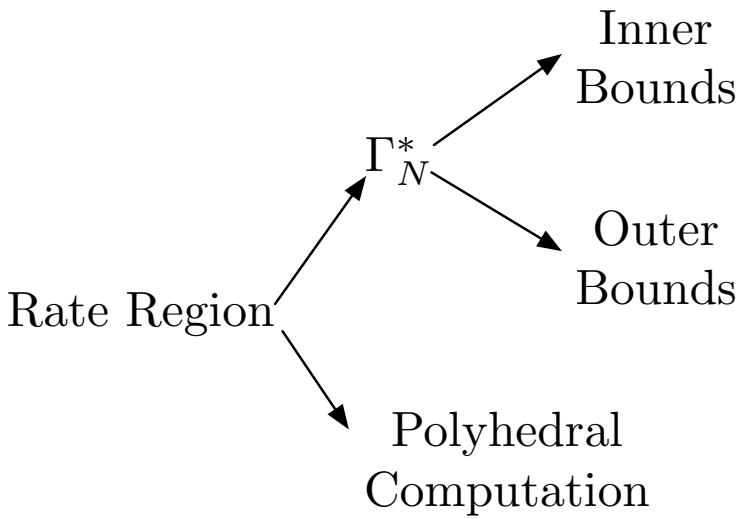
$$\mathcal{R}_{in} = \text{Ex}(\text{proj}_{U_e}(\Gamma_N^{In} \cap \mathcal{L}_{12345}))$$

- It becomes general polyhedral computations:
  - Initial polyhedra $\rightarrow$ $\cap$ constraints $\rightarrow$ projections



Problem Decomposition

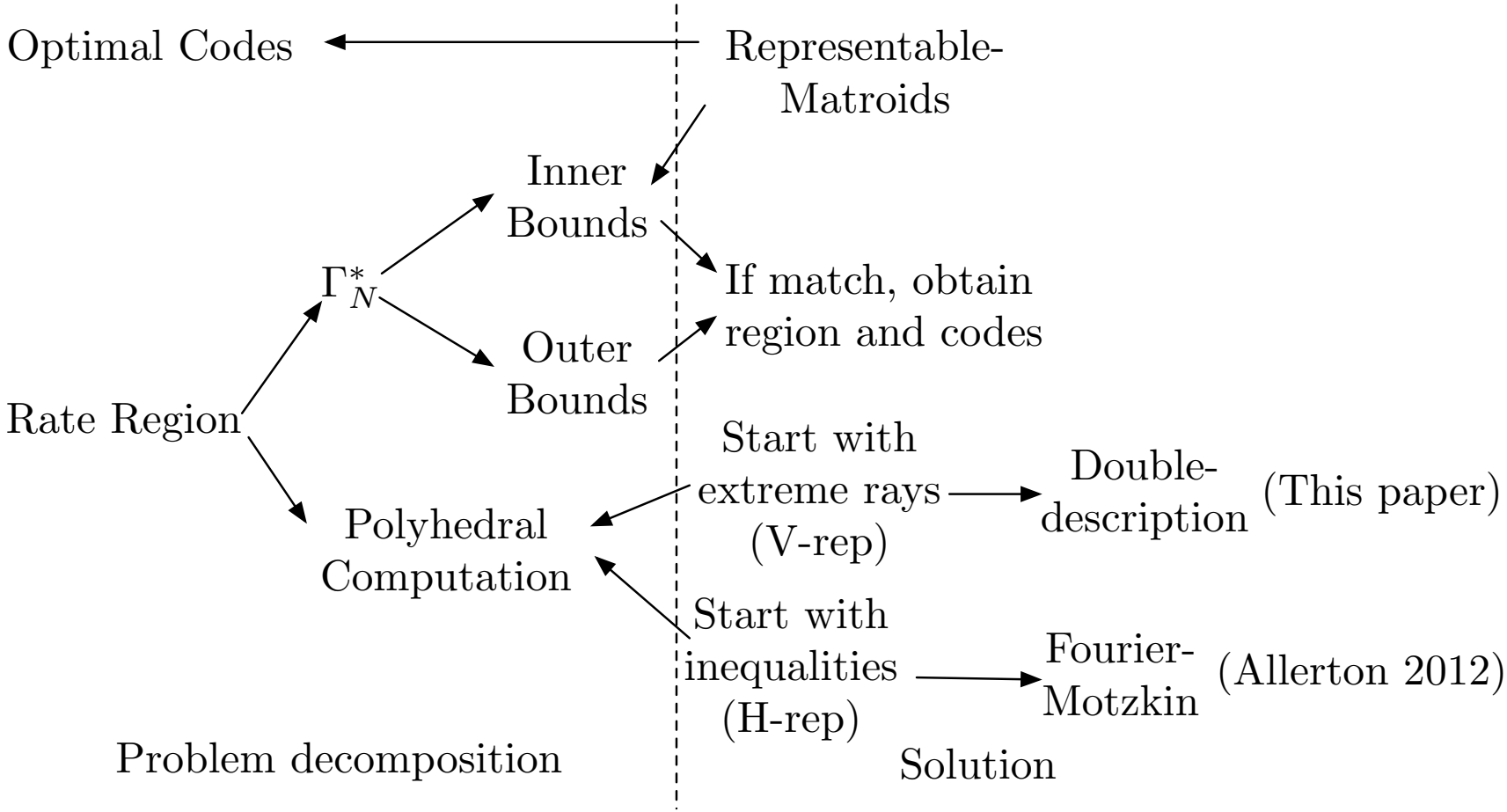
- Our problem can be decomposed as:



