A new computational approach for determining rate regions and optimal codes for coded networks

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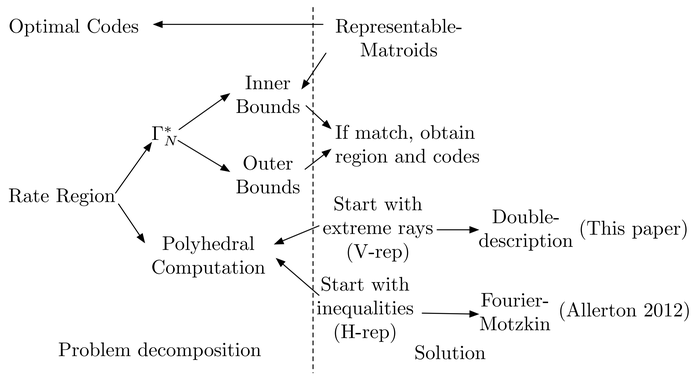
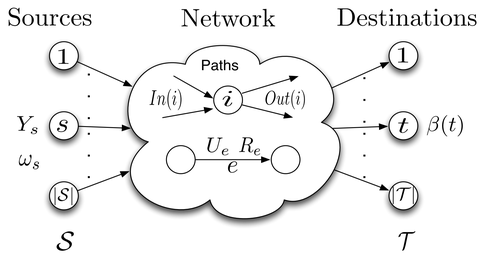
1. Problem

- What algorithms are most effective for computing rate regions that are achievable by linear codes for multisource multicast netwroks?



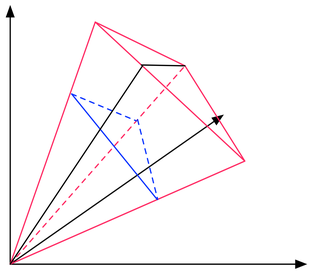


Network constraints:

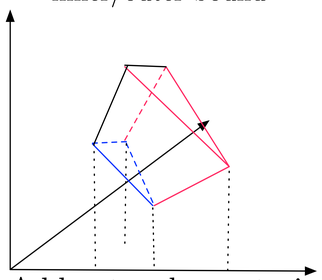
$$\mathcal{L}_1 = \{\mathbf{h} \in \Gamma_N^* : h_{Y_s} \geq \omega_s\}$$
$$\mathcal{L}_2 = \{\mathbf{h} \in \Gamma_N^* : h_{Y_\mathcal{S}} = \Sigma_{s\in\mathcal{S}} h_{Y_s}\}$$
$$\mathcal{L}_3 = \{\mathbf{h} \in \Gamma_N^* : h_{X_{\mathrm{Out}(k)}|Y_s} = 0\}$$
$$\mathcal{L}_4 = \{\mathbf{h} \in \Gamma_N^* : h_{X_{\mathrm{Out}(i)}|X_{\mathrm{In}(i)}} = 0\}$$
$$\mathcal{L}_5 = \{\mathbf{h} \in \Gamma_N^* : h_{Y_{\beta(t)}|U_{\mathrm{In}(t)}} = 0\}$$

$$\mathcal{R} = \mathrm{Ex}(\mathrm{proj}_{U_e}(\overline{\mathrm{con}(\Gamma_N^* \cap \mathcal{L}_{234})} \cap \mathcal{L}_{15})) \text{ [1]}$$

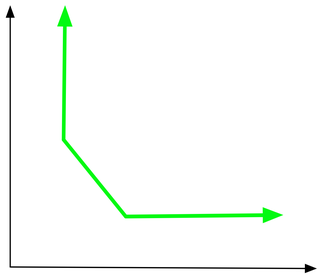
2. Computation demonstration



Substitute Γ_N^* with polyhedral inner/outer bound



Add network constraints and fix source entropies

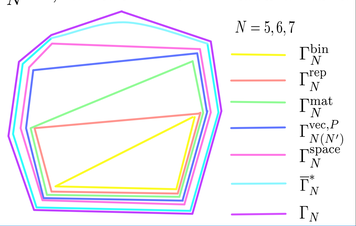


Project onto rate variables to obtain rate region

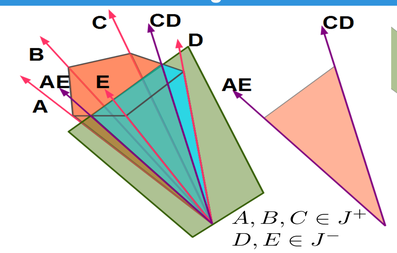
3. Bounds on region of entropic vectors

- Rank of representable matroid = scaled entropic vector: $\mathbf{u} \sim \mathcal{U}(GF(q)^{1\times M})$, create $\mathbf{X} = \mathbf{uA} \Rightarrow h_\mathcal{A} = r(\mathcal{A}) \log_2 q, \mathcal{A} \subseteq \mathbf{X}$, where $\mathbf{A}$ is the matrix representation and $r(\mathcal{A})$ is the linear rank of associated columns in $\mathbf{A}$.
- Inner bound of Γ_N^*: Γ_N^q, conic hull of $GF(q)$-rep. matroids.
- Projecting rep. matroid: $\Gamma_{N(N')}^{\mathrm{vec},P} \Pi_N^P \Gamma_{N'}^q \subseteq \Gamma_N^{\mathrm{space}}$, where $N' > N$ and P is a partition of $\{1,\ldots,N'\}$ with $|P| = N$.

