Symmetry in Network Coding

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***Abstract*—We establish connections between graph theoretic symmetry, symmetries of network codes, and symmetries of rate regions for k-unicast network coding and multi-source network coding. We identify a group we call the network symmetry group as the common thread between these notions of symmetry and characterize it as a subgroup of the automorphism group of a directed cyclic graph appropriately constructed from the underlying network's directed acyclic graph. Such a characterization allows one to obtain the network symmetry group using algorithms for computing automorphism groups of graphs. We discuss connections to generalizations of Chen and Yeung's partition symmetrical entropy functions and how knowledge of the network symmetry group can be utilized to reduce the complexity of computing the LP outer bounds on network coding capacity as well as the complexity of polyhedral projection for computing rate regions.**

***Index Terms*—network coding, symmetry, graph automorphism**

I. Introduction

Many important pragmatic problems, including the efficient transfer of information over networks, the design of efficient distributed information storage systems, and the design of streaming media systems, have been shown to involve determining the capacity region of an abstracted network under multi-source multi-sink network coding (MSNC). Yan et al. [1] provided an implicit characterization of the rate region of MSNC over directed acyclic graphs in terms of the entropy function region Γ_n^*, however, the problems of characterizing Γ_n^* and its closure $\overline{\Gamma_n^*}$ remain open to date. Nonetheless, [1] provides a method, in principle, for at least bounding the MSNC capacity region of a network by substituting in known polyhedral inner and outer bounds for Γ_n^* [2]–[5]. When the resulting inner and outer bounds match, the capacity region has been determined.

The notion of network symmetry can be helpful for determining these bounds and capacity regions in at least two ways. First of all, certain inner and outer bounds can be shown to match, yielding exact calculation of the rate region, when the network has certain symmetries. In this vein, Chen and Yeung [6] have shown that certain symmetrical parts of $\bar{\Gamma}_N^*$ fixed under the action of symmetric groups defined by certain types of partitions are equivalent to the same symmetrical parts of the Shannon outer bound Γ_N. A second important way that network symmetry can be helpful in determining rate regions for networks under MSNC is via the reduction of complexity of computing their polyhedral bounds.

Bearing this in mind, the first goal of this work is to formalize the notion of structural symmetry in MSNC problems. A natural way to obtain such a formalization is via groups, and with each instance of multi-source network coding we will associate a subgroup of S_n (group of all permutations of n symbols) called the *network symmetry group* (NSG) that contains information about the symmetries of that instance. §II is devoted to the preliminaries and the definition of NSG. The connections between NSGs and symmetries of network codes and rate regions are discussed in §III. Secondly, we provide means to compute this group given an instance of a multi-source network coding problem. This is achieved through a graph theoretic characterization of the NSG, specifically, as a subgroup of automorphism group of a directed cyclic graph called the *dual circulation graph* that we construct from the directed acyclic graph underlying the MSNC instance. The k-unicast network coding problem (k-UNC), which is a special case of MSNC problem, has a simpler characterization of NSGs, which is covered in §IV, followed by the more general characterization of NSG for MSNC in §V. We then provide pointers to algorithms for computing automorphism groups of graphs that can be readily used for computing NSGs. Finally, in §VII we discuss connection to Chen and Yeung's partition symmetrical entropy functions [6] and use of NSGs in polyhedral projection to compute polyhedral bounds on rate regions and other applications.

II. Network Symmetry Group

We begin by defining the problems of interest for this work.

Definition 1. *An instance of multisource network coding problem (MSNC) is described by the tuple $(\mathcal{G}, \mathcal{S}, \mathcal{T}, \beta)$ where $\mathcal{G} = (V, E)$ is a directed acyclic graph, $\mathcal{S}, \mathcal{T} \subseteq V$ are the sets of source and sink nodes respectively, $\mathcal{S} \cap \mathcal{T} = \emptyset$, and $\beta : \mathcal{T} \to 2^{\mathcal{S}} \setminus \emptyset$ is a map giving the demands for each of the sink nodes.*

The directed acyclic graphs considered in this work are assumed to be simple (i.e. there exists at most one directed edge between any $u, v \in V$).

Definition 2. *An instance of MSNC problem is an instance of k-unicast problem if $|\beta(t)| = 1$, $\forall t \in \mathcal{T}$, $|\mathcal{S}| = |\mathcal{T}| = k$ and β is a bijection between $\mathcal{T}$ and $\mathcal{S}$.*

With each source node $s \in \mathcal{S}$ and edge $e \in E$ we associate discrete random variables X_s and X_e respectively. Altogether, between the edge and source random variables, we have a set of $n = |E| + |\mathcal{S}|$ random variables, collected into the set $\mathcal{X}_n$. For each edge $e = (u, v) \in E$, define $\mathsf{head}(e) = v$ and $\mathsf{tail}(e) = u$. For each node $v \in V$ we define sets $\mathsf{In}(v), \mathsf{Out}(v) \subseteq \mathcal{X}_n$. For $v \in V \setminus \mathcal{S}$, $\mathsf{In}(v)$ is the collection of random variables associated with edges $e \in E$

