Distributed Scalar Quantizers for Subband Allocation

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***Abstract*—Efficient downlink resource allocation (e.g., subbands in OFDMA/LTE) requires channel state information (e.g., subband gains) local to each user be transmitted to the base station (BS). Lossy encoding of the relevant state may result in suboptimal resource allocations by the BS, the performance cost of which may be captured by a suitable distortion measure. This problem is an indirect distributed lossy source coding problem with the function to be computed representing the optimal resource allocation, and the distortion measuring the cost of suboptimal allocations. In this paper we investigate the use of distributed scalar quantizers for lossy encoding of state, where the BS wishes to compute the index of the user with the largest gain on each subband. We prove the superiority of a heterogeneous (across users) quantizer design over the optimal homogeneous quantizer design, even though the source variables are i.i.d.**

***Index Terms*—Quantization, rateless coding, adaptive modulation and coding, resource allocation**

I. Introduction

In a multiuser OFDMA system, each of the available subbands may be assigned to a user. Once this assignment is done, the BS can select an appropriate modulation and coding scheme to approach the capacity of the channel on the subband between the BS and the selected user. The BS needs to know, for each subband, the user with the best channel gain and the value of the best gain; the other users' channel gains are irrelevant. Further, if the BS utilizes a *rateless* code on the subbands then it only needs to compute the index of the user with the best channel gain and not the actual value of the channel gain. A distinguishing feature of our approach is to model resource allocation problems as a chief estimating officer (CEO) (i.e., an indirect distributed lossy source coding) problem [1], by capturing the resource allocation decision as a function of the state variables, and representing the cost of suboptimal allocations via a suitable distortion measure. The associated rate-distortion (R-D) function gives the fundamental overhead-performance tradeoffs in the resource allocation problem, where the overhead is the sum rate of the messages sent to the BS, and the performance cost is the subband gain lost due to a suboptimal allocation.

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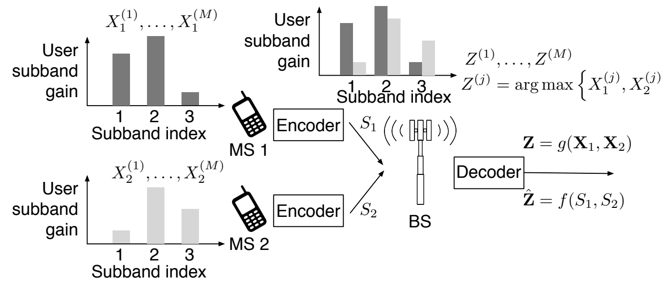

Fig. 1. The BS wishes to compute the index ($\arg\max$) of the user with the largest gain on each subband. The users encode their local gains across subbands using SQs.

In this work, we consider the design of distributed scalar quantizers (SQs) as an achievable scheme (i.e., realizable code) for encoding state for the allocation of subbands across users. Traditionally, quantizers are designed to minimize the reproduction distortion when using a finite-rate code for a continuous random source. For the resource allocation problem, what needs to be recovered at the decoder is a function of the sources instead of the sources directly. Additionally, the distributed nature of the sources greatly complicates the code design process. For example, if N mobile stations (MSs) were *centrally* located, then $\log_2 N$ bits would suffice to communicate which user has the best channel to the BS.

Recent work by Misra et al. considered the problem of distributed functional scalar quantization (DFSQ) [2]. By focusing on the high-rate regime and assuming a mean squared error (MSE) distortion, the authors are able to make several approximations to obtain distortion expressions that are optimal asymptotically (i.e., as the rate goes to infinity). We assume a different distortion measure, derive an exact expression for the distortion as a function of the quantizer parameters, and derive necessary conditions for optimal parameters. Moreover, our results hold for all rates.

Our focus on the use of SQs as an achievable scheme is motivated by several results concerning the optimality of a layered architecture of quantization followed by entropy coding. Zamir et al. considered the distributed encoding and centralized decoding of continuous valued sources and established that *lattice* quantization followed by Slepian-Wolf (SW) encoding is optimal asymptotically in rate [3]. When the sources are Gaussian and the distortion is MSE, local vector quantizers followed by SW coding is optimal, not just asymptotically [4]. For discrete valued random variables, *scalar* quantization with block entropy encoding is optimal [5]. Each of the problem models considered in [3]–[5] can be understood as an instance of indirect distributed lossy source coding for the identity function.