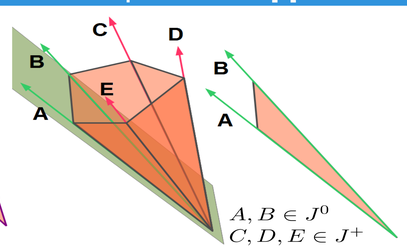
Γ_N^{bin}: binary rep. matroid inner bound;
Γ_N^{rep}: rep. matroid inner bound;
Γ_N^{mat}: matroid inner bound;
$\Gamma_{N(N',P)}^{\mathrm{vec},q}$: vector $GF(q)$-representable inner bound;
$\Gamma_N^{\mathrm{space}}$: subspace arrangements inner bound;
$\bar{\Gamma}_N^*$: closure of Γ_N^*;
Γ_N: Shannon outer bound.



4. Inserting constraints: double description method [3]



Applying a generic equality constraint to a polyhedral cone: $\mathcal{O}(n)$

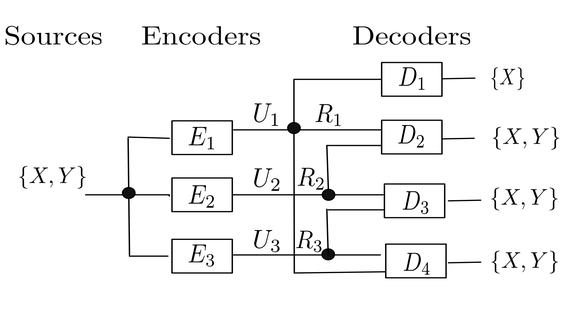


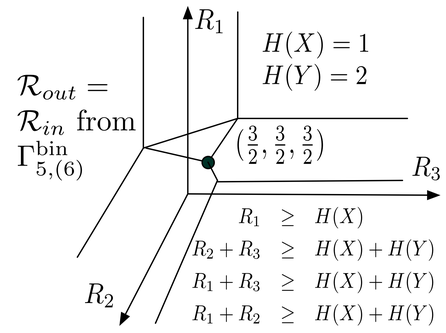
Applying network constraint to any polyhedral bound $\subseteq \Gamma_N$: $\mathcal{O}(n)$

5. Proposed computational approach

	Operation	Path I		Path II
Step 1	Initial polyhedral cone	H-rep: a list of m inequalities	Rep. Conversion	V-rep: a list of m extreme rays
Step 2	Add k equ./ineq. constraints	Append $\mathcal{O}(1)$	Rep. Conversion	k iterations of double description $\mathcal{O}(mk)$
Step 3	Project onto a subset of variables	FM eliminate n variables $\mathcal{O}(m^{2^n})$	Rep. Conversion	Delete unnecessary corrdinates $\mathcal{O}(1)$

6. Computed rate region of an MDCS [4] example

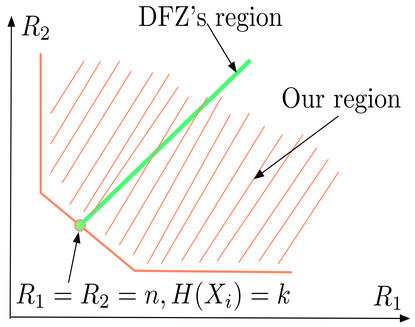




7. General code construction to achieve any point in the rate region

1. Express given point in the rate region as conic combination of (projected) matroid ranks
   - Consider $(H(X), H(Y), H(U_1), H(U_2), H(U_3))$ $(1, 2, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}) = \frac{1}{2}(1,1,1,1,1) + \frac{1}{2}(1,3,2,2,2)$.
2. Approximate coefficients using rationals, block length of sources = LCM of the denominators
   - Coefficients are $\frac{1}{2}, \frac{1}{2}$. Block length = LCM of denominators = 2.
3. LCM $\times$ each rational coefficient = repetition times of associated basic solution
   - Repeat the solution $\mathbb{C}_1$ associated with $(1,1,1,1,1)$ once and solution $\mathbb{C}_2$ associated with $(1,3,2,2,2)$ once.
4. Find partition of source digits and corresponding basic solutions
5. Arrange basic solutions with proper repetition times in a block diagonal matrix
   - Source vector $v = [X_1, X_2, Y_1, Y_2, Y_3, Y_4]$. One partition of v is $\{\{X_1, Y_3\}, \{X_2 Y_1 Y_2 Y_4\}\}$ with first element mapped to $\mathbb{C}_1$ and second mapped to $\mathbb{C}_2$, where we choose $\mathbb{C}_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $\mathbb{C}_2 = \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right]$.
   - Final solution $\mathbb{C} = \begin{bmatrix} \mathbb{C}_1 & \mathbb{O} \\ \mathbb{O} & \mathbb{C}_2 \end{bmatrix}$.
6. Rearrange the order of source vector and columns of the solution
   - Rearrange columns of $\mathbb{C}$ to get $U_1 U_2 U_3 = v \times \left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$.

8. Results



Comparison with DFZ's work in [2]

1. Representable matroids give both inner bound on Γ_N^* and optimal codes to achieve the rate region obtained from this inner bound
2. A linear code construction from representations of matroids
3. Inspired by double description method, a new efficient computation approach is proposed to calculate rate regions, especially for inner bound
4. For MDCS [4], rate regions for the 100 2-level and 3-level 3-encoder configurations are obtained, including some results not previously known
5. Binary linear codes suffice for the entire rate regions for all 100 cases
6. Computer-aided proofs are realistic in network information theory provided we exploit the special structure of the problems.

Selected references

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3. K. Fukuda and A. Prodon. Double description method revisited. Combinatorics and computer science, pages 91–111, 1996.
4. K. P. Hau. Multilevel diversity coding with independent data streams. Master's thesis, The Chinese University of Hong Kong, June 1995.

Software information

Please contact us for the software or visit our webpage: http://www.ece.drexel.edu/walsh/aspitrg/software.html

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