s.t. $\mathsf{head}(e) = v$. For $s \in \mathcal{S}$, $\mathsf{In}(s)$ is the set $\{X_s\}$. For each $v \in V \setminus \mathcal{T}$, $\mathsf{Out(v)}$ is the collection of random variables associated with edges $e \in E$ s.t. $\mathsf{tail}(e) = v$ whereas for $t \in \mathcal{T}$, $\mathsf{Out}(t)$ is same as $\beta(t)$. Let $\mathcal{H}_n \triangleq \mathbb{R}^{2^n-1}$ and $\mathcal{H}'_n \triangleq \mathbb{R}^{2^n-1+n}$. The network coding constraints for MSNC problem [1] can be classified into 3 sets. The first set $\mathcal{L}_1$ contains the source independence constraint

$$\sum_{s\in\mathcal{S}} h_s = h_{\mathcal{S}}. \tag{1}$$

The second set $\mathcal{L}_2$ contains node constraints

$$h_{\mathsf{In}(i)} = h_{\mathsf{In}(i)\cup\mathsf{Out}(i))},\ i \in V. \tag{2}$$

Let the subsets of $\mathcal{L}_2$ associated with $s \in \mathcal{S}$,$v \in V \setminus (\mathcal{S} \cup \mathcal{T})$ and $t \in \mathcal{T}$ be denoted as $\mathcal{L}_2^1$, $\mathcal{L}_2^2$ and $\mathcal{L}_2^3$ respectively. The third type of constraints is the rate constraints on information rates of edge random variables. For each edge $e \in E$ and source $s \in \mathcal{S}$ we have rate constraints

$$h_e \leq R_e, \quad h_s \geq \omega_s. \tag{3}$$

Let the set of all rate constraints be denoted as $\mathcal{L}_3$. Define $\mathcal{L}_{123}$ as, $\mathcal{L}_{123} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$. When considering rate regions, we purposefully remain agnostic w.r.t. the application (i.e. information flow vs. information storage) by allowing both source entropies and edge rates to be variables, which leads to an extended formulation of the rate region. This allows us to develop a unified framework for symmetry in MSNC. Let $\mathcal{L}_{123}^{\cap}$ denote the intersection of halfspaces and hyperplanes in $\mathcal{H}'_n$ corresponding to constraints in the set $\mathcal{L}_{123}$. Similarly, let $\mathcal{L}'$ and $\mathcal{L}''$ denote such intersection for $\mathcal{L}_1 \cup \mathcal{L}_2^1 \cup \mathcal{L}_2^2$ and for $\mathcal{L}_3 \cup \mathcal{L}_2^3$ respectively. Under this extended formulation, the exact rate region in [1] can be restated as

$$\mathcal{R}_* = \mathsf{proj}_{\boldsymbol{\omega},\boldsymbol{r}}(\overline{\mathsf{con}(\Gamma_n^* \cap \mathcal{L}')} \cap \mathcal{L}'') \tag{4}$$

where $\mathsf{proj}_{\boldsymbol{\omega},\boldsymbol{r}}(\cdot)$ corresponds to linear projection onto $\boldsymbol{\omega} = [\omega_s | s \in \mathcal{S}]$ and $\boldsymbol{r} = [R_e | e \in \mathcal{E}]$ and each set it viewed as a subset of $\mathbb{R}^{2^n-1+n}$ with variables not appearing in its definition being unconstrained. A polyhedral outer (inner) bound $\mathcal{R}_{\mathsf{out}}$ ($\mathcal{R}_{\mathsf{in}}$) on exact rate region can be specified in terms of a generic polyhedral outer (inner) bound $\Gamma_n^{\mathsf{out}} \supseteq \overline{\Gamma_n^*}$ ($\Gamma_n^{\mathsf{in}} \subseteq \overline{\Gamma_n^*}$) on $\overline{\Gamma_n^*}$ as

$$\mathcal{R}_k = \mathsf{proj}_{\boldsymbol{\omega},\boldsymbol{r}}(\Gamma_n^k \cap \mathcal{L}_{123}^{\cap}), \quad k \in \{\mathsf{in}, \mathsf{out}\}. \tag{5}$$

Having reviewed the preliminaries of network coding, we move on to define terminology related to groups and symmetry of directed graphs. Subgroups are denoted using '$\leq$' and $[n] \triangleq \{1, \ldots, n\}$. First, we consider the action of a finite group on a set.

Definition 3. *The (left) group action of a group $G \leq S_n$ on set S is the map $\phi : G \times S \to S : (g, x) \mapsto \phi(g, x)$ that satisfies:*

G1 $\phi((g \circ h), x) = \phi(g, \phi(h, x)),\ \forall g, h \in G, x \in S$
G2 $\phi(e, x) = x,\ \forall x \in S$

For simplicity, denote $\phi(g, x)$ as x^g and for a set $X \subseteq S$, denote $\phi(g, X)$ as X^g. For $X \subseteq S$, an element $g \in G$ is said to *stabilize X setwise* if $X^g = X$. The collection of all group elements $g \in G$ that setwise stabilize a subset $X \subseteq S$ forms a group called *stabilizer subgroup*, denoted as G_X.