Our main result is a proof of the superiority of a simple and natural heterogeneous scalar quantizer (HetSQ) design over the optimal homogeneous scalar quantizer (HomSQ) design for computing the optimal user index under our given distortion measure. Superiority means the HetSQ achieves the same distortion with lower rate as the best HomSQ. Our result is surprising in that HetSQ are seen to achieve significant rate gains over optimal HomSQ even when the state variables are i.i.d. For a uniform discrete distribution we show numerically that our HetSQ is both significantly better than the optimal HomSQ, and moreover very close to the fundamental limit.

II. Problem Model

We focus our attention on the subband allocation problem for a single BS with two MSs as depicted in Fig. 1. At the i-th MS, the local state $\mathbf{X}_i$ is a vector of downlink channel capacities, which we model as random variables that are i.i.d. across users. Having observed its local state, the i-th MS sends a message $S_i \in \{1, \ldots, 2^{nR_i}\}$ at rate R_i to the BS. If the BS had direct access to the local state at each MS, it could compute the optimal subband allocation $\mathbf{Z} = g(\mathbf{X}_1, \mathbf{X}_2)$. Instead, the BS has to compute the subband allocation $\hat{\mathbf{Z}} = f(S_1, S_2)$ based on the messages it has received from the MSs. The performance of the system is measured with a distortion function $d(\boldsymbol{Z}, \hat{\boldsymbol{Z}})$.

If we assume the BS utilizes a rateless code then the only information the BS needs is which MS has the better subband; the optimal resource allocation function is naturally

$$Z^{(j)} = \arg\max_i \{X_i^{(j)} : i = 1, 2\}. \quad (1)$$

The BS estimates which user has largest gain on each subband and uses a rateless code (e.g., Hybrid ARQ) to find the coding rate on each subband. This is contrasted with the conventional adaptive modulation and coding (AMC), where the BS needs to estimate both the best user and the max gain on each subband. The distortion is the difference between the $\max$ rate for the MS with the actual $\arg\max$ index and the rate for the MS with estimated $\arg\max$

$$d(Z, \hat{Z}) = \frac{1}{M} \sum_{j=1}^{M} \left(X_{Z^{(j)}}^{(j)} - X_{\hat{Z}^{(j)}}^{(j)} \right) \quad (2)$$

where M is the number of available subbands. Observe that by definition, the quantity $X_{Z^{(j)}}^{(j)} - X_{\hat{Z}^{(j)}}^{(j)}$ is always non-negative.

III. Optimal Scalar Quantizer Design

In this section, we consider the design of SQs for a *single* subband; to handle multiple subbands, the MSs can utilize a quantizer per subband. We first consider the case where both MSs are using the same quantizer and derive an expression for the resulting distortion. Using this expression, we pose two non-linear optimization problems: first, minimize distortion for a given number of levels, and; second, minimize distortion for a given number of levels subject to a constraint on the entropy of the quantizer output. We provide first order necessary conditions for the optimal quantizer for both non-linear optimizations. We then argue that the same distortion performance can be achieved with a smaller sum rate on the uplink by utilizing different quantizers at each MS. We show that the design of the HetSQ can be accomplished via the same design procedure as for the HomSQ.

Let X_1 and X_2 be the channel capacity for the two users and let Z be the index of the user with maximum channel capacity. We assume that the random variables X_1, X_2 are i.i.d. with common PDF $f(x)$, CDF $F(x)$, and support set $\mathcal{X} \subseteq \mathbb{R}_+$.

A. Homogeneous Scalar Quantizers

Normally, a SQ is specified as a set of *decision boundaries* and *reconstruction levels* [6]. For the resource allocation problem, we do not need the BS to produce estimates for X_1, X_2, or even X_Z (i.e., the value of the maximum channel capacity). We can therefore specify the quantizer with just a set of decision boundaries $\{\ell_k : k = 0, \ldots, K\}$ which divide the support set $\mathcal{X}$ into K intervals

$$\mathcal{L}_k = [\ell_{k-1}, \ell_k] \quad k = 1, \ldots, K \quad (3)$$

where $\ell_0 \triangleq \inf \mathcal{X}$ and $\ell_K \triangleq \sup \mathcal{X}$. Let $S_i \in \{1, \ldots, K\}$ indicate the interval in which user i's observed capacity lies. The resource controller will pick user i if $S_i > S_j$ and will randomly pick a user if $S_i = S_j$; we denote the channel capacity so obtained as $X_{\hat{Z}}$.

