An equivalence between network coding and index coding

—A paper by Effros, Rouayheb, Langberg [1]

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Outline

1. Network coding (NC) intro
2. Index coding (IC) intro
3. Mapping network coding to index coding
4. Equivalence proofs

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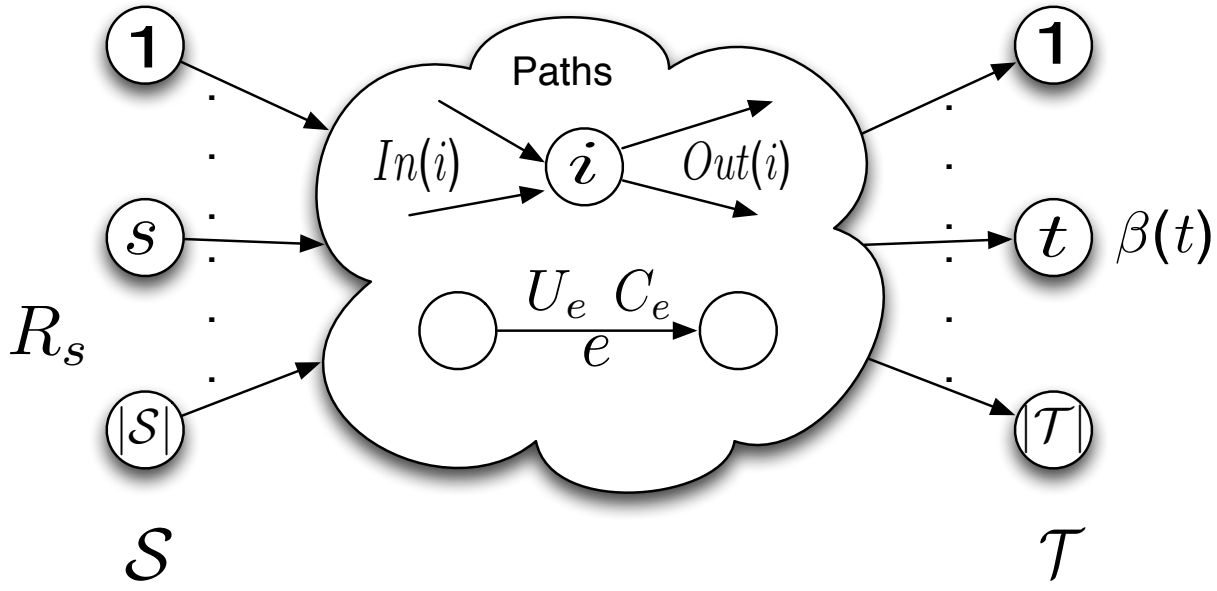
Network coding (NC) model

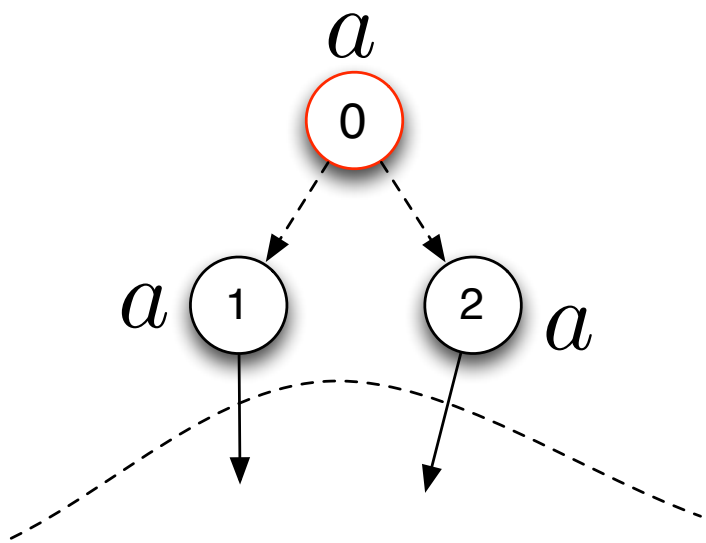


- Directed acyclic network $G = (V, E)$
- Source s with rate R_s, edge e with capacity C_e, sink node t requires $\beta(t)$
- Independent sources
- Variables on out-going edges are function of variables on in-coming edges
- Goal: transmit information from source to sink via network codes