The basic group action that we consider in this work is that of subgroup of S_n on $\mathcal{X}_n$, or more properly, its indices $\mathcal{S} \cup \mathcal{E}$: such a group acts on $\mathcal{X}_n$ by permuting subscripts of random variables in $\mathcal{X}_n$ via the map $\phi : (g, X_i) \mapsto X_{i^g}$. This fundamental group action induces an action on several other sets that are defined using $\mathcal{X}_n$. One such induced action is the action on the power set $2^{\mathcal{X}_n}$ of all subsets of $\mathcal{X}_n$ via the map $\psi : (g, A) \mapsto \{\phi(g, X_i) \mid X_i \in A\}$, for any $A \in 2^{\mathcal{X}_n}$. Let $\hat{\mathcal{L}}$ be the set of all linear constraints of the form specified in equations (1), (2) and (3) amongst entropies of subsets of random variables in $\mathcal{X}_n$ and $\boldsymbol{\omega}, \boldsymbol{r}$ that arise from MSNC instances. $G \leq S_n$ acts on each constraint $\mathcal{C} \in \hat{\mathcal{L}}$ via the map $\pi : (g, \mathcal{C}) \mapsto \mathcal{C}'$ where $\mathcal{C}'$ is is obtained from $\mathcal{C}$ by replacing joint entropies h_A of sets $A \subseteq 2^{\mathcal{X}_n}$ appearing in $\mathcal{C}$ by joint entropy $\psi(A)$ and by replacing rate variables R_e and ω_s by rate variable R_{e^g} and ω_{s^g}, respectively. For example, let $g = (1,2)(3,4)(7,8) \in G \leq S_8$, $\mathcal{S} = [|\mathcal{S}|]$, $E = [|\mathcal{S}| + |E|] \setminus [|\mathcal{S}|]$ and $\mathcal{C}$ be the constraint $h_{\{1,3\}} = h_{\{1,3,4,8\}}$. Then $\pi(g, \mathcal{C})$ or $\mathcal{C}^g$ is the constraint $h_{\{2,4\}} = h_{\{2,4,3,7\}}$ and if $\mathcal{C}$ is the constraint $h_8 \leq R_8$ then $\mathcal{C}^g$ is the constraint $h_7 \leq R_7$. Note that $\mathcal{L}_{123} \subset \hat{\mathcal{L}}$.

We define the network symmetry group (NSG) as follows.

Definition 4. *The network symmetry group $G^{\mathcal{I}}$ of a MSNC instance $\mathcal{I} = (\mathcal{G} = (V, E), \mathcal{S}, \mathcal{T}, \beta)$ is the subgroup of S_n, $n = |\mathcal{S}| + |E|$, that stabilizes $\mathcal{L}_{123}$ setwise under its induced action on $\hat{\mathcal{L}}$.*

When defined this way, $G^{\mathcal{I}}$ also stabilizes sets $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ setwise. Furthermore, since $\mathcal{L}_1$ is stablized setwise, subsets $\mathcal{L}_2^1, \mathcal{L}_2^2$ and $\mathcal{L}_2^3$ of constraints in $\mathcal{L}_2$ are also stabilized setwise. A natural question now arises: given an instance $\mathcal{I}$ of MSNC problem, how does one compute the NSG $G^{\mathcal{I}}$? We answer this question in sections IV and V, using a graph theoretic characterization of $G^{\mathcal{I}}$ involving two more concepts below.

Definition 5. *An automorphism σ of a directed graph $\mathcal{G} = (V, E)$ is a bijection $\sigma : V \to V$ s.t. if $(u, v) \in E$ then $(\sigma(u), \sigma(v)) \in E$.*

Definition 6. *([7], pg. 265) The line graph of a directed graph $\mathcal{G} = (V, E)$ is the directed graph $\mathcal{G}^* = (E, P)$ where $P \triangleq \{(e_1, e_2) \mid e_1 = (u, v), e_2 = (z, w) \in E$ and $v = z\}$*

III. Symmetries of codes and rate regions

In this section we consider action of NSG $G^{\mathcal{I}}$ of a MSNC instance $\mathcal{I}$ on network codes for $\mathcal{I}$ and rate region [1] associated with $\mathcal{I}$.

Definition 7. *A network code $(\{f_e\}, \{g_t\})$ for a MSNC instance $\mathcal{I}$ is an assignment of a function f_e to each edge $e \in E$ and a function g_t to each sink $t \in \mathcal{T}$.*

When we say that a network code $(\{f_e\}, \{g_t\})$ satisfies $\mathcal{L}_{123}$, we mean that source random variables and edge random variables created by $\{f_e\}$ and $\{g_t\}$ satisfy constraints in $\mathcal{L}_{123}$.

As mentioned in previous section, $G^{\mathcal{I}}$ stabilizes the subset of $\mathcal{L}_2$ associated with $t \in \mathcal{T}$ setwise. With slight abuse of notation, we shall refer to this permutation as $\pi : \mathcal{T} \to \mathcal{T}$. $G^{\mathcal{I}}$ acts on a network code via the map $\delta : (g, (\{f_i\}, \{g_j\})) \mapsto (\{f_{i^g}\}, \{g_{\pi(t)}\})$. The definition of the NSG then implies the following theorem, which links the NSG to symmetries of network codes.