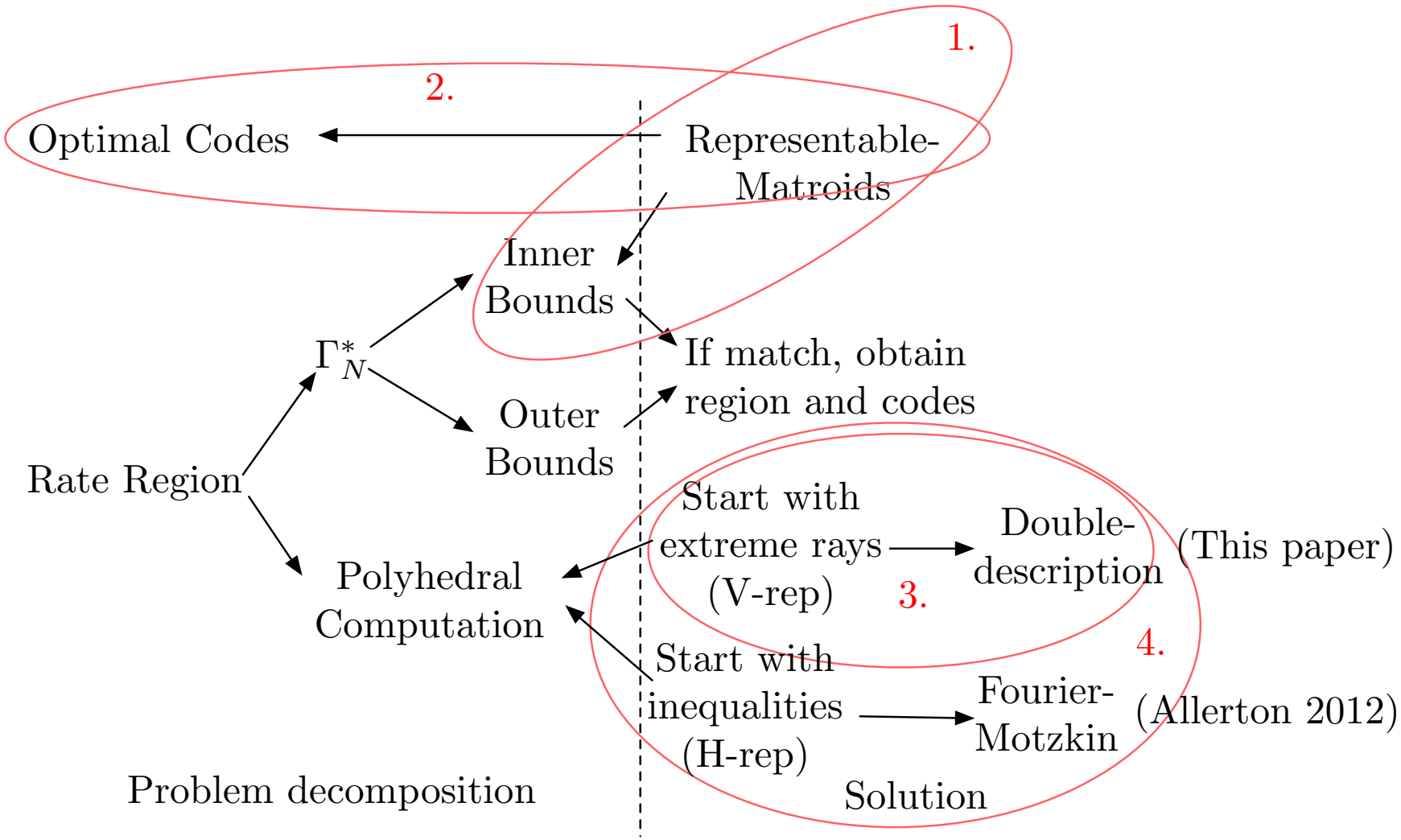
Problem decomposition

Proposed solution

- Proposed solutions

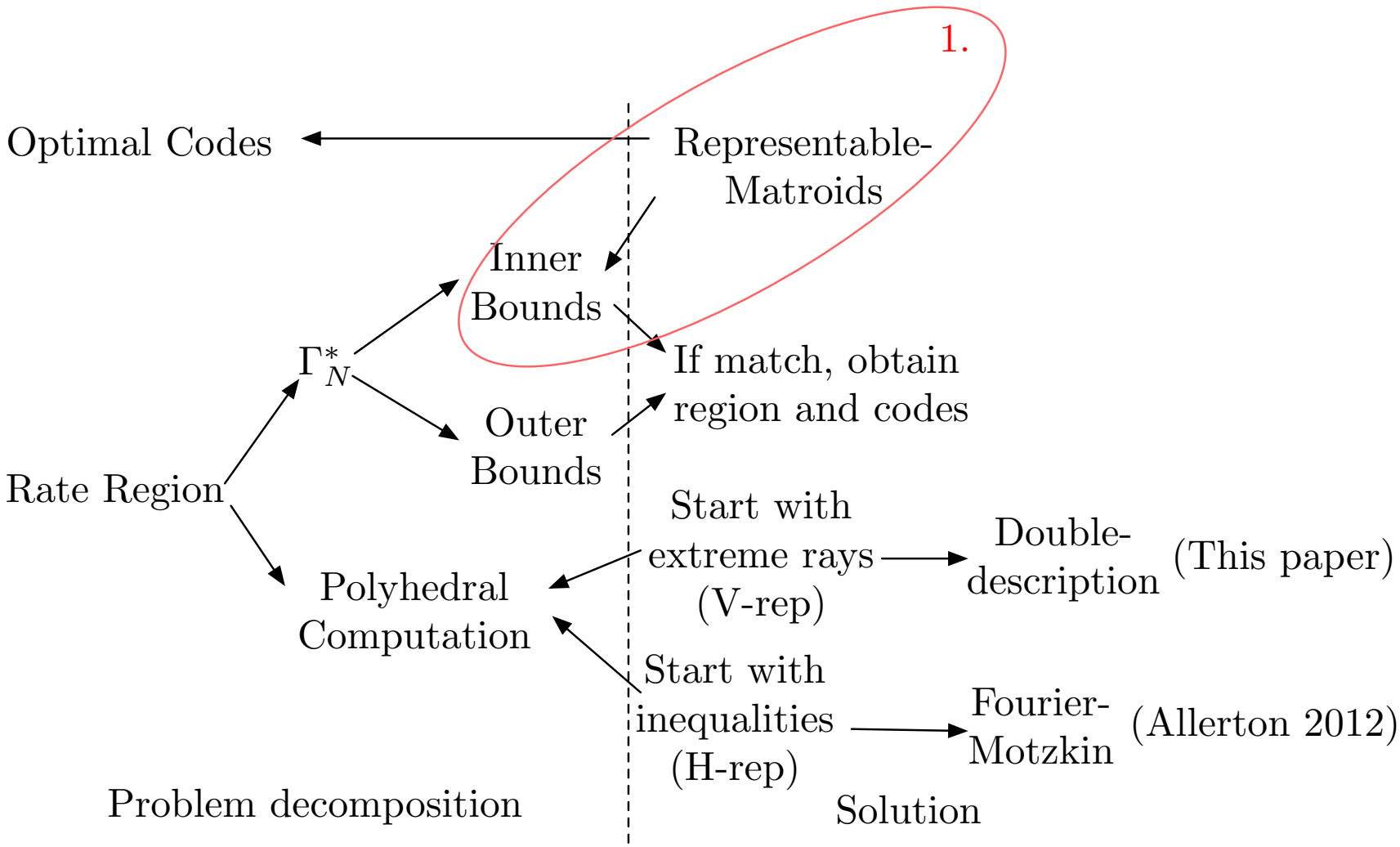


Outline


1.
2.
Optimal Codes
Representable-
Matroids
Inner
Bounds
Γ^*_N
If match, obtain
region and codes
Outer
Bounds
Rate Region
Start with
extreme rays
(V-rep)
Double-
description
(This paper)
3.
Polyhedral
Computation
4.
Start with
inequalities
(H-rep)
Fourier-
Motzkin
(Allerton 2012)
Problem decomposition
Solution

Outline

1. Bounding Γ_N^* from the Inside: Representable matroids

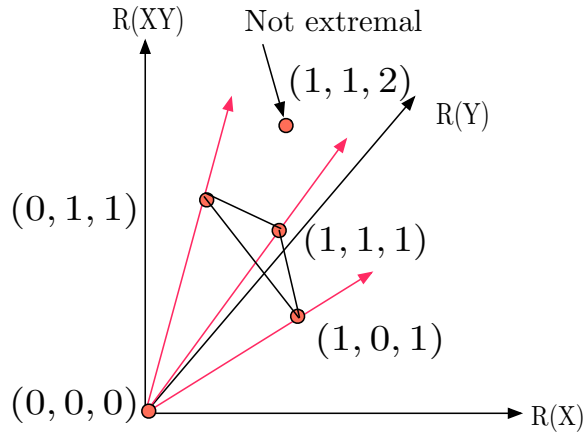


Bounding Γ_N^* from the Inside: Representable matroids

- **Matroids**: $r \in \Gamma_N \cap \mathbb{Z}^{2^N-1}, r(\mathcal{A}) \leq |\mathcal{A}|$
- $GF(q)$-**Representable Matroids**: $\exists \mathbf{A} \in GF(q)^{M \times N}$ s.t. $r(\mathcal{A}) = \text{rank}(\mathbf{A}_{:,\mathcal{A}})$
- Rank of rep. matroid = scaled entropic vector: $\mathbf{u} \sim \mathcal{U}(GF(q)^{1 \times M})$

$$\mathbf{X} = \mathbf{uA} \Rightarrow h_{\mathcal{A}} = r(\mathcal{A}) \log_2 q$$

- Inner bound of Γ_N^*: Γ_N^q conic hull of $GF(q)$-rep. matroids. (See right for $N = 2$)
- Representability $\Leftrightarrow$ No forbidden minors:
  - e.g.: $GF(2)$ rep. $\Leftrightarrow$ no $U(2,4)$ minor (Tutte)
- To obtain V-rep of Γ_N^q, see right.