Any network has associated multiple unicast network

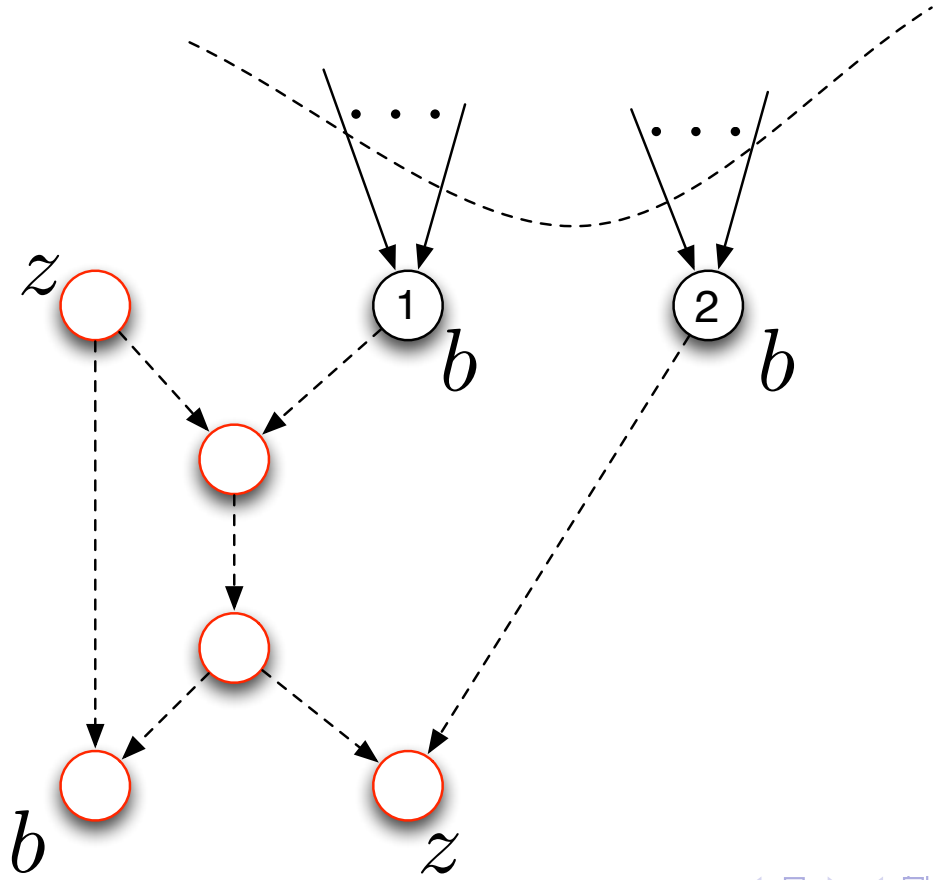
In [2], it is shown that for any multi-multicast network, there exists an associated multiple unicast network such that one is solvable if and only if the other is solvable.

- Construction
  - If a source message a is transmitted by more than one sources, add a new super node to transmit a to those sources.



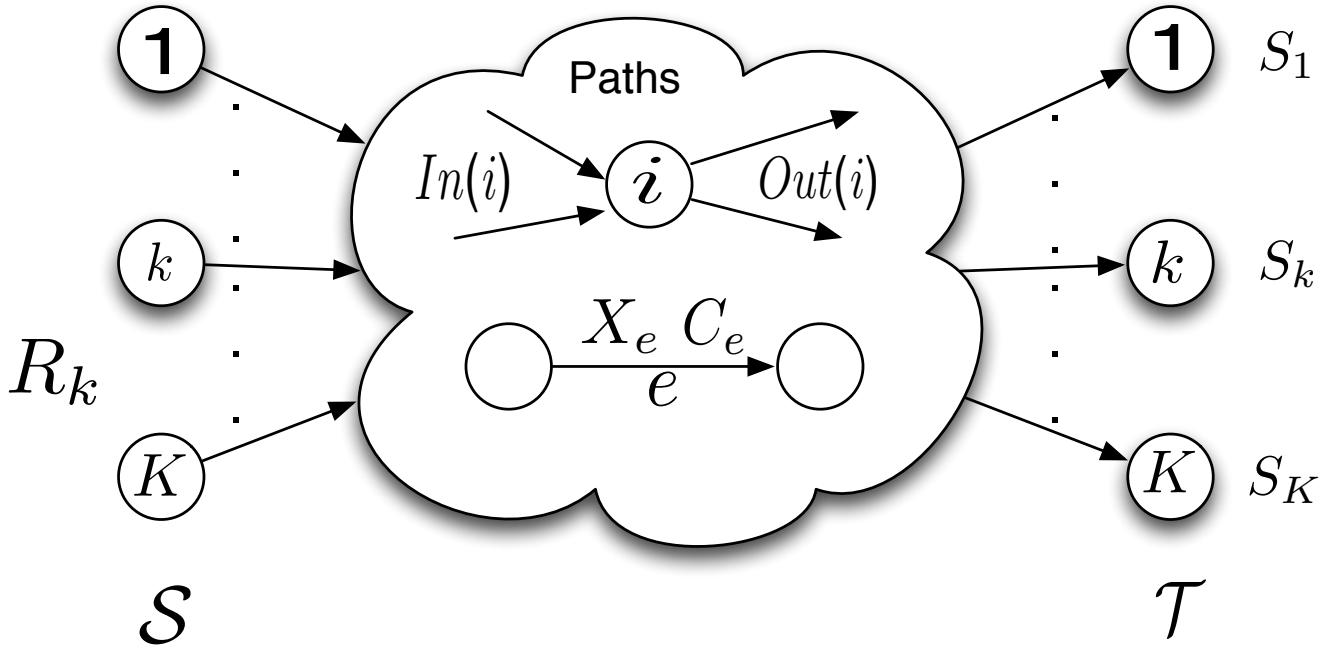
Any network has associated multiple unicast network