Theorem 1. *Let $G^{\mathcal{I}}$ be the NSG associated with MSNC instance $\mathcal{I}$. Then, for any $g \in G^{\mathcal{I}}$. If network code $(\{f_i\}, \{g_t\})$ satisfies $\mathcal{L}_{123}$, so does $(\{f_{i^g}\}, \{g_{\pi(t)}\})$ for every $g \in G^{\mathcal{I}}$.*

A pair $\boldsymbol{\omega}, \boldsymbol{r}$ of source information rates and edge rates vectors is achievable if there exists a network code satisfying $\mathcal{L}_{123}$ corresponding to it. The next theorem relates NSGs to symmetries of the rate region.

Theorem 2. *If $\boldsymbol{\omega}, \boldsymbol{r}$ is an achievable source information rates and edge rates vector pair for a MSNC instance $\mathcal{I}$, so are $[\omega_{s^g} | s \in \mathcal{S}], [R_{e^g} | e \in E]$ for every $g \in G^{\mathcal{I}}$.*

Proof: Let $(\{f_i\}, \{g_t\})$ be the network code that achieves $\boldsymbol{\omega}, \boldsymbol{r}$. From theorem 1 network code $(\{f_{i^g}\}, \{g_{\pi(t)}\})$ achieves $[\omega_{s^g} | s \in \mathcal{S}], [R_{e^g} | e \in E]$, for each $g \in G^{\mathcal{I}}$. ■

IV. Symmetries in k-unicast network conding

We consider the problem of computing $G^{\mathcal{I}}$ corresponding to an instance $\mathcal{I}$ of k-UNC problem.

Definition 8. *The circulation graph of an instance $(\mathcal{G}, \mathcal{S}, \mathcal{T}, \beta)$ of k-UNC problem is a directed graph $\mathcal{G}_c = (V_c, E_c)$ such that $V_c = V$ and $E_c = E \cup \{(t, \beta(t)) \mid t \in \mathcal{T}\}$*

We call the set $F_c \triangleq \{(t, \beta(t)) \mid t \in \mathcal{T}\} \subseteq E_c$ the *feedback edge set*. The line graph of circulation graph $\mathcal{G}_c$ will henceforth be called the *dual circulation graph* and denoted as $\mathcal{G}_c^*$. For example, the circulation graph and dual circulation graph corresponding to the butterfly network (Fig. 1) are shown in Fig. 1 and Fig 2 respectively. The construction of the dual circulation of graph is reminiscent of the construction of a guessing game associated with a k-UNC instance (eg. in fig. 2 by deleting vertices associated with X_4, X_7, X_8, X_2, X_5 and X_6 and replacing them with pairs of directed edges $X_1 \leftrightarrow X_3, X_2 \leftrightarrow X_1$, $X_3 \leftrightarrow X_2$), where the main objective is to determine solvability of a network coding instance (see eg. [8], [9] for related definitions and constructions). For the purpose of this work, it suffices to use a construction that captures the symmetries of network codes and rate regions with order of the graph so constructed being reasonable in terms of, e.g., the number of variables in the original k-UNC instance.

With each edge $e \in E_c \backslash F_c$ we associate corresponding edge random variable X_e and with each edge $f = (t, \beta(t)) \in F_c$ of the circulation graph, we associate a source random variable $X_{\beta(t)}$. In this setup, any automorphism of the dual circulation graph induces a permutation on $\mathcal{X}_n$. We will denote the group of automorphisms of $\mathcal{G}_c^*$ as $\mathsf{Aut}(\mathcal{G}_c^*)$ and treat it as a group of permutations of $\mathcal{X}_n$. Let $G_{F_c} \leq S_n$ be the group of permutations of $\mathcal{X}_n$ that stabilize the subset of random

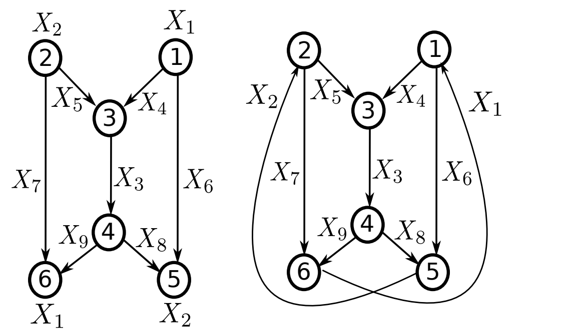

Figure 1: (left) The 2-UNC butterfly instance $\mathcal{I}$, (right) associated circulation graph $\mathcal{G}_c$

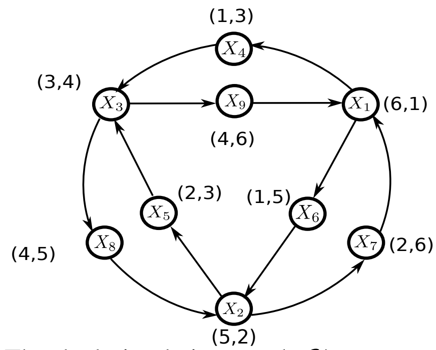

Figure 2: The dual circulation graph $\mathcal{G}_c^*$ corresponding to $\mathcal{I}$

variables associated edges in F_c (the source random variables) setwise. The following theorem states that the NSG of a k-UNC instance is $G_{F_c} \cap \mathsf{Aut}(\mathcal{G}_c^*)$. The proof relies on the fact that the edges in the dual circulation graph correspond to directed paths in of length 2 in the circulation graph.