List of non-isomorphic matroids

↓

Forbidden minor exclusion

↓

Add permutations

↓

Conic hull

↓

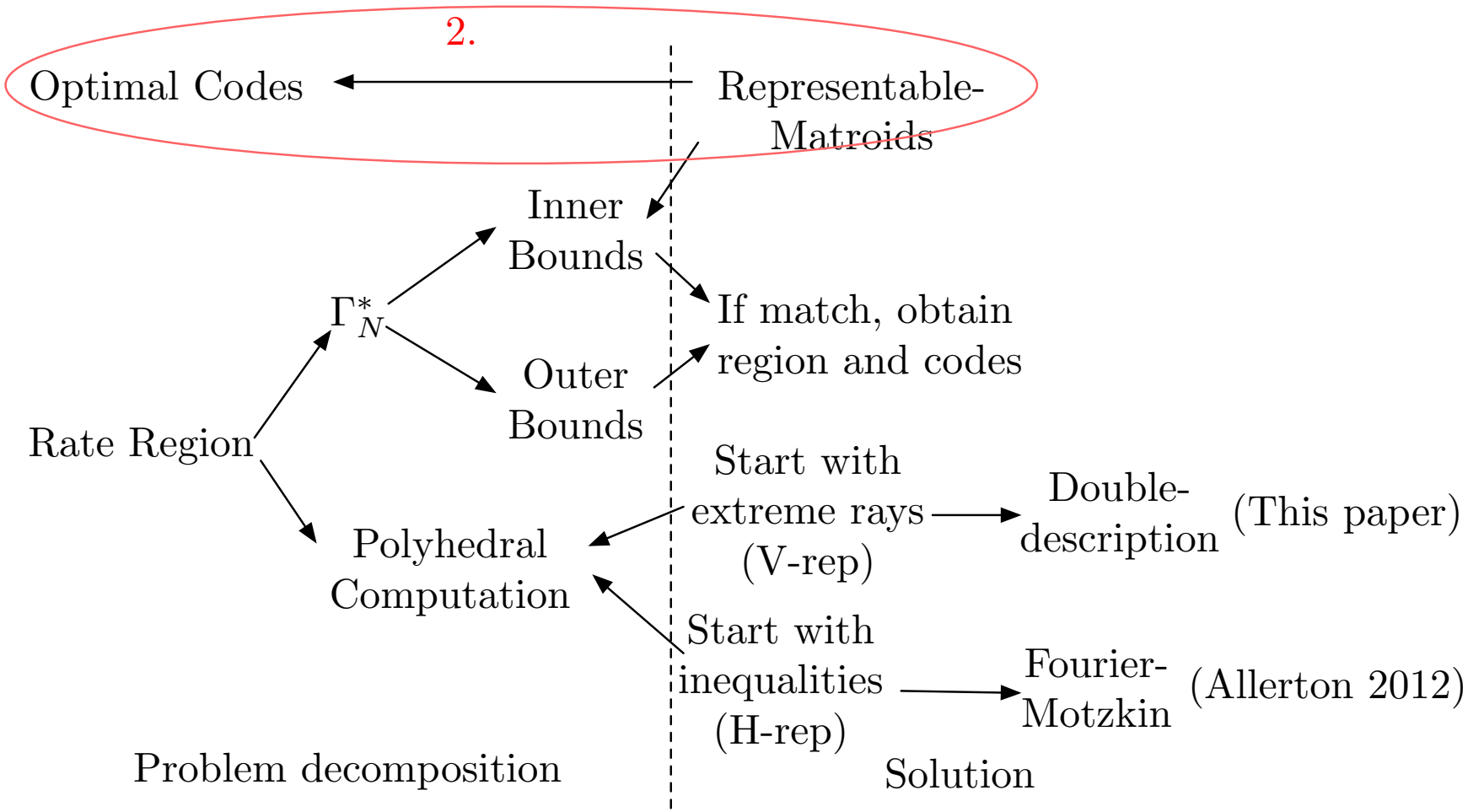
V-rep of rep. matroid inner bound

Bounding Γ^*_N from the Inside: Vector representable matroids

- Generalize: a (vector) random variable can be mapped with multiple columns in the rep. matrix to obtain a better inner bound for lower dimension.

