Matroid Theory

with a focus on operations

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Outline

Matroid

- Matroid Review
- Dual
- Contraction & Deletion
- Excluded Minors

Matroid Review

Definition

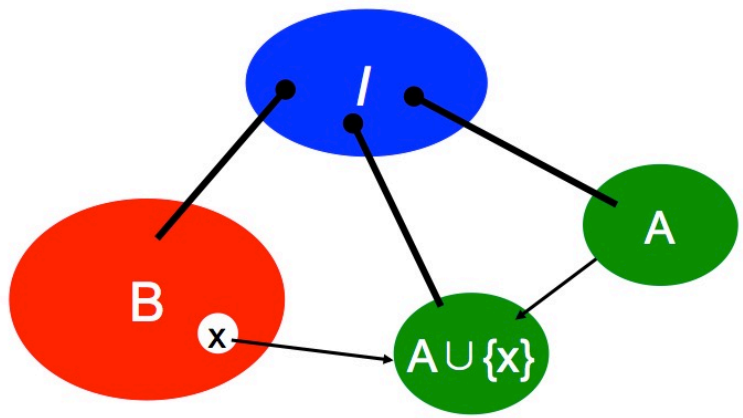
Definition

A matroid M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E having the following three properties:

1. $\phi \in \mathcal{I}$;
2. Hereditary:
   If $I \in \mathcal{I}$ *and* $I' \subseteq I$, *then* $I' \in \mathcal{I}$;
3. Augmentation:
   If I_1 *and* I_2 *are in* $\mathcal{I}$ *and* $|I_1| < |I_2|$, *then there is an element* e *of* $I_2 - I_1$ *such that* $I_1 \cup e \in \mathcal{I}$.

Demo

Augmentation



Matroid Review

Base/Basis

Bases $\mathcal{B}(M)$ of a matroid are the maximal independent sets.

Properties

- Same cardinality for all bases;
- The matroid is in the span of any base;

Consider

$$A = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right] \end{array}$$

Bases: {1,2,3}, {1,2,4}, {1,2,5}, {1,3,4}, {1,4,5}, {2,3,4}, {2,3,5}, {3,4,5}

Matroid Review

Base/Basis

A collection of subsets $\mathcal{B} \subseteq 2^E$ of a ground set E are the bases of a matroid if and only if:

- $\mathcal{B}$ is non-empty;
- base exchange: If $B_1, B_2 \in \mathcal{B}$, for any $x \in (B_1 - B_2)$, there is an element $y \in (B_2 - B_1)$ such that $(B_1 - x) \cup y \in \mathcal{B}$.

Consider

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Bases: {1,2,3}, {1,2,4}, {1,2,5}, {1,3,4}, {1,4,5}, {2,3,4}, {2,3,5}, {3,4,5}

Matroid Review

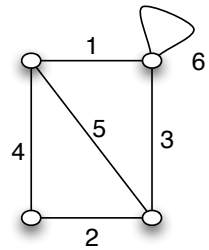
Circuit

The circuits $\mathcal{C}(M)$ of a matroid are the minimal dependent sets.

Property

- If any one element is deleted from a circuit, it will become an independent set.
- Independent sets $\mathcal{I}(M)$ does not contain any element in $\mathcal{C}(M)$.

Consider the graph



Circuits: {1,2,3,4}, {1,3,5}, {2,4,5}, {6}

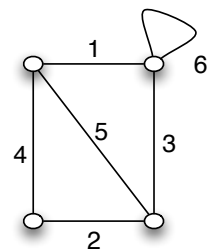
Matroid Review

Circuit

A collection of sets $\mathcal{C}$ is the collection of circuits of a matroid if and only if:

- $\phi \notin \mathcal{C}$
- if $C_1, C_2 \in \mathcal{C}$ with $C_1 \subseteq C_2$ then $C_1 = C_2$
- if $C_1, C_2 \in \mathcal{C}, C_1 \neq C_2$ and $e \in C_1 \cap C_2$ then there is some $C_3 \in C$ with $C_3 \subseteq (C_1 \cup C_2) - e$.