- Construction (cont.)
  - If a source message b is required by more than one sink nodes, add 5 new nodes to let b is required by only one.
  - Iterations may be needed for the cases where more than 2 sinks requiring same message



Multi-multicast to multi-unicast

Thus, W.O.L.G, we can only consider the multi-unicast networks.

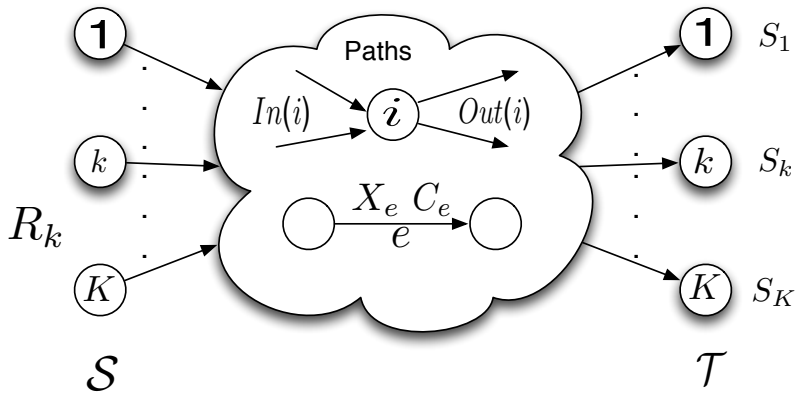


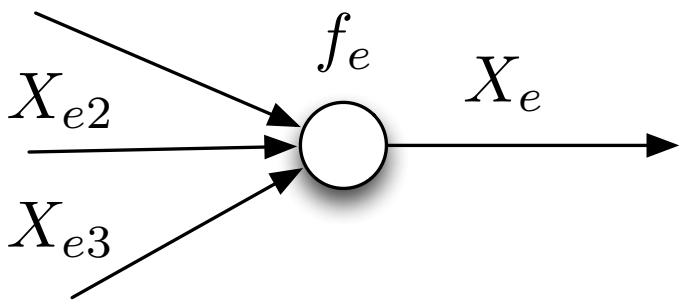
- Source with index k has rate R_k, edge e with capacity C_e, sink node with index k requires Y_k
- Independent sources
- Variables on out-going edges are function of variables on in-coming edges
- Goal: K source/sink pairs wish to communicate

Achievable rate tuple in multi-unicast NC

Rate tuple $R = (R_1, \ldots, R_K)$ is *achievable with block length n (or (R, n)-feasible) if* $\exists \{S_k\}, \{X_e\}$*:*

- *Rate: Source Y_k independent and uniform over* $[2^{R_k n}]$
- *Edge capacity: Edge X_e with support* $[2^{C_e n}]$
- *Functionality: For edge e, is function of incoming edges,* $X_e = f_e(X_{e1}, X_{e2}, \ldots)$
- *Decoding: Sink t_k, requires S_k*





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4. Equivalence proofs

Index coding (IC) motivation

- A set of packets need to be delivered to a set of users
- Users may have some information already (side information)
- Broadcast channel is used by the server
- Goal: minimum transmission times