Outline

2. Representable matroids determine linear codes: example

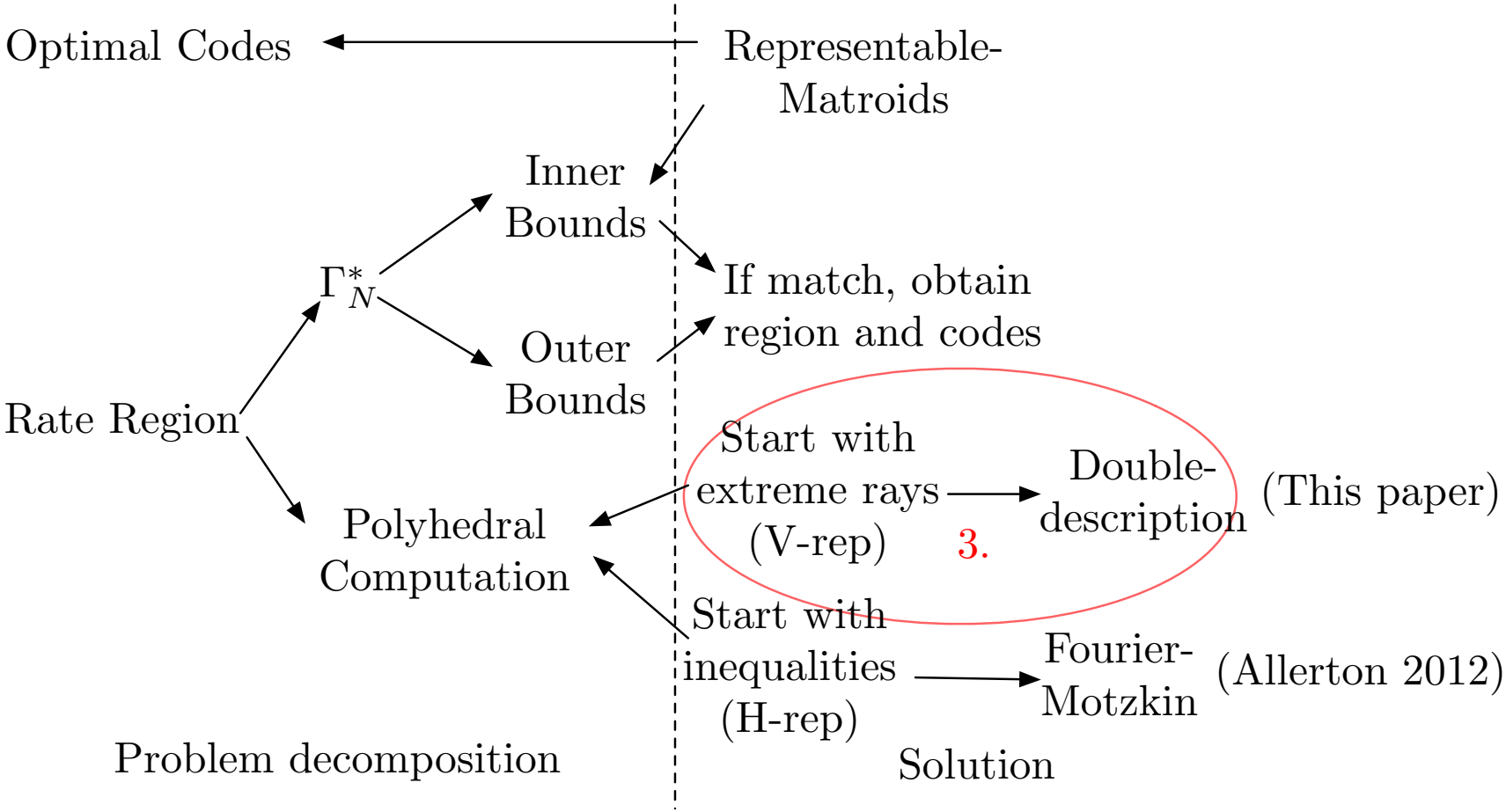


Representable matroids determine linear codes

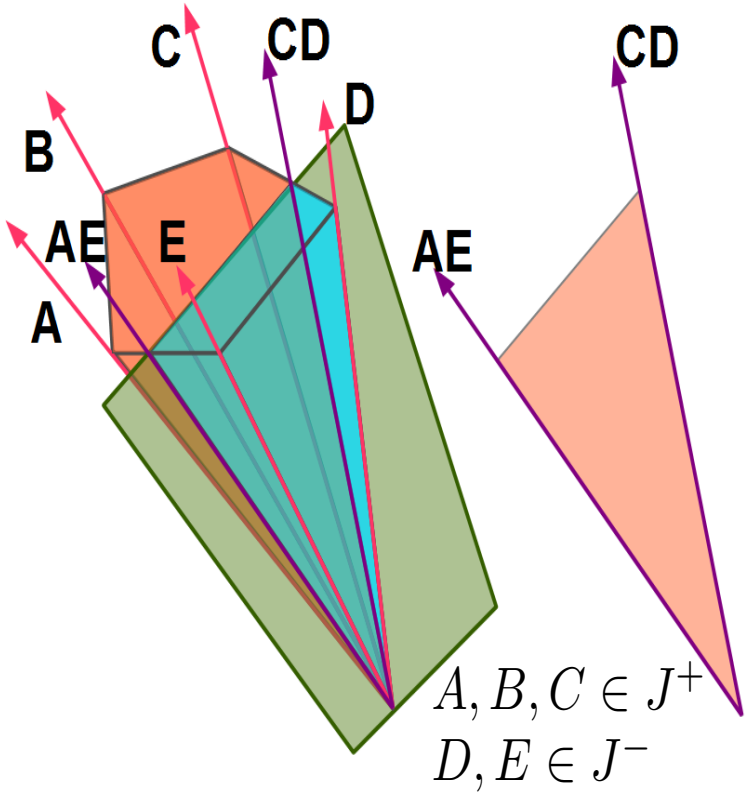
- A point in the rate region $\rightarrow$ expressed as conic combination of (projections of) rep. matroid ranks $\rightarrow$ each rank determines one code $\rightarrow$ put all solutions (with some repetitions if necessary) in a block diagonal matrix $\rightarrow$ final solution
- Interested? Details in poster and paper.