Consider the graph



Circuits: {1,2,3,4}, {1,3,5}, {2,4,5}, {6}

Matroid Review

Rank Function

Rank: maximun number of linearly independent vectors.
For any base of a matroid, $r(B) = r(M)$.
Formally, a rank function is $r : 2^E \to \mathbb{N} \cup \{0\}$, for which

- For $X \subseteq E, 0 \leq r(X) \leq |X|$
- If $X \subseteq Y \subseteq E, r(X) \leq r(Y)$
- $\forall X, Y \subseteq E, r(X \cap Y) + r(X \cup Y) \leq r(X) + r(Y)$

Consider

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Bases: {1,2,3}, {1,2,4}, {1,2,5}, {1,3,4}, {1,4,5}, {2,3,4}, {2,3,5}, {3,4,5};
Rank: $r(M) = r(B) = 3$

Matroid Review

Closure

Given a rank function of a matroid, the closure operation $\mathrm{cl} : 2^{\mathrm{E}} \rightarrow 2^{\mathrm{E}}$ is defined as

$$\mathrm{cl}\ X = \{x \in E | r(X \cup x) = r(X)\}.$$

Closure is a span of a subspace.
$\mathrm{cl}\ B = E(M)$;
Independent Set:
$\mathcal{I} = \{X \subseteq E | x \notin \mathrm{cl}(X - x) \forall x \in X\}$

A subset X of $E(M)$ is said to be a flat if $\mathrm{cl}\ \mathrm{X} = \mathrm{X}$.
Consider

$$A = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right] \end{array}$$

cl {1,2}={1,2,} $\rightarrow$ flat
cl {1,3}={1,3,5}

Matroid Review

Closure

A function $\mathrm{cl} : 2^{\mathrm{E}} \to 2^{\mathrm{E}}$ is closure for a matroid $M = (E, \mathcal{I})$ iff

- If $X \subseteq E$ then $X \subseteq \mathrm{cl}X$
- If $X \subseteq Y \subseteq E$, then $\mathrm{cl}X \subseteq \mathrm{cl}\,Y$
- If $X \subseteq E$ then $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}X$
- If $X \subseteq E$ and $x \in E$ and $y \in \mathrm{cl}(X \cup x) - \mathrm{cl}(X)$ then $x \in \mathrm{cl}(X \cup y)$

Consider

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

cl {1,2}={1,2,} $\to$ flat

cl {1,3}={1,3,5}

Matroid Review

Greedy Algorithm

Ground set E, $\mathcal{I}$ is a collection of subsets of E. A mapping function $\omega : E \to \mathbf{R}$. For $X \in \mathcal{I}$

$$W(X) = \sum_{x \in X} \omega(x)$$

An optimization solution in $\mathcal{I}$ is a greedy process:

1. $X_0 = 0$, $j = 0$
2. If $\exists e_{j+1} \in (E - X_j)$ such that $X_j \cup e_{j+1} \in \mathcal{I}$ and e_{j+1} has the max (min) weight among the elements left, then $X_{j+1} = X_j \cup e_{j+1}$, $j = j + 1$, repeat. Otherwise, stop.

Such a collection $\mathcal{I}$ is the collection of independent sets iff

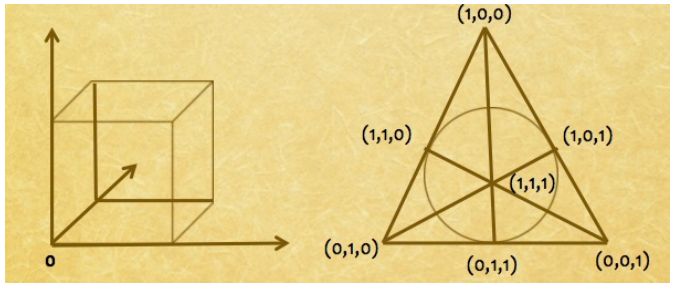
1. $\phi \in \mathcal{I}$
2. If $I_1 \in \mathcal{I}$ and $I_2 \subset I_1$, then $I_2 \in \mathcal{I}$
3. For all weight mapping functions, the above greedy algorithm is able to find an optimization solution in $\mathcal{I}$.