Wants: M_1
Has: M_2, M_4

Wants: M_2
Has: M_1, M_3

M_1, M_2, M_3, M_4

Wants: M_3
Has: M_2, M_4

Wants: M_4
Has: M_1

Index coding motivation

- Traditional way: send M_1, M_2, M_3, M_4 respectively
- Coding: send $M_2 + M_3, M_1 + M_4$ respectively
- Side information is utilized
- Transmission times are reduced

Wants: M_1
Has: M_2, M_4

Wants: M_2
Has: M_1, M_3

M_1, M_2, M_3, M_4

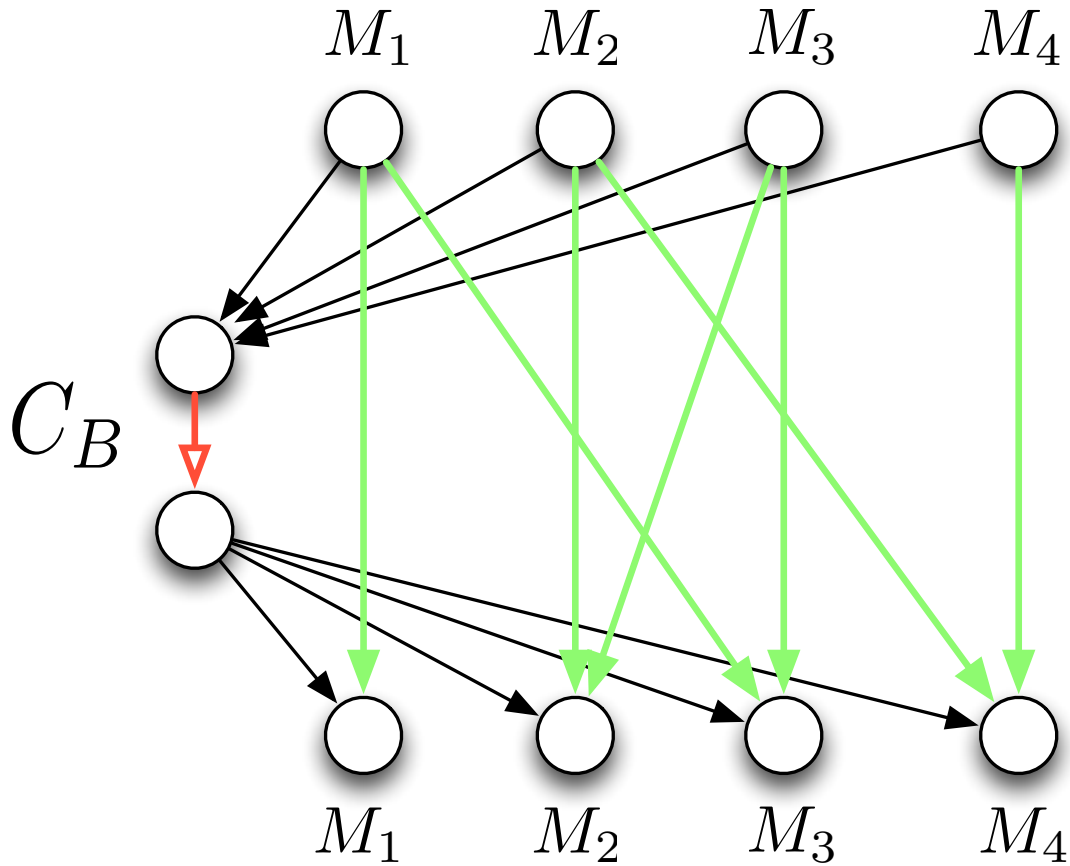
Wants: M_3
Has: M_2, M_4

Wants: M_4
Has: M_1

Index coding model

The above example can be represented as a typical index coding model:

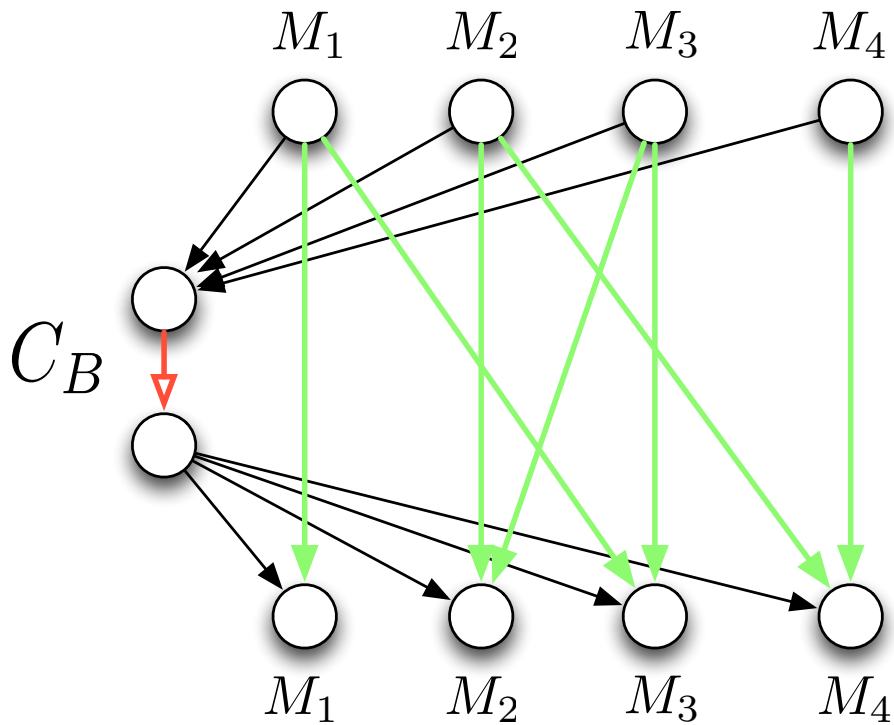
- Sources, sinks with requirements and side information, broadcasting node with coding
- Broadcast link has capacity C_B, others unlimited



Achievable rate in IC

Rate tuple $R = (R_1, \ldots, R_K)$ is *achievable with block length n (or (R, n)-feasible) if* $\exists \{M_k\}, \{X_B\}$*:*

- *Rate: Source M_k independent and uniform over* $[2^{R_k n}]$
  - *Edge capacity: Edge X_B with support* $[2^{C_B n}]$
  - *Functionality: For edge B, is function of incoming edges,* $X_B = f_B(M_1, M_2, \ldots)$
  - *Decoding: Sink t_k, requires M_k*



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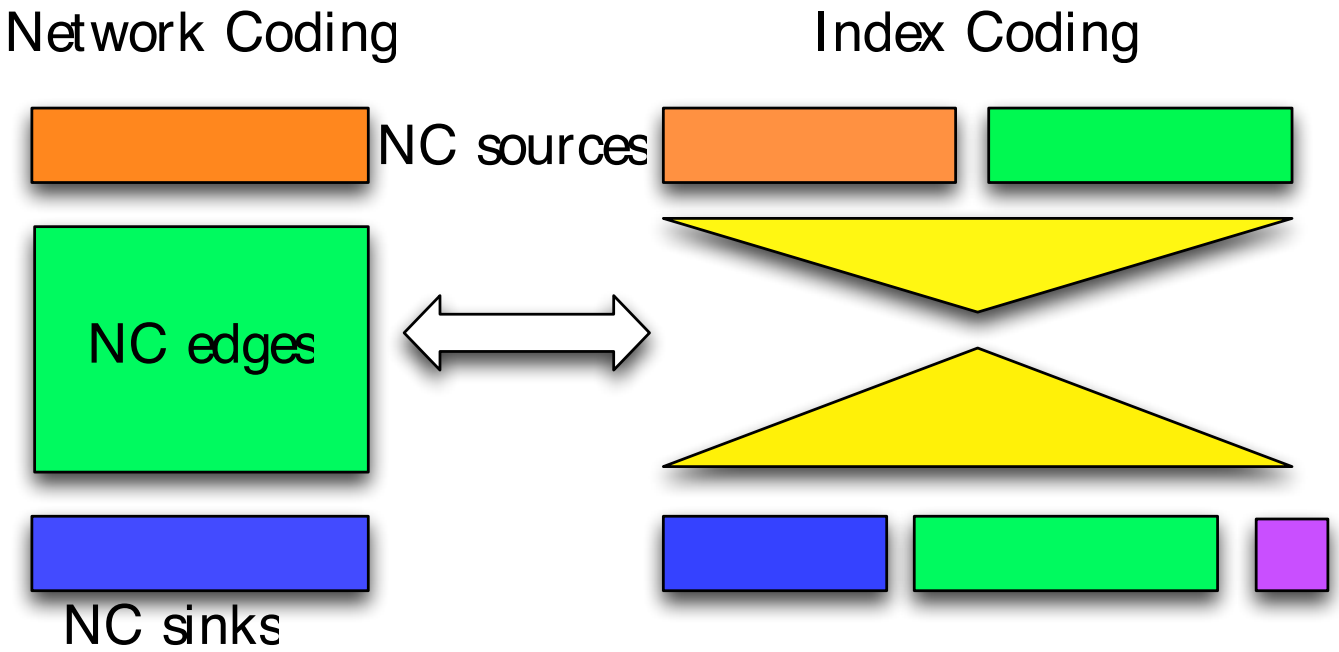
Reduce NC to IC