Lemma 1. *The expected distortion as a function of the decision boundaries* $\{\ell_k : k = 0, \ldots, K\}$ *is*

$$D(\boldsymbol{\ell}) \triangleq \mathbb{E}\left[X_Z - X_{\hat{Z}}\right] = \sum_{k=1}^{K} \int_{\ell_{k-1}}^{\ell_k} \left(2F(x) - F(\ell_{k-1}) - F(\ell_k)\right) x f(x)\, \mathrm{d}x. \quad (4)$$

Proof: The ℓ_i's should be chosen to minimize

$$\mathbb{E}\left[X_Z - X_{\hat{Z}}\right] = \mathbb{E}\left[\mathbb{E}\left[X_Z - X_{\hat{Z}} \mid S_1, S_2\right]\right]. \quad (5)$$

Whenever $S_1 \neq S_2$, the BS correctly picks the maximum, hence, the only non-zero contribution comes from the case that $S_1 = S_2$. In this case, for $x \in \mathcal{L}_k$

$$F_{X_Z|S_1,S_2}(x \mid k, k) = \frac{(F(x) - F(\ell_{k-1}))^2}{(F(\ell_k) - F(\ell_{k-1}))^2} \quad (6)$$

$$f_{X_Z|S_1,S_2}(x \mid k, k) = \frac{2(F(x) - F(\ell_{k-1}))f(x)}{(F(\ell_k) - F(\ell_{k-1}))^2} \quad (7)$$

When $S_1 = S_2 = k$, $\mathbb{P}\left(Z = 1 \mid S_1 = k, S_2 = k\right) = \mathbb{P}\left(Z = 2 \mid S_1 = k, S_2 = k\right) = 1/2$ and the BS allocates the channel by flipping a coin. By randomly selecting either MS, $X_{\hat{Z}}$ will have the same distribution as X_i; that is

$$f_{X_{\hat{Z}}|S_1,S_2}(x|S_1,S_2) = \frac{f(x)}{F(\ell_k) - F(\ell_{k-1})}. \tag{8}$$

Finally,

$$\mathbb{P}\left(S_1 = k, S_2 = k\right) = \left(F(\ell_k) - F(\ell_{k-1})\right)^2. \tag{9}$$

■

a) Minimum Distortion: For a given number of intervals K, the decision boundaries $\{\ell_k : k = 0, \ldots, K\}$ that minimize the expected distortion are given by the solution to the following non-linear optimization:

$$\begin{aligned} \underset{\boldsymbol{\ell}}{\text{minimize}} \quad & D(\boldsymbol{\ell}) \\ \text{subject to} \quad & \ell_{k-1} \leq \ell_k \quad k = 1, \ldots, K. \end{aligned} \tag{10}$$

Theorem 1. *If $\{\ell_k^* : k = 0, \ldots, K\}$ is an optimal solution to (10) then there exists $\mu_k^* \geq 0$ for $k = 1, \ldots, K$ such that*

$$f(\ell_k) \int_{\mathcal{D}_k} (\ell_k - x) f(x)\, \mathrm{d}x - \mu_k + \mu_{k+1} = 0 \tag{11a}$$

$$\mu_k^*(\ell_{k-1}^* - \ell_k^*) = 0. \tag{11b}$$

Proof: The Lagrangian associated with this problem is

$$L(\boldsymbol{\ell}, \boldsymbol{\mu}) = D(\boldsymbol{\ell}) + \sum_{k=1}^{K} \mu_k(\ell_{k-1} - \ell_k) \tag{12}$$

Taking the derivative w.r.t. ℓ_i gives

$$\frac{\partial L(\boldsymbol{\ell}, \boldsymbol{\mu})}{\partial \ell_k} = \frac{\partial D(\boldsymbol{\ell})}{\partial \ell_k} - \mu_k + \mu_{k+1} \tag{13}$$

where

$$\frac{\partial D(\boldsymbol{\ell})}{\partial \ell_k} = f(\ell_k) \int_{\mathcal{D}_k} (\ell_k - x) f(x)\, \mathrm{d}x \tag{14}$$

and $\mathcal{D}_k = \mathcal{L}_k \cup \mathcal{L}_{k+1} = [\ell_{k-1}, \ell_{k+1}]$. ■

Remark. In §IV, we solved for the optimal decision boundaries by setting all the Lagrange multipliers to zero and solving (11a). Depending upon the distribution, (11a) can be solved exactly or with a non-linear solver.

b) Entropy-constrained minimum distortion: The interval S_i that the i-th user's observed capacity lies in is a discrete random variable with probability mass function given by

$$p_k \triangleq \mathbb{P}\left(S_i = k\right) = F(\ell_k) - F(\ell_{k-1}) \quad k = 1, \ldots, K \tag{15}$$

and the entropy of S_i is $H(S_i) = -\sum_{k=1}^{K} p_k \log_2 p_k$. The total rate needed for the two users to report their intervals is

$$R_H(\boldsymbol{\ell}) \triangleq H(S_1) + H(S_2) = 2H(S) \tag{16}$$

by the i.i.d. assumption of the channel capacities and the homogeneity of the quantizers.