Representable Matroids

Binary

A matroid that is isomorphic to a vector matroid, where all elements take values from field $\mathbb{F}$, is called $\mathbb{F}$-representable.

Fano matroid is only representable over field of characteristic 2 (Binary).



Binary matroids have excluded minor: $U_{2,4}$

Representable Matroids

Ternary

Ternary matroids are those who can be represented over field of characteristic 3.

Regular Matroids

- Representable over every field
- Both Binary and Ternary

$U_{2,k}$ is $\mathbb{F}$-representable if and only if $|\mathbb{F}| \geq k-1$. $|U_{2,4}|$ is 3-representable (Ternary).

$$B = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Ternary matroids have excluded minor: $U_{2,5}, U_{3,5}, F_7$ *and* F_7^*.

Representable Matroids

Graphic Matroid

In a graph $G = (V, E)$, let $\mathcal{C} \subseteq 2^E$ be the sets of the graph cycles. Then $\mathcal{C}$ forms a set of circuits for a matroid with ground set E. The matroid $M(G)$ derived in this manner is called a cycle matroid (Graphic Matroid).

Excluded minors for graphic matroids:
$U_{2,4}, F_7, F_7^*, M^*(K_5), M^*(K_{3,3})$

K_5 matroid and $K_{3,3}$ matroid:

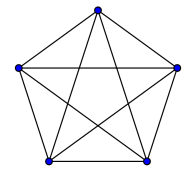


Figure: K_5 matroid

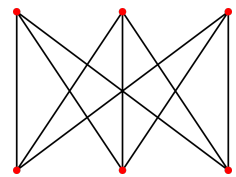


Figure: $K_{3,3}$ matroid

Representable Matroids

Graphic

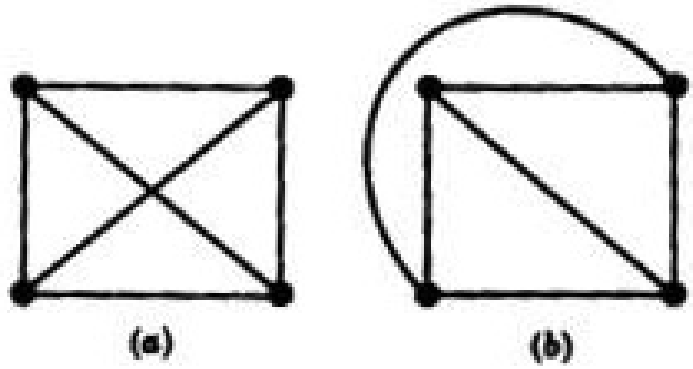
Graphic Matroids:

- Independent sets: no cycles are contained
- Rank function: $r(X) = |V(G[X])| - w(G[X])$
- Flat: contains no cycle that is contained in but exactly one edge
- Hyperplane: $E(G) - H$ is a minimal set of edges whose removal from G increases the number of connected components

Representable Matroids

Cographic and Planar

A matroid is called cographic if its dual is graphic.
Ex: $M^*(K_5)$ and $M^*(K_{3,3})$
A both graphic and cographic matroid is called planar, isomorphic to a cycle matroid derived from a planar. In a planar, all edges can be drawn on a plane without intersections.



Excluded minors for cographic:
$U_{2,4}, F_7, F_7^*, M(K_5), M(K_{3,3})$
Excluded minors for planar:
those excluded minors of graphic and cographic

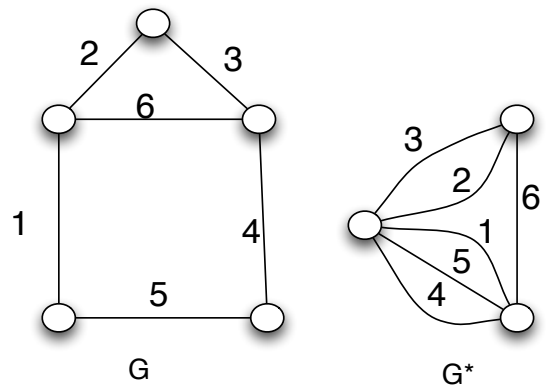
Dual

Definition

The dual matroid M^* to a matroid M is the matroid with bases that are complements of the bases of M.

$$\mathcal{B}(M^*) = \{E - B | B \in \mathcal{B}(M)\}.$$

- $(M^*)^* = M$ and X is a base if and only if $E - X$ is a cobase.