Theorem 3. *The network symmetry group of a k-UNC instance is $G^{\mathcal{I}} = G_{F_c} \cap \mathsf{Aut}(\mathcal{G}_c^*)$, the group of permutations of $\mathcal{X}_n$ induced by the subgroup of automorphism group of its dual circulation graph that setwise stabilizes the feedback edge set.*

Proof: If a permutation $\sigma \in G^{\mathcal{I}}$ is not in G_{F_c} then the source independence constraint does not remain fixed. Hence, $G^{\mathcal{I}} \leq G_{F_c}$. Now we prove that $G^{\mathcal{I}} \leq \mathsf{Aut}(\mathcal{G}_c^*)$ i.e. if $\sigma \in G^{\mathcal{I}}$, then $\sigma \in \mathsf{Aut}(\mathcal{G}_c^*)$. Since a permutation $\sigma \in G^{\mathcal{I}}$ fixes $\mathcal{L}_{12}$, for every constraint $\mathcal{C} \in \mathcal{L}_{12}$, $\mathcal{C}^{\sigma}$ is also a constraint in $\mathcal{L}_{12}$. It follows that for $e_1, e_2 \in \mathcal{G}_c$, if (e_1, e_2) forms a directed path in $\mathcal{G}_c$, then $(\sigma(e_1), \sigma(e_2))$ also forms a directed path in $\mathcal{G}_c$, implying that $\sigma \in \mathsf{Aut}(\mathcal{G}_c^*)$. Hence $G^{\mathcal{I}} \leq G_{F_c} \cap \mathsf{Aut}(\mathcal{G}_c^*)$. Conversely, consider a permutation $\sigma \in G_{F_c} \cap \mathsf{Aut}(\mathcal{G}_c^*)$. $\mathcal{L}_1$ is fixed under σ since $\sigma \in G_{F_c}$. $\mathcal{L}_3$ is fixed by definition of automorphism. As for $\mathcal{L}_2$, if (e_1, e_2) form a directed path in $\mathcal{G}_c$ then $(\sigma(e_1), \sigma(e_2))$ also forms a directed path in $\mathcal{G}_c$. Hence, to preserve length 2 directed paths $\mathsf{In}(i) \times \mathsf{Out}(i)$ through a node $i \in V$, $\sigma(\mathsf{In}(i)) = \mathsf{In}(j)$ and $\sigma(\mathsf{Out}(i)) = \mathsf{Out}(j)$ for some $j \in V$, i.e. a permutation is induced on V. Hence, $\mathcal{L}_{123}$ remains setwise fixed implying $G_{F_c} \cap \mathsf{Aut}(\mathcal{G}_c^*) \leq G^{\mathcal{I}}$. ■

For example, the NSG $G^{\mathcal{I}}$ for Fig. 1 is a subgroup of S_9 of order 2, with the only non-trivial automorphism being $(1,2)(3)(4,5)(6,7)(8,9)$which is understood as a permutation of subscripts $i \in [n]$ of $X_i \in \mathcal{X}_n$ in cycle decomposition notation.

V. Symmetries in MSNC

The characterization of NSGs for MSNC instances follows same rough procedure as that of k-UNC instances. However, it is complicated by the fact that there exist sources that are demanded by more than one sink nodes. To create the circulation graph $\mathcal{G}_c$ from $\mathcal{G}$ we add some new vertices and edges to $\mathcal{G}$. Specifically, we add a set of vertices $\mathcal{S}' \triangleq \{s' \mid s \in \mathcal{S}\}$, sets of edges $F_c \triangleq \{(t, j') \mid j \in \beta(t), t \in \mathcal{T}\}$ and $E_{\mathcal{S}} \triangleq \{(s', s) \mid s' \in \mathcal{S}'$ and $s \in \mathcal{S}\}$

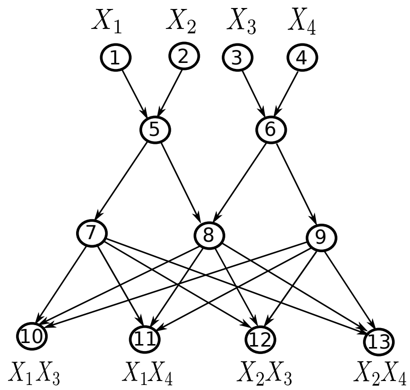

Figure 3: M-network: an instance of MSNC problem

Definition 9. *The circulation graph of a MSNC instance is a directed graph* $\mathcal{G}_c = (V \cup \mathcal{S}', E \cup E_{\mathcal{S}} \cup F_c)$.

In order to characterize NSG $G^{\mathcal{I}}$ of a MSNC instance $\mathcal{I}$, we associate a random variable in $\mathcal{X}_n$ with each eage in $E \cup E_{\mathcal{S}}$ of the circulation graph. We do not associate any random variables with edges in F_c. Similar to k-UNC instances, the dual circulation graph of a MSNC instance is obtained as the line graph of the circulation graph and is denoted as $\mathcal{G}_c^*$. In this setup, we cannot directly treat $\mathsf{Aut}(\mathcal{G}_c^*)$ as a group that permutes $\mathcal{X}_n$. Instead, we restrict our attention to automorphisms $\sigma \in \mathsf{Aut}(\mathcal{G}_c^*)$ that stabilize $E_{\mathcal{S}}$ setwise.