We now consider the problem of minimizing the distortion subject to an upper limit on the sum rate.

$$\begin{aligned} \underset{\boldsymbol{\ell}}{\text{minimize}} \quad & D(\boldsymbol{\ell}) \\ \text{subject to} \quad & R_H(\boldsymbol{\ell}) \leq R_0 \\ & \ell_{k-1} \leq \ell_k \quad k = 1, \ldots, K \end{aligned} \tag{17}$$

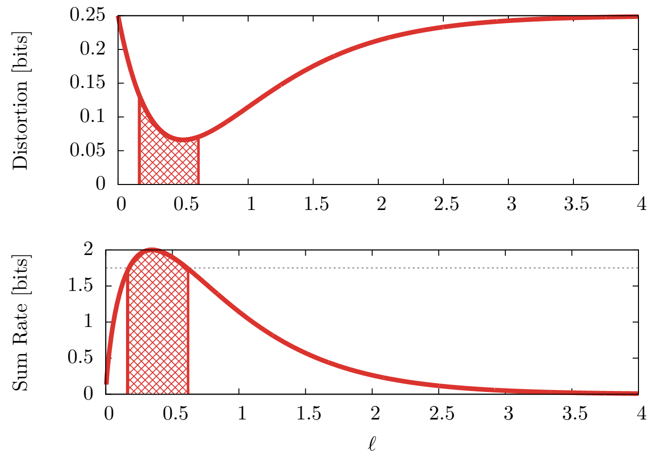

Fig. 2. Plots of $D(\ell)$ and $R(\ell)$ as functions of ℓ. For $R(\ell) \leq 1.75$, the set of feasible ℓ is seen to be non-convex.

In general, this problem is *not* convex. To see this, consider $X_i \sim \mathrm{Exp}(\lambda)$ and a single threshold ℓ (two intervals: $[0, \ell), [\ell, \infty)$). Fig. 2 shows a plot of $D(\ell)$ (top) and $R(\ell)$ (bottom) as ℓ is swept from $\inf \mathcal{X}$ to $\sup \mathcal{X}$. For $R_0 = 1.75$ bits, the range of *infeasible* ℓ is shown as a filled area under the rate and distortion curves and we see that the set of feasible ℓ is non-convex.

Theorem 2. *If $\{\ell_k^* : k = 0, \ldots, K\}$ is an optimal solution to (17), then there exists $\mu_k^* \geq 0$ for $k = 1, \ldots, K$ and $\mu_R \geq 0$ such that*

$$f(\ell_i) \left(\int_{\mathcal{D}_i^*} (\ell_i^* - x) f(x)\, \mathrm{d}x + 2\mu_R \log_2 \left(\frac{p_{i+1}^*}{p_i^*} \right) \right) - \mu_i + \mu_{i+1} = 0 \tag{18a}$$

$$\mu_i^*(\ell_{i-1}^* - \ell_i^*) = 0 \text{ and } \mu_R^*(R_H(\boldsymbol{\ell}^*) - R_0) = 0. \tag{18b}$$

Proof: The Lagrangian associated with this problem is

$$L(\boldsymbol{\ell}, \boldsymbol{\mu}) = D(\boldsymbol{\ell}) + \mu_R(R_H(\boldsymbol{\ell}) - r) + \sum_{k=1}^{K} \mu_k(\ell_{k-1} - \ell_k) \tag{19}$$

Taking the derivative w.r.t. ℓ_i gives

$$\frac{\partial L(\boldsymbol{\ell}, \boldsymbol{\mu})}{\partial \ell_i} = \frac{\partial D(\boldsymbol{\ell})}{\partial \ell_i} + \mu_R \frac{\partial R_H(\boldsymbol{\ell})}{\partial \ell_i} - \mu_i + \mu_{i+1} \tag{20}$$

where

$$\frac{\partial R_H(\boldsymbol{\ell})}{\partial \ell_i} = 2 f(\ell_i) \log_2 \left(\frac{p_{i+1}}{p_i} \right). \tag{21}$$

■

Remark. Solving for the optimal entropy constrained quantizer is more difficult than solving for the minimum distortion quantizer. Depending upon the given values of R_0 and K, the decision boundaries may collapse and the associated Lagrange multipliers need no longer be identically zero. A general solution technique for (18) is beyond the scope of the present paper; generalizations to both Lloyd's and Max's algorithms for entropy constrained quantizer design are presented in [7].