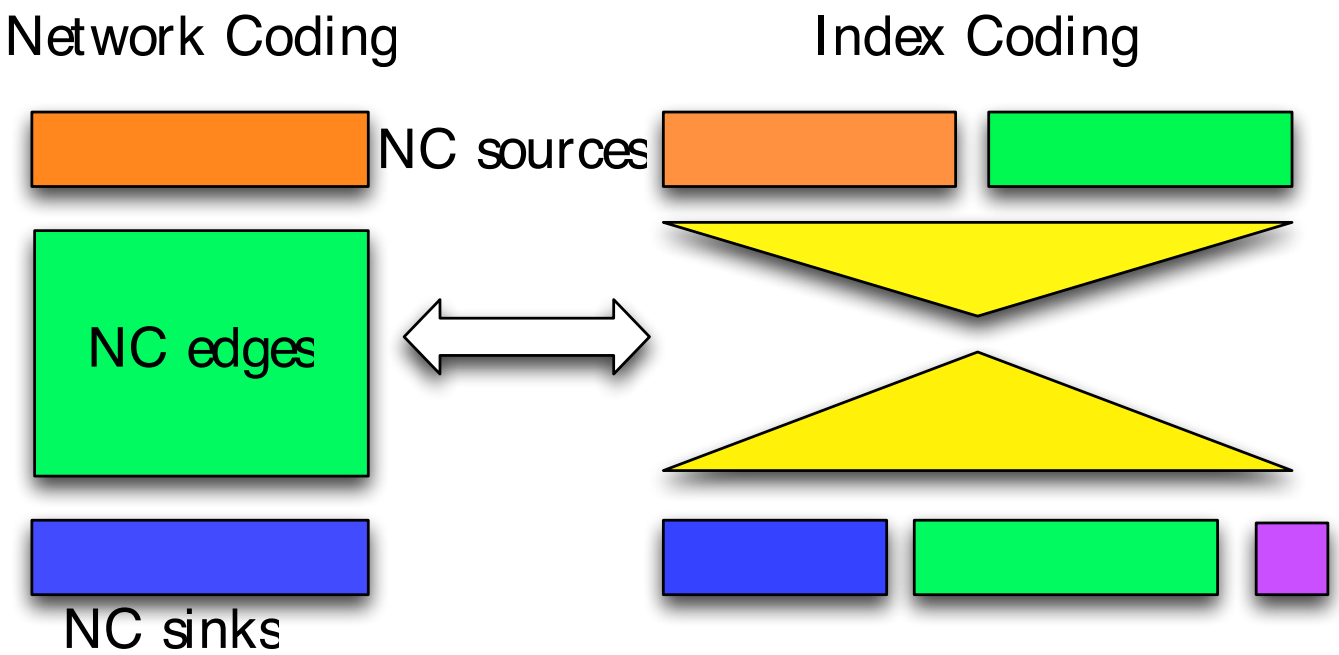


Instances in IC:

- Sources: NC sources and NC edges
- Sinks: NC sinks, NC edges and a special sink representing all edges are function of all NC sources
- For NC edge e, sink t_e in IC wants X_e and has as side information all IC sources incoming to e in NC

IC encodes topology of NC in its sinks!

Reduce NC to IC



Reduction	NC	IC
Sources	$S_1, \ldots S_K$	$\{S'_k\}, \{S'_e\}$
Sinks	$t_1, \ldots t_K$	$\{t'_k\}, \{t'_e\}, t_{all}$
Capacity	C_e	$C_B = \sum C_e$
Rate	$R_1, \ldots R_K$	$R'_k = R_k, R'_e = C_e$

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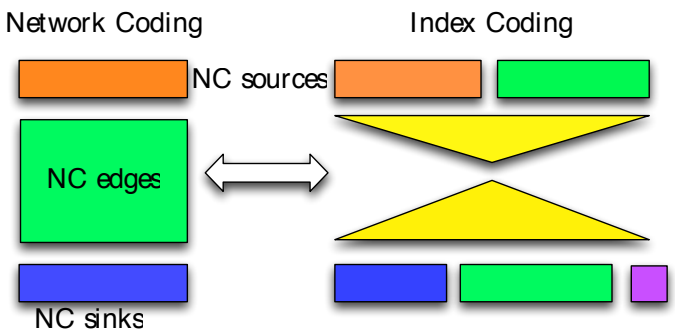
Equivalence between NC and IC

Theorem

For any NC with rate tuple R, one can construct an IC with rate R' such that for any block length n: NC is (R, n)-feasible iff IC is (R', n)-feasible.



