Network Fundamentals

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May 14, 2012

Outline

Mappings

Point-to-point communication network—Directed Graph $G(V, E)$:

- V: set of nodes in networks
- E: set of channels
- Assume: $|E| < \infty$
- Unique source node: s
- Inflow and outflow: $In(i) \& Out(i)$
- Capacity (Rate Constraints): $\mathbf{R} = (R_e, e \in E)$

Definitions

Definition

Routing (Store-Forward): information replicated and forwarded at certain intermediate nodes.
Network Coding: information processed at intermediate nodes. Output is a function of input.

Wireless/Satallite Communication

Properties:

1. All the output channels have the same capacity;
2. The same symbol is sent on each of the output channel.

Examples

Two nodes exchange information.

Fact

Network Coding is not necessary at a node if the node has only one input channel and the capacity of each output chanel is the same as the input channel.

Butterfly Network

Another example network to show advantage of network coding:

Example

Two source nodes and two sink nodes butterfly network with capacity of 1 on each edge.

Source Separation

Definition

Source Separation: Coding independent information sources separately.

Compressing two independent data: $H(X, Y) = H(X) + H(Y)$

Note:

Source separation does not guarantee optimality in multiple information sources case, the problem cannot always be decomposed into a number of single-source problems. (Butterfly Network)

Max-flow and Min-cut

Flow $\mathbf{F} = (F_e : e \in E)$:

- $0 \leq F_e \leq R_e$; $F_+(i) = \sum_{e \in In(i)} F_e = F_-(i) = \sum_{e \in Out(i)} F_e$

Definition

Max Flow: $\max \mathbf{F}$ w.r.t $\mathbf{R}$.
Cut: a cut between node s and node t is a subset U of V s.t $s \in U$ and $t \notin U$.
Cut edges:
$E_U = \{e \in E : e \in Out(i) \cap In(j) \ for\ some\ i \in U \& j \notin U\}$.
Cut capacity: $\sum_{e \in E_U} R_e$; minimum is called Min-cut.

Max-flow Min-cut Theorem

Theorem

Let G be a graph with source node s, sink node t and rate constraints $\mathbf{R}$. Then the value of a max-flow from node s to node t is equal to the capacity of a min-cut between the two nodes.

Example

Traffic flow between two points in a city. (Ford Fulkerson Algorithm to get the max-flow and min-cut)

Max-flow Bound Theorem

Theorem

For a graph G with rate constraints $\mathbf{R}$, if ω is achievable, then

$$\omega \leq \min_{l} maxflow(t_l).$$

Note: Only converse proof at this moment.

General Class of Network Codes

Definitions

A general class of network codes $(n,(\eta_e : e \in E),\tau)$: K transactions mapping; Index sets; Encoding functions; Decoding Functions.

Achievability of information rate: For a graph G with rate constraints $\mathbf{R}$, an information rate $\omega \geq 0$ is asymptotically achievable if for any $\epsilon > 0$, there exists for sufficiently large n an $(n,(\eta_e : e \in E),\tau)$ network code on G such that

$$n^{-1} \log_2 \eta_e \leq R_e + \epsilon$$

for all $e \in E$ and

$$\tau \geq \omega - \epsilon.$$

Sketch of Proof of Max-flow Bound

Proof.

Converse proof sketch:

Assume code exists and ω is achievable; consider any sink t_l and cut U;

Define all information know by node j: $w_j(x)$; $w_{t_l}(x)$ is a function of $\tilde{f}_k(x), k \in \cup_{e \in E_U} T_e$;

x can be determined at node t_l, we have $H(X) \leq \sum_{e \in E_U} \log_2 \eta_e$;

Thus $\omega - \epsilon \leq \tau \leq \sum_{e \in E_U} R_e + |E_U|\epsilon$

Minimization of right hand and ω hold for all l, so $w \leq \min_l \text{ maxflow}(t_l)$. □

Linear Network Coding

Definition

Linear Network Coding: Linear function (matrix) map between input and output. Local description and Global description.

Example

Random Linear Network Coding: $E_i = C_1^i B_1 + \cdots + C_k^i B_k$.
Applications in Peer-to-Peer networks.

Q&A

Thank you! Next Step:

Review of matroid theory;
Relations between matroid and network, matorid and coding.

"Information Theory and Network Coding", Raymond Yeung, Chap 17,18&19

"Networks,Matroids, and Non-Shannon Information Inequalities", R. Dougherty, C. Freiling and K. Zeger, Trans. Information Theory, Vol. 53, No. 6 June 2007