We conclude with some observations about the R-D curve for entropy-constrained quantizers. For a given K, suppose $\boldsymbol{\ell}^*$ is a solution to (10). If $R_0 \geq R(\boldsymbol{\ell}^*)$, then the rate constraint in (17) is not active and $\boldsymbol{\ell}^*$ is also a solution to (17) for the same K. On the other hand, if $R_0 < R(\boldsymbol{\ell}^*)$ then the rate constraint in (17) is active and $\boldsymbol{\ell}^*$ is infeasible for (17) [7]. Next, consider the R-D curve for a N-level entropy-constrained quantizer and the sequence of R-D points given by (10) for $K = 1, \ldots, N$. These R-D points all lie on the R-D curve for the N-level entropy-constrained quantizer.

B. Heterogeneous Scalar Quantizers

It is somewhat intuitive to suppose that because the channel capacities are i.i.d., the quantizers at each MS should be identical. For symmetric functions (e.g., $\max$), Misra et al. consider only the design of the quantizer for a single user [2]. When the function is *not* symmetric (e.g., $\arg\max$ as in our case), the assumption of HomSQ is in fact not true. To show this, the following lemmas will be needed.

Lemma 2. *Let X be a random variable and $a, b, c, d \in \mathbb{R}$ such that $a \leq c \leq b \leq d$. Then*

$$\mathbb{E}\left[X \mid X \in [a,b]\right] \leq \mathbb{E}\left[X \mid X \in [c,d]\right]. \quad (22)$$

Proof: Omitted for brevity. ■

Lemma 3. *Let X and Y be i.i.d. random variables, $Z = \max\{X, Y\}$, and $a \leq c \leq b \leq d$. Then*

$$\mathbb{E}\left[Z - Y \mid X \in [a,b], Y \in [c,d]\right] \leq \mathbb{E}\left[Z - X \mid X \in [a,b], Y \in [c,d]\right]. \quad (23)$$

Proof: Omitted for brevity. ■

Lemma 4. *Let X and Y be i.i.d. random variables with common PDF $f(\cdot)$ and CDF $F(\cdot)$, $Z = \max\{X, Y\}$, and $a \leq c \leq b \leq d$. Then*

$$\mathbb{P}\left(X \in [a,b], Y \in [c,d]\right) \mathbb{E}\left[Z - Y \mid X \in [a,b], Y \in [c,d]\right] = \int_c^b x(2F(x) - F(c) - F(b)) f(x)\,\mathrm{d}x. \quad (24)$$

Proof: Omitted for brevity. ■

Theorem 3. *For an optimal HomSQ $\boldsymbol{\ell}^*$ that achieves a distortion $D(\boldsymbol{\ell}^*)$, there exists a HetSQ that achieves the same distortion but at a lower rate.*

Proof: Given an optimal HomSQ $\boldsymbol{\ell}^*$, we will construct a HetSQ by assigning the odd indexed decision boundaries to user one and the even indexed decision boundaries to user two. Both quantizers are also assigned $\inf \mathcal{X}$ and $\sup \mathcal{X}$ as decision boundaries. The intervals at the two quantizers satisfy the staggered requirements of Lemma 4, with user one and user two alternating the roles of X and Y in the statement of the lemma. Repeated application of Lemma 4 to the quantization intervals of the HetSQ results in an expression for the heterogeneous quantizers distortion that is identical to (4). Hence, the HetSQ achieves the same distortion. The reduction in rate follows from observing that a users quantization intervals for the HetSQ is the union of disjoint events[1] and the subadditivity of the function $f(t) = -t \log t$. ■

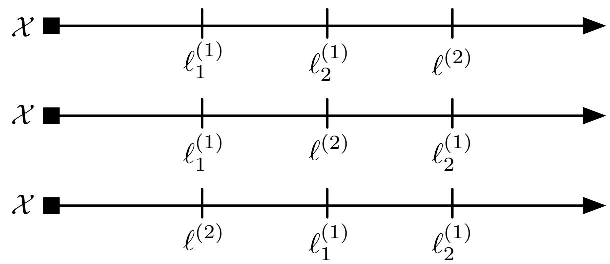

Fig. 3. Possible orderings when using three total thresholds across two users

Theorem 4. *For a HetSQ, if there exists an quantization interval for a user that is completely contained in the quantization interval for another user, then the quantizer's rate can be lowered without increasing its distortion.*

Proof Sketch: Our proof strategy is based on the following "thinning" argument: If such an interval exists, it can be combined with either the preceding or following quantization interval at the user. Which of the intervals it is combined with will depend on the conditional expected value of the capacity.