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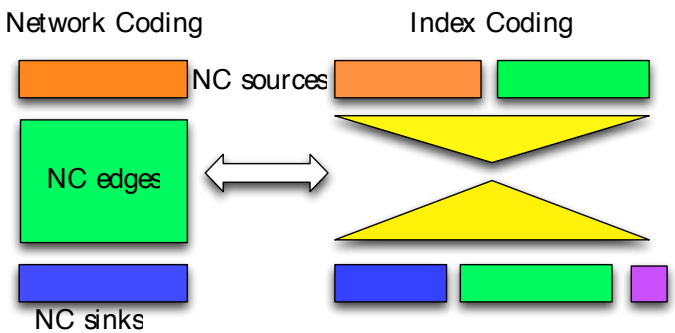
Proof: NC→IC

Basic idea: simulate NC solution in IC decoding.

- Each edge e in NC has a global function $F_e(S_1, \ldots S_K) = X_e$
- Recall X_e has support $[2^{C_e n}]$, $C_B = \sum C_e$ and X_B has support $[2^{C_B n}]$
- X_B can be divided into chunks to associated with e
- Let $X_B(e) = S'_e + F_e(S'_1, \ldots, S'_K)$



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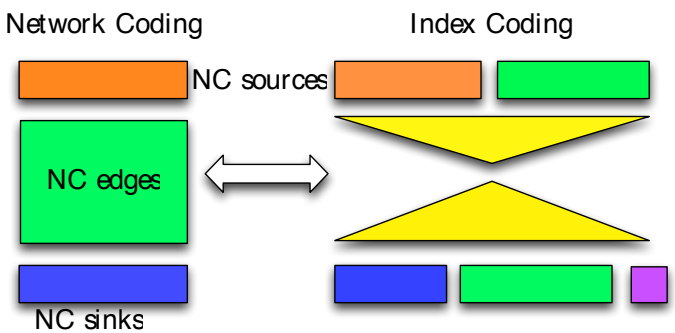
Proof: NC→IC (cont.)

- Decoding:
  - consider t'_e: wants S'_e and has $\{S'_a\}$ for edges $a \in In(e)$, also receives X_B
  - for each a, compute $X_B(a) - S'_a = F_a(S'_1, \ldots, S'_K)$
  - use local function f_e to compute $f_e(F_{a1}(S'_1, \ldots, S'_K) \ldots) = F_e(S'_1, \ldots, S'_K)$
  - Then decode $S'_e = X_B(e) - F_e(S'_1, \ldots, S'_K)$
  - Same process for other sinks



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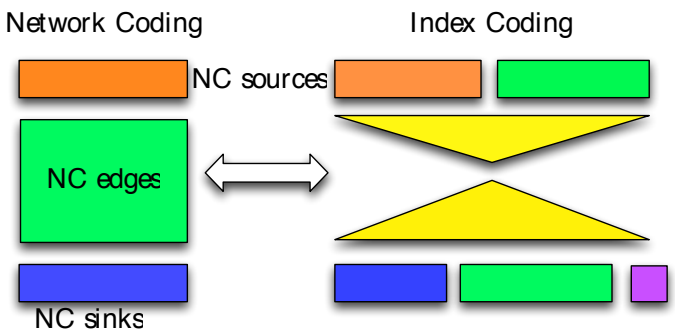
Capacity	C_e	$C_B = \sum C_e$
Rate	$R_1, \ldots R_K$	$R'_k = R_k, R'_e = C_e$

Proof: IC→NC

- Encoding function $f_B(\{S'_k\}, \{S'_e\})$
- t_{all} requires all S'_E decodable given any S'_K and X_B
- Crucial property: for any S'_K, $\exists S'_E$ such that $X_B = f_B(S'_K, S'_E) = \sigma$
- To define NC we need $X_B = \sigma$



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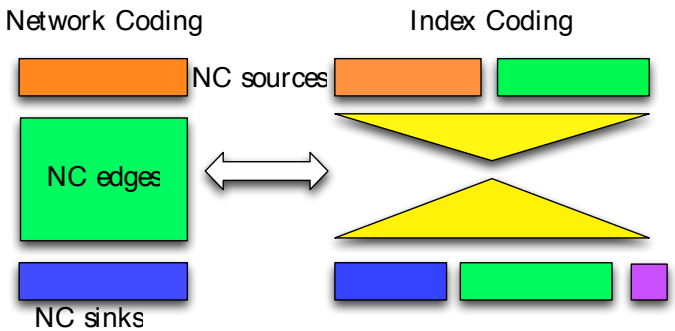
Proof: IC→NC (cont.)