Lemma 1. *If $G \leq \mathsf{Aut}(\mathcal{G}_c^*)$ stabilizes $E_{\mathcal{S}}$ setwise then it also stabilizes F_c and E setwise.*

Proof: Let $\sigma(u, v) = (w, z)$ for $(u, v), (w, z) \in E_{\mathcal{S}}, \sigma \in G$ and let $(d, u) \in F_c$. Then, $\sigma(d, u) = (d', w) \in F_c$ for some sink node $d' \in \mathcal{T}$, in order to preserve directed path (d, u, v). As an implication, E is also setwise stabilized.■

From lemma 1, $E \cup E_{\mathcal{S}}$ is setwise fixed by such G. Thus, any member of automorphism group of the dual circulation graph that setwise stabilizes E_S *induces* a permutation on $\mathcal{X}_n$. The set of all such induced permutations forms a group. We denote this group as G_{E_S} which can now be treated as a group of permutations of $\mathcal{X}_n$. Alternatively, $G^{\mathcal{I}}$ can be treated as a group of permutations of $E \cup E_{\mathcal{S}}$.

Lemma 2. *Every $\sigma \in G^{\mathcal{I}}$ induces a permutation on $V \cup \mathcal{S}'$, that stabilizes $\mathcal{S}'$ and $\mathcal{T}$ setwise.*

Proof: A $\sigma \in G^{\mathcal{I}}$ stabilizes set of source random variables setwise. Hence, if a constraint $\mathcal{C}$ is associated with $t \in \mathcal{T}$ then, $\mathcal{C}^{\sigma}$ must also be associated with some $t' \in \mathcal{T}$ (although $t = t'$ is possible). Secondly, since $\mathcal{S}' = \{\mathsf{tail}(e) \mid e \in E_{\mathcal{S}}\}$, i.e. it contains nodes that are tails of edges associated with source random variables, a permutation is induced on $\mathcal{S}'$. As for $V \setminus (\mathcal{S}' \cup \mathcal{T})$, the permutation can be constructed from the permutation of $\mathcal{L}_2$ brought about by σ. ■

We now show that NSG $G^{\mathcal{I}}$ is same as G_{E_S}.

Theorem 4. *The network symmetry group of a MSNC instance is the group of permutations induced onto $E \cup E_S$ by the subgroup of $\mathsf{Aut}(\mathcal{G}_c^*)$ that stabilizes E_S setwise.*

Proof: Since we associated random variables with each edge in $E \cup E_S$, a permutation of $\mathcal{X}_n$ induces a permutation of $E \cup E_S$. If a permutation induced on $E \cup E_S$ by $\sigma \in G^{\mathcal{I}}$ does not stabilize E_S setwise, then source independence constraint does not remain fixed under such a permutation, contradicting the fact that $\sigma \in G^{\mathcal{I}}$. We now show that every permutation of $E_S \cup E$ induced by $\sigma \in G^{\mathcal{I}}$, is also induced by some $\sigma' \in \mathsf{Aut}(\mathcal{G}_c^*)$ that stabilizes $E_{\mathcal{S}}$ setwise, by explicitly constructing one such $\sigma' \in \mathsf{Aut}(\mathcal{G}_c^*)$. From lemma 2 we know that $\sigma \in \mathcal{G}^{\mathcal{I}}$ induces a permutation of $\mathcal{S}'$ in addition to permutation of the set $\mathcal{T} \subset V$ of sink nodes. We will refer to it as $\sigma(s')$ and $\sigma(d)$ for $s' \in \mathcal{S}'$ and $d \in \mathcal{T}$ respectively.

$$\sigma'(u, v) = \begin{cases} \sigma(u, v) & \text{if } (u, v) \in E_S \cup E \\ (\sigma(u), \sigma(v)) & \text{if } (u, v) \in F_c. \end{cases} \quad (6)$$

Since σ preserves $\mathcal{L}_2$, for $e_1, e_2 \in E \cup F$, if (e_1, e_2) forms a directed path in $\mathcal{G}_c$, then $(\sigma'(e_1), \sigma'(e_2))$ also forms a directed path in $\mathcal{G}_c$. The remaining directed paths of type $(d, s', s), d \in \mathcal{T}, s' \in \mathcal{S}', s \in \mathcal{S}$ are preserved by construction of σ'. Hence, $\sigma' \in \mathsf{Aut}(\mathcal{G}_c^*)$. Conversely, we must show that $\mathcal{L}_{123}$ is stabilized setwise under action of G_{E_S}. By definition, G_{E_S} fixes the source independence constraint and stabilizes $\mathcal{L}_1$. As for $\mathcal{L}_2$, if $\sigma \in G_{E_S}$ and (e_1, e_2) form a directed path in $\mathcal{G}_c$ for $e_1, e_2 \in E \cup E_{\mathcal{S}}$, then $(\sigma(e_1), \sigma(e_2))$ also forms a directed path in $\mathcal{G}_c$. Hence, to preserve length 2 directed paths $\mathsf{In}(i) \times \mathsf{Out}(i)$ through a node $i \in V$, $\sigma(\mathsf{In}(i)) = \mathsf{In}(j)$ and $\sigma(\mathsf{Out}(i)) = \mathsf{Out}(j)$ for some $j \in V$, i.e. a permutation is induced on V. Thus, $\mathcal{L}_2$ is stabilized setwise. By Lemma 1, $\mathcal{L}_3$ is also stabilized setwise. Hence $\sigma \in G^{\mathcal{I}}$. ■