Outline

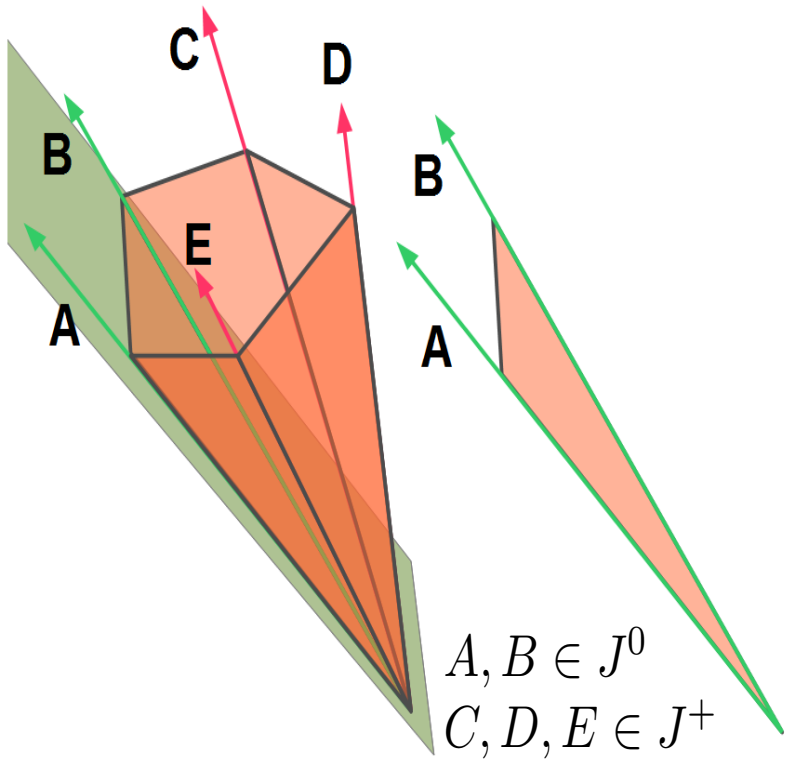
3. Use double description step to apply constraints