- Consider edge e in NC, we need local encoding function $f_e(X_{a1}, \ldots) = X_e$
- We have decoding function for t'_e in IC: $g'_e(S'_{a1}, \ldots, X_B) = S'_e$
- Let $f_e(X_{a1}, \ldots) = g'_e(X_{a1}, \ldots, \sigma)$
- Similarly for sink k in NC, we need decoding function $g_k(X_{a1}, \ldots) = S_k$
- We have $g'_k(S_{a1}, \ldots f_B) = S'_k$, and then let $g_k(X_{a1}, \ldots) = g'_k(X_{a1}, \ldots, \sigma)$
- Let $S_l = S'_l, X_e = S'_e$, can show the correctness of enc. (dec.)



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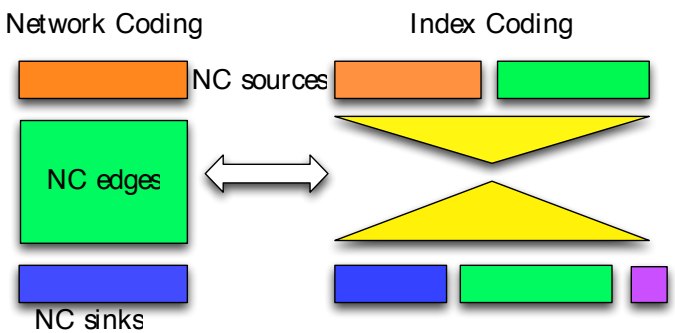
Sinks	$t_1, \ldots t_K$	$\{t'_k\}, \{t_e{}'\}, t_{all}$
Capacity	C_e	$C_B = \sum C_e$
Rate	$R_1, \ldots R_K$	$R'_k = R_k, R'_e = C_e$

Proof: IC→NC (cont.)

- Encoding:
  -

$$
\begin{aligned}
X_e &= f_e(X_{a1}, \ldots) \\
&= g'_e(X_{a1}, \ldots . \sigma) \\
&= g'_e(X_{a1}, \ldots . f_B(S'_I, S'_E)) \\
&= g'_e(S'_{a1}, \ldots . f_B(S'_I, S'_E)) \\
&= S'_e
\end{aligned}
$$



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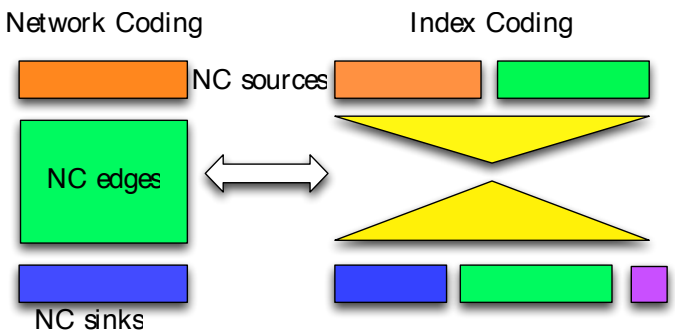
Proof: IC→NC (cont.)

- Decoding:
  -

$$\begin{aligned}\hat{S}_i &= g_i(X_{a1}, \ldots) \\ &= g_i'(X_{a1}, \ldots . \sigma) \\ &= g_i'(S_{a1}', \ldots . f_B(S_I', S_E')) \\ &= S_i' \\ &= S_i\end{aligned}$$



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Sinks	$t_1, \ldots t_K$	$\{t_k'\}, \{t_e{}'\}, t_{all}$
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Reference

1. Michelle Effros and Salim Y. El Rouayheb and Michael Langberg, "An Equivalence between Network Coding and Index Coding", arXive: abs/1211.6660, Nov 2012
2. Randall Dougherty and Kenneth Zeger, "Nonreversibility and Equivalent Constructions of Multiple-Unicast Networks", IEEE transactions on Information Theory, vol. 52, no. 11, Nov 2006