Dual

Independent Sets

- X is independent if and only if $E - X$ is cospanning
- X is spanning if and only if $E - X$ is coindependent

Dual

Circuits

- X is a circuit if and only if $E - X$ is a cohyperplane
- X is a hyperplane if and only if $E - X$ is a cocircuit

Dual

Rank Function

- Rank function of the dual matroid M^*:

$$r^*(X) = |X| + r(E - X) - r(M).$$

Dual

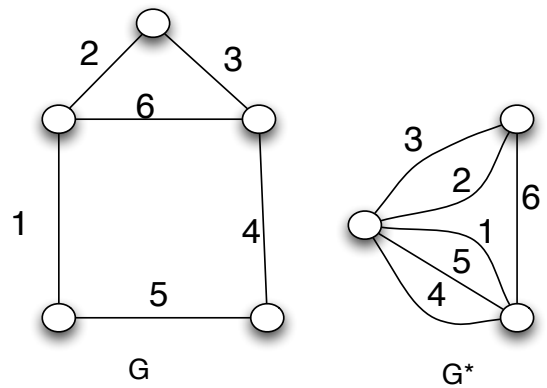
Representable Case

- $C = \left[\mathbf{I}_{r(M)} \ \mathbf{A}_{r(M)\times(|E|-r(M))}\right]$
- $C^* = \left[-\mathbf{A}^T \ \mathbf{I}_{|E|-r(M)}\right]$

Dual

Graphic Case

- Circuits X of $M^*(G)$ are cocircuits of $M(G)$; $E - X$ is a hyperplane in M(G)
- Complement of a Hyperplane is a minimal collection of edges (Bonds) whose removal from G increases the number of connected components
- Circuits of $M^*(G)$ are the bonds of G



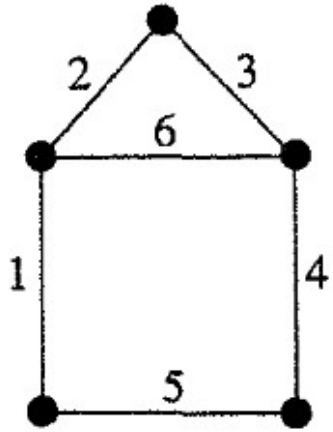
Contraction & Deletion

Definition

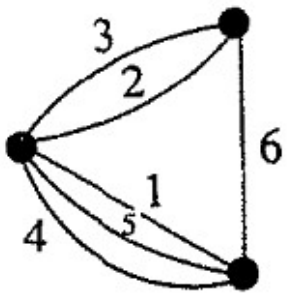
- Restriction is a shrink of domain; Deletion a subset T is a restriction on (E-T).

$$M\backslash T = M|(E - T)$$

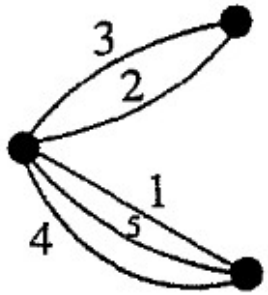
$\mathcal{I}(M\backslash T) = \{I \subseteq T : I \in \mathcal{I}(M)\}$
$\mathcal{C}(M\backslash T) = \{C \subseteq T : C \in \mathcal{C}(M)\}$
$r_{M\backslash T}(X) = r_M(X)$
Example.