For example, for the M-network in Fig. 3, the subgroup of the automorphism group of the dual circulation graph of the circulation graph in Fig. 4 that stabilizes $E_{\mathcal{S}}$, is a subgroup of S_{32} of order 8 with 3 generators. To help visualize: the cycle decomposition of the permutations of the set of (subscripts

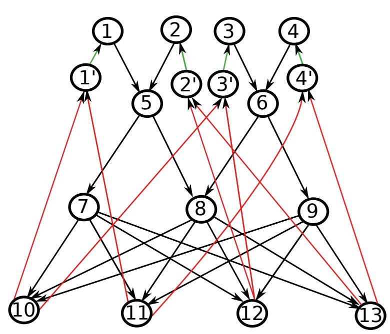

Figure 4: Circulation graph $\mathcal{G}_c$ corresponding to M-network. Edges in $E_{\mathcal{S}}$ are green while edges in F_c are red

of) source random variables induced by the generators are: $(1,3)(2,4)$, $(3,4)$, and $(1,2)$.

VI. Computing $\mathsf{Aut}(\mathcal{G}_c^*)$

Several software tools can be utilized to follow the method presented in this paper for computing the network symmetry group based on stabilizer subgroups of automorphism groups of graphs. One of the first softwares successful in practice for computing automorphism groups of (di)graphs is McKay's `nauty` [10] (**N**o **AUT**omorphisms, **Y**es?). The underlying algorithm essentially performs canonical labeling of the given (di)graph and as a byproduct, automorphism group is computed. McKay's algorithm is based on partition refinement and traversal of a search tree, where each node of the tree corresponds to an ordered partition of the vertices of the graph while leaves correspond to discrete partitions (where each block is a singleton) of vertices. This structure readily allows us to start with pre-defined partition of vertices of the graph that the resultant group of automorphisms must respect. This is useful for computing stabilizer subgroups of $\mathsf{Aut}(\mathcal{G}_c^*)$ eg. subgroup that stabilizes feedback edge set in case of k-UNC instances and the subgroup that stabilizes $E_{\mathcal{S}}$ in case of MSNC instances. Several improvements on the basic algorithm can be found in literature and are implemented as `saucy` [**?**], `Bliss` [**?**] and `Traces` [10]. For computing network symmetry groups of butterfly network and M-network, we used `SAGE` [11] which makes use of an implementation of McKay's algorithm called `N.I.C.E` (**N**ice **I**somorphism **C**heck **E**ngine) by Miller [12]. For computing network symmetry groups of butterfly network and M-network, we used `SAGE` [11] which makes use of an implementation of McKay's algorithm called `N.I.C.E` (**N**ice **I**somorphism **C**heck **E**ngine) by Miller [12].

VII. Conclusion & Applications of NSGs

In this paper we introduced the network symmetry group for a network coding problem, showed how to compute it using graph automorphism algorithms, and linked it with symmetries in the rate region and its network codes. In this final section of the paper, we discuss some directions for future work that demonstrate the utility of the NSG in applications.

Knowledge of the symmetry groups of polyhedra, which are specified as a subgroup of S_d or general linear group $GL(\mathbb{R},d)$ for a polyhedron $\mathcal{P} \subseteq \mathbb{R}^d$, can be used to substantially reduce the complexity of solving linear programs (LPs) (see [13], [14]). The main result in [14] states that the solution of a symmmetric LP exists in a lower dimensional polyhedral subset of the given polyhedron, specifically those points in $\mathcal{P}$ that are fixed (i.e. they map to themselves) under the action of the symmetry group of $\mathcal{P}$ on $\mathbb{R}^d$. Furthermore, symmetry can be exploited in polyhedral representation conversion [15]. NSGs can be interpreted as a subgroup of S_{2^n+n-1} acting on $\mathcal{H}'_n$ by permuting the standard basis vectors. By definition, for both $k \in \{\mathsf{In}, \mathsf{Out}\}$ the NSG stabilizes $\Gamma_n^k \cap \mathcal{L}_{123}^{\cap}$ setwise, provided the selected Γ_n^k, like Γ_n^*, is setwise invariant under permutation of $\mathcal{X}_n$. Owing to this and the action it also induces on $\boldsymbol{\omega}, \boldsymbol{r}$ the NSG is a polyhedral symmetry group of both cones (before and after projection) in the projection. For a detailed discussion of the exploitation of knowledge of the NSG for computing polyhedral bounds on rate regions, see [16].