As an example, consider the following scenario with a total of three thresholds, where (w.l.o.g.) two thresholds (three bins) are assigned to user one and a single threshold (two bins) two user two. Fig. 3 shows the three possible orderings of the thresholds. We focus our attention on the first order $\ell_1^{(1)} \leq \ell_1^{(2)} \leq \ell^{(2)}$ and consider the six different possible outcomes for (S_1, S_2). For the cases $(S_1 = 0, S_2 = 0), (S_1 = 2, S_2 = 0), (S_1 = 2, S_2 = 1)$, the assumptions of Lemma 4 hold and we can apply the lemma. For the cases $(S_1 = 0, S_2 = 1), (S_1 = 1, S_2 = 1)$, we can trivially see that the conditional distortion is identically zero. For the case $(S_1 = 1, S_2 = 0)$, the distortion will depend on the output of the detector which in turn depends on $\mathbb{E}\left[X_1 \mid S_1\right] \lesseqgtr \mathbb{E}\left[X_2 \mid S_2\right]$. Suppose $\mathbb{E}\left[X_1 \mid S_1\right] \geq \mathbb{E}\left[X_2 \mid S_2\right]$, then it can be shown that $\mathbb{E}\left[X_Z - X_{\hat{Z}}\right]$ is independent of $\ell_2^{(1)}$. In a similar manner, if $\mathbb{E}\left[X_1 \mid S_1\right] \leq \mathbb{E}\left[X_2 \mid S_2\right]$, then it can be shown that $\mathbb{E}\left[X_Z - X_{\hat{Z}}\right]$ is independent of $\ell_1^{(1)}$. If $(\ell_1^{(1)}, \ell_2^{(1)}, \ell^{(2)})$ is HetSQ such that $\ell_1^{(1)} \leq \ell_1^{(2)} \leq \ell^{(2)}$ or $\ell^{(2)} \leq \ell_1^{(1)} \leq \ell_1^{(2)}$, then $D(\ell_1^{(1)}, \ell_2^{(1)}, \ell^{(2)}) = D(\ell^{(1)}, \ell^{(2)})$. Further, it can be shown in these cases that $R_{NH}(\ell_1^{(1)}, \ell_2^{(1)}, \ell^{(2)}) \geq R_{NH}(\ell^{(1)}, \ell^{(2)})$. We conclude that we can improve the rate for the HetSQ corresponding to these orderings of thresholds *without* increasing the distortion.

This process of merging extraneous quantization intervals can be repeated until no interval that satisfies the statement of the theorem can be found. ■

The previous two theorems suggests Algorithm 1 for the design of HetSQs. While the proposed scheme may not

[1] with the possible exception of the first and last interval

Algorithm 1 Design procedure for HetSQ

1: Select the total number desired bins K
2: Solve (11) for the optimal thresholds
3: Assign the odd indexed thresholds to user one and the even indexed thresholds to user two

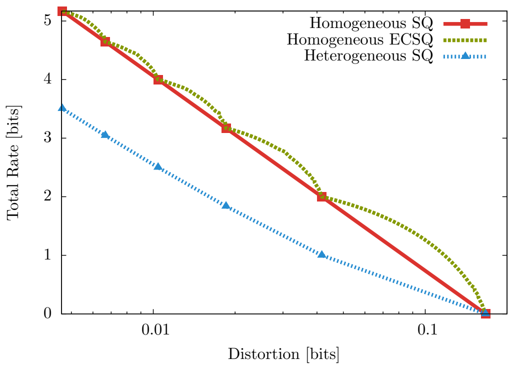

Fig. 4. SQ R-D curve for channel capacity distributed Uniform$(0, 1)$.

be optimal, it does not satisfy the sufficient condition for suboptimality of Theorem 4.

IV. Results

In this section, we consider two different continuous distributions for the channel capacity and compare the performance of HomSQ, homogeneous entropy-constrained scalar quantization (ECSQ), and HetSQ. We also show results for a discrete distribution for channel capacity in order to gauge the performance of the SQs relative to fundamental limit given by the R-D function.