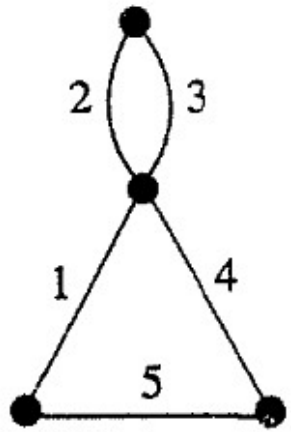
G



G*



G*\6



(G*\6)*

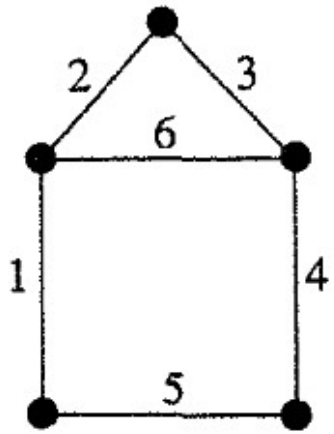
Contraction & Deletion

Definition & Graphic case

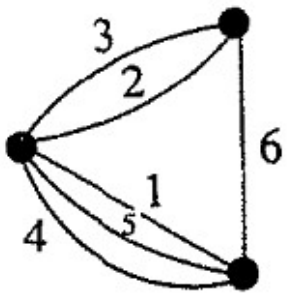
- Contraction is dual operation of deletion.

$$M/T = (M^* \setminus T)^*$$

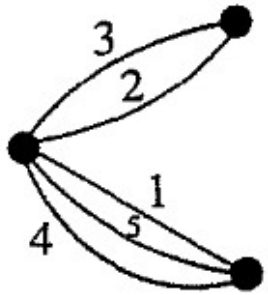
In a graphic matroid, $M(G)/T = M(G/T)$, which means the matroid contracting T is the matroid on the subgraph obtained from G by contracting edges T.
Example.



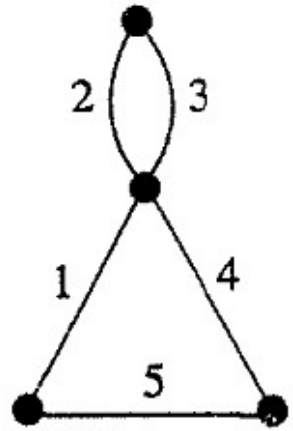
G



G^*



$G^* \setminus 6$

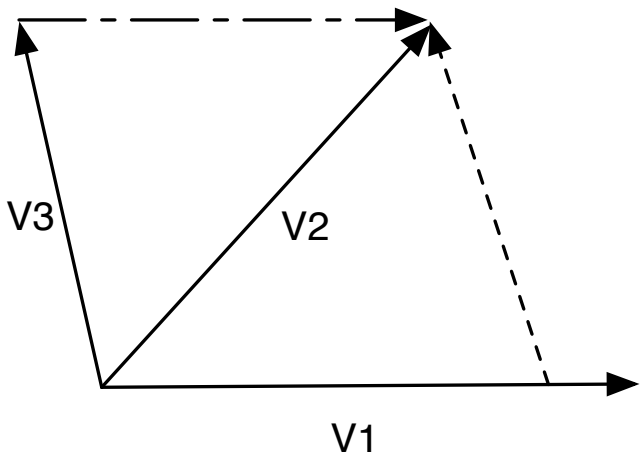


$(G^* \setminus 6)^*$

Contraction & Deletion

Representable case

Intuitively, contraction means delete elements (edges) and merge the corresponding end points into single points if viewed in a graph. While deletion of an element does not do the combination.
In Vectors, contracting an element means remove the corresponding component from the other vectors.
Contracting T from a representable matroid: Find a base $B \supseteq T$ and represent other vectors with B, then delete T.



Contraction & Deletion

Rank Function

- Rank function:

$$r_{M\setminus T}(X) = r_M(X)$$

$$r_{M/T}(X) = r_M(X \cup T) - r_M(T)$$

Also compare these with dual operation, since the above equation can be obtained from $r_{M\setminus T}$ and the following one

$$r^*(X) = |X| + r(E - X) - r(E)$$

Contraction & Deletion

Independent sets

- Independent sets

$$\mathcal{I}(M \backslash T) = \{I \subseteq E - T : I \in \mathcal{I}(M)\}$$

$$\mathcal{I}(M/T) = \{I \subseteq E - T : I \cup B_T \in \mathcal{I}(M)\} = \{I \in \mathcal{I}(M) : I \cap \mathrm{cl}_M(X) = \phi\}$$

where B_T is the base of $M|T$. $M|T$ has a base B such that $B \cup I \in \mathcal{I}$.