Use double description [3] step to apply constraints



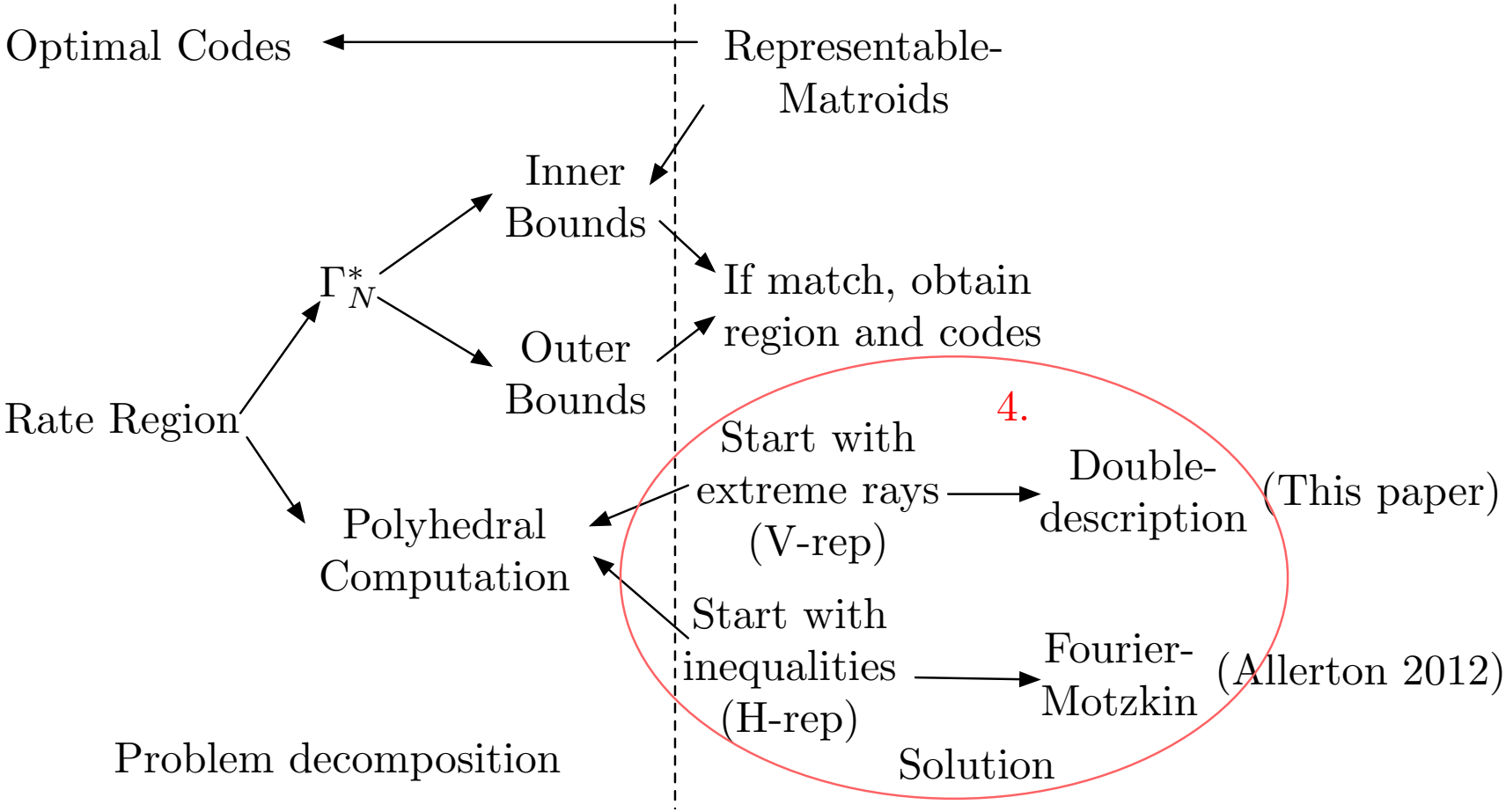
- Left: in general, there are new rays generated by J^+ and J^-.



- Right: special case, there are no new rays generated, since $J^- = \emptyset$.

Outline

4. Various computation techniques and new results



Computation techniques comparison

Recall the computation procedure: Initial polyhedra $\rightarrow \cap$ constraints $\rightarrow$ projections.

	Operation	Path I		Path II
Step 1	Initial polyhedral cone	H-rep: a list of m inequalities	Rep. Conversion	V-rep: a list of m extreme rays
Step 2	Add k equ./ineq. constraints	Append $\mathcal{O}(1)$	Rep. Conversion	k iterations of double description $\mathcal{O}(mk)$
Step 3	Project onto a subset of variables	FM eliminate n variables $\mathcal{O}(m^{2^n})$	Rep. Conversion	Delete unnecessary corrdinates $\mathcal{O}(1)$

Summary

1. Representable matroids $\rightarrow$ inner bound on Γ_N^*+ optimal codes
2. Double description method $\rightarrow$ a new efficient computation approach is proposed to calculate rate regions, especially for inner bound
3. For MDCS [4], exact rate regions for the 100 2-level and 3-level 3-encoder configurations are obtained, including some results not previously known
4. Binary linear codes suffice for the entire rate regions for all 100 cases
5. Computer-aided proofs are realistic in network information theory provided we exploit the special structure of the problems.

References

1. R. W. Yeung. Information Theory and Network Coding. Springer, 2008.
2. R. Dougherty, C. Freiling, and K. Zeger. Networks, matroids, and non- shannon information inequalities. Information Theory, IEEE Transac- tions on, 53(6):1949–1969, 2007.
3. K. Fukuda and A. Prodon. Double description method revisited. Combinatorics and computer science, pages 91–111, 1996.
4. K. P. Hau. Multilevel diversity coding with independent data streams. Master's thesis, The Chinese University of Hong Kong, June 1995.

Thank you

Thank you! Any questions?