A. Uniform Channel Capacity

We first consider the case where each MSs channel capacity is Uniform(a, b). We have

$$\ell_k^* = \frac{aK + (b-a)k}{K}, \ \mu_k^* = 0 \quad k \in 1, \dots, K \tag{25}$$

as a solution to (11). As expected, the optimal quantizer for a uniform distribution is uniform. Substituting into the expressions for distortion and rate we obtain

$$\mathbb{E}\left[X_Z - X_{\hat{Z}}\right] = \frac{b-a}{6K^2}, \ H(S) = \log_2 K. \tag{26}$$

Fig. 4 shows the expected distortion versus the rate for uniformly distributed channel capacity. For the homogeneous ECSQ, the number of intervals was fixed at $K = 6$ and R_0 was swept from 0 to $2\log_2 6 = 5.170$ bits. We see that the R-D points for (10) for $K = 1, \dots, 6$ lie on the ECSQ curve as expected. Non-intuitively, the ECSQ curve is *worse* than time-sharing between the solutions of (10) for different values of K; these results are similar to results reported in [7] for quantization of a uniform source with mean squared error distortion. Finally, we observe that performance of the HetSQ is markedly better than the HomSQ.

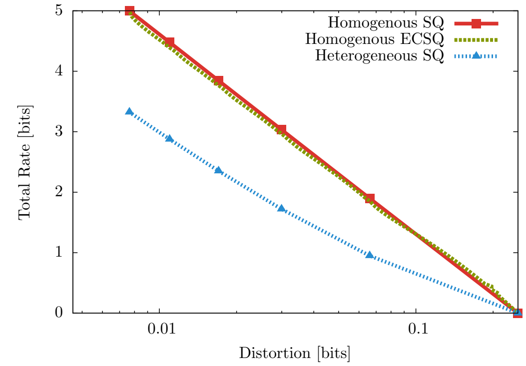

Fig. 5. SQ R-D curve for channel capacity distributed Exponential(2).

This performance gain does come at a cost in terms of fairness. To demonstrate this, consider the optimal HetSQ when using two intervals at each MSs, for which the optimal thresholds are given as $\ell^{(1)*} = {}^{2a+b}/_{3}$ and $\ell^{(2)*} = {}^{a+2b}/_{3}$. Observe that $S_1 \sim$ Bernoulli$(1/3)$ and $S_2 \sim$ Bernoulli$(2/3)$ independent of a and b and therefore

$$R_1 + R_2 = H(S_1) + H(S_2) = \frac{2}{3}\log_2\left(\frac{27}{4}\right) \approx 1.837. \tag{27}$$

We get the same distortion as for identical quantizers with $K = 3$ which requires $2\log_2 3 \approx 3.170$ bits. By Lemma 4, we see that user two is awarded the channel with probability $\mathbb{P}(S_1 = 0, S_2 = 0) + \mathbb{P}(S_1 = 1, S_2 = 0) + \mathbb{P}(S_1 = 1, S_2 = 1) = {}^5/_9$ while user one is awarded the channel with probability ${}^4/_9$. If we use three total thresholds (i.e., $K = 4$) with two thresholds at user one and one threshold at user two, then user one and user two are awarded the channel with equal probability. However, user one's rate is $R_1 = 1.5$ bits while user two's rate is $R_2 = 1$ bit.

B. Exponential Channel Capacity

We now consider the scenario where the channel capacity is distributed Exponential(λ). If we define $w_i^* \triangleq \lambda \ell_i^*$, then (11a) can be written as

$$w_k^* = \frac{-e^{-w_{k-1}^*}(1 + w_{k-1}^*) + e^{-w_{k+1}^*}(1 + w_{k+1}^*)}{\left(e^{-w_{k+1}^*} - e^{-w_{k-1}^*}\right)} \tag{28}$$

and the optimal quantizer for $X_1, X_2 \sim$ Exponential(λ) can be appropriate scaling of the optimal quantizer for $X_1, X_2 \sim$ Exponential(1). Observe that the above simplifies for the boundary conditions $\ell_0^* = 0$ and $\ell_K^* = \infty$

$$w_1^* = \frac{1 - e^{-w_2^*}(1 + w_2^*)}{\left(1 - e^{-w_2^*}\right)}, \ w_{K-1}^* = 1 + w_{K-2}^* \tag{29}$$

Fig. 5 shows the expected distortion versus the rate for exponentially distributed channel capacity. The parameter λ was chosen so that the expected channel capacities for Fig. 4 and Fig. 5 are the same. For the homogeneous ECSQ, the number of intervals was fixed at $K = 6$ and R_0 was swept from 0 to $2\log_2 6 = 5.170$ bits. As with the uniformly distributed channel capacity, we see that the R-D points for