Contraction & Deletion

Bases

- Bases: $\mathcal{B}(M \backslash T)$ is the set of maximal members of $\{B - T : B \in \mathcal{B}(M)\}$;

$$\mathcal{B}(M/T) = \{B' \subseteq E - T : B' \cup B_T \in \mathcal{B}(M)\}$$

Contraction & Deletion

Circuits

- Circuits:

$$\mathcal{C}(M \backslash T) = \{C \subseteq E - T : C \in \mathcal{C}(M)\}$$

$$\mathcal{C}(M / T) = \min\{C - T \subseteq E - T : C \in \mathcal{C}(M)\}$$

Contraction & Deletion

Closure

- Closure:

$$\mathrm{cl}_{M\setminus T}(X) = \mathrm{cl}_M(X) - T$$

$$\mathrm{cl}_{M/T}(X) = \mathrm{cl}_M(X \cup T) - T$$

The closures also show the difference between deletion and contraction. That is, deletion does not affect the relationships between remaining elements.

Contraction & Deletion

Other theorems

- $M \backslash T = M / T$ iff $r(T) + r(E - T) = r(M)$, T is a separator of M.
- $M \backslash e = M / e$ iff e is a loop or a coloop of M
- In a matroid M, let T_1 and T_2 be disjoint subsets of $E(M)$. Then TFAE:
  - $(M \backslash T_1) \backslash T_2 = M \backslash (T_1 \cup T_2) = (M \backslash T_2) \backslash T_1$
  - $(M / T_1) / T_2 = M / (T_1 \cup T_2) = (M / T_2) / T_1$
  - $(M \backslash T_1) / T_2 = (M / T_2) \backslash T_1$

Excluded Minors

Definition

Definition

Minor: If a matroid M' can be obtained by operating combination of restrictions and contractions on a matroid M, then M' is called a minor of M.

Examples: $U_{2,4}$ is excluded by binary; $F_7 \& F_7^*$ are excluded by ternary.

Excluded Minors

Summary

Relationships between various classes of matroids:

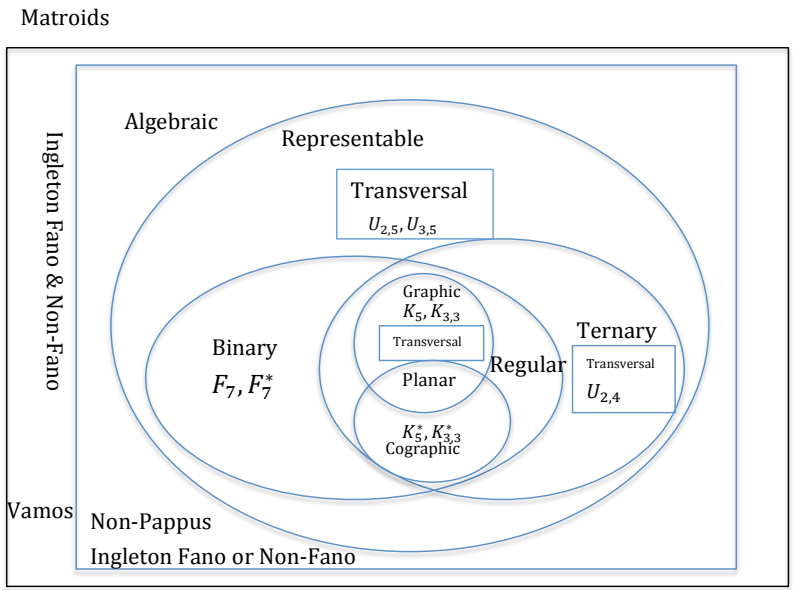


Figure: Relationships between various classes matroids: $U_{2,4}, F_7$, etc. in the graph are examples of the matroid class in which they lie. If an example is not contained in one class, it is called an excluded minor of this class.