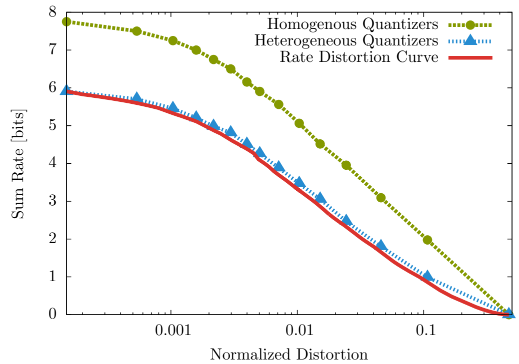

Fig. 6. Comparison of HomSQ and HetSQ to R-D function for a uniform distribution on the LTE defined CQI values. The distortion (average capacity loss) has been normalized by the average channel capacity.

(10) for $K = 1, \ldots, 6$ lie on the ECSQ curve as expected. Unlike uniformly distributed channel capacity, the differences between the ECSQ curve and time-sharing between the solutions of (10) for different values of K are almost non-existent. For higher values of R_0, the ECSQ curve is marginally better than time-sharing. Again, the HetSQ is markedly better than the HomSQ. Note that for the same rate, exponentially distributed channel capacity has a higher distortion as expected since the support set for Uniform$(0,1)$ is contained within the support set of Exponential(2).

As above, consider the optimal HetSQ when using two intervals at each MSs, for which the optimal thresholds are given as $w_1^* = 1 + W(-2e^{-2})$ and $w_2^* = 2 + W(-2e^{-2})$ where $W(x)$ is the Lambert W function. Observe that $S_1 \sim$ Bernoulli$(1-e^{-w_1^*}) \approx 0.448$ and $S_2 \sim$ Bernoulli$(1-e^{-w_2^*}) \approx 0.797$ and therefore

$$R_1 + R_2 = H(S_1) + H(S_2) \approx 1.720. \qquad (30)$$

We get the same distortion as for identical quantizers with $K = 3$ which requires 3.033 bits. User two is awarded the channel with probability $\mathbb{P}(S_1 = 0, S_2 = 0) + \mathbb{P}(S_1 = 1, S_2 = 0) + \mathbb{P}(S_1 = 1, S_2 = 1) \approx 0.560$ while user one is awarded the channel with probability 0.440.

C. LTE CQI

For the previous examples, we were able to gauge the performance improvement of HetSQ over HomSQ. Since these distributions are continuous, we do not know how these schemes perform compared to the optimal R-D tradeoffs. The LTE standard defines 16 channel quality indicator (CQI) levels that MSs use to inform the BS of the channel rate [8]. The rate region is known for independent sources as the Berger-Tung inner and outer bounds coincide due to the assumption of independence across users. This allows us to compute the R-D function for an assumed distribution on these CQI values and we can gauge the performance of SQs relative to the optimal R-D tradeoffs.

Fig. 6 shows the R-D tradeoffs for both HomSQ and HetSQ, along with the R-D function for the associated indirect distributed lossy source coding problem. Unlike before, the support set for this problem is finite and zero distortion can be achieved with sum rate $R_1 + R_2 \le 8$ bits; the $D = 0$ point has been omitted for the use of $\log$ scaling. We observe from Fig. 6 that HetSQ are very close to the fundamental limit for computing $\arg\max$; on average, the HetSQ achieves a rate within 0.124 bits of the R-D function. HomSQ, on the other hand, achieve a rate within 1.6224 bits on average from the R-D function. For the case of discrete sources, when the BS wishes to recover both sources (as opposed to a function of them), it has been shown that SQs followed by SW encoding is optimal [5].

V. Conclusions and Future Work

In this paper, we have considered the design of SQs for distributed function computation in the context of a resource allocation problem. We proposed a natural, non-quadratic distortion measure and provided an exact expression as a function of the quantizer parameters. We have argued that HetSQ are more efficient than HomSQ by achieving the same distortion at a lower rate. We considered several example capacity distributions, demonstrating that the performance of HetSQ *may* be close to fundamental limits.

We restricted our focus to SQs for a single subband. It is known that vector quantizers are more efficient than SQs, even when the source outputs being blocked into vectors are independent [6], [9]. As mentioned previously, recent results have shown that local vector quantizers followed by SW encoding is optimal for certain two-terminal problems with continuous distributions [3], [4]. This motivates the consideration of vector quantization in future work and to handle the case with multiple subbands.

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