Given the network symmetry group $G^{\mathcal{I}}$, let Σ_p be the subgroup of S_n that setwise stabilizes each block in the partition p of $\mathcal{X}_n$ which corresponds to the orbits of $G^{\mathcal{I}}$ on $\mathcal{X}_n$. Chen and Yeung [6] considered the action of Σ_p on $\mathcal{H}_n$ and defined the partition symmetrical entropy function region (Ψ_p^*) and polymatroidal region (Ψ_p) which are those points in the respective regions that are fixed under the action of Σ_p. Similar regions ($\Psi_{\mathcal{I}}^*$ and $\Psi_{\mathcal{I}}$ resp.) can be defined corresponding to $G^{\mathcal{I}} \leq \Sigma_p$ as can Ψ_p^k and $\Psi_{\mathcal{I}}^k$ from Γ_n^k for $k \in \{\mathsf{In}, \mathsf{Out}\}$. When it comes to network coding applications, usually $G^{\mathcal{I}} < \Sigma_p$, which means that there are group elements in Σ_p which do not leave the constraints $\mathcal{L}_{123}$ fixed. This occurs, for instance, in both networks in Fig. 1 and Fig. 3. In these instances, even though the groups $G^{\mathcal{I}}$ and Σ_p induce the same partition p of $\mathcal{X}_n$, and hence are associated with the same fixed subspace $\mathrm{Fix}_{G^{\mathcal{I}}}(\mathbb{R}^{|\mathcal{X}_n|}) = \mathrm{Fix}_{\Sigma_p}(\mathbb{R}^{|\mathcal{X}_n|})$ in the rate region coordinates (i.e. after projection), on the power sets, and hence in entropy coordinates (before projection), $\mathrm{Fix}_{G^{\mathcal{I}}}(\mathbb{R}^{|2^{\mathcal{X}_n}\setminus\{\emptyset\}|}) \neq \mathrm{Fix}_{\Sigma_p}(\mathbb{R}^{|2^{\mathcal{X}_n}\setminus\{\emptyset\}|})$. As such, while it is certain that $\mathcal{R}_k \cap \mathrm{Fix}_{\Sigma_p}(\mathbb{R}^{|\mathcal{X}_n|}) = \mathsf{proj}_{\boldsymbol{\omega},\boldsymbol{r}} \Psi_{\mathcal{I}}^k \cap \mathcal{L}_{123}^{\cap}$, in general it will be possible that $\mathcal{R}_k \cap \mathrm{Fix}_{\Sigma_p}(\mathbb{R}^{|\mathcal{X}_n|}) \supsetneq \mathsf{proj}_{\boldsymbol{\omega},\boldsymbol{r}} \Psi_p^k \cap \mathcal{L}_{123}^{\cap}$. This is the reason why it is the NSG $G^{\mathcal{I}}$ and its fixed subspace of the entropy region $\Psi_{\mathcal{I}}^k$, rather than Ψ_p^k, that is the appropriate notion of entropy region symmetry when it comes to network codes, for $k \in \{\mathsf{In}, \mathsf{Out}\}$.

For instance, for the butterfly network in Fig. 1, the dimension of Ψ_p is $3^4 \times 2 = 162$ (see [6]) while the dimension of $\Psi_{\mathcal{I}}$ is $\frac{1}{2}(511+31) = 271$, by observing in Burnside's lemma that the only non-empty subsets of $\mathcal{X}_n$ fixed under $(1,2)(3)(4,5)(6,7)(8,9)$ are the ones that can be obtained as union of some collection of blocks in the partition p.

Finally, NSGs are useful in algorithms for isomorph free exhaustive generation [17] of network codes, enhancing e.g. [4], as they can be exploited to list only those codes that are different up to these additional known symmetries.

Acknowledgments

NSF support under CCF-1016588 and CCF-1421828 is gratefully acknowledged.

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Appendix A
Additional Examples

The three generators of subgroup of $\mathsf{Aut}(\mathcal{G}_c^*)$ that setwise stabilizes $E_{\mathcal{S}}$ for M-network, written as permutations of edges of circulation graph associated with M-network are:

$$\begin{aligned}
g_1 =&((10,1'),(10,3'))((11,1'),(12,3'))((11,4'),(12,2'))\\
&((13,2'),(13,4'))((1',1),(3',3))((2',2),(4',4))\\
&((1,5),(3,6))((2,5),(4,6))((5,7),(6,9))((5,8),(6,8))\\
&((7,10),(9,10))((7,11),(9,12))((7,12),(9,11))\\
&((7,13),(9,13))((8,11),(8,12))\\
g_2 =&((10,1'),(11,1'))((10,3'),(11,4'))((12,2'),(13,2'))\\
&((12,3'),(13,3'))((3',3),(4',4))((3,6),(4,6))\\
&((7,10),(7,11))((7,12),(7,13))((8,10),(8,11))\\
&((8,12),(8,13))((9,10),(9,11))((9,12),(9,13))\\
g_3 =&((10,1'),(12,2'))((10,3'),(12,3'))((11,1'),(13,2'))\\
&((11,4'),(13,4'))((1',1),(2',2))((1,5),(2,5))\\
&((7,10),(7,12))((7,11),(7,13))((8,10),(8,12))\\
&((8,11),(8,13))((9,10),(9,12))((9,11),(9,13))
\end{aligned} \tag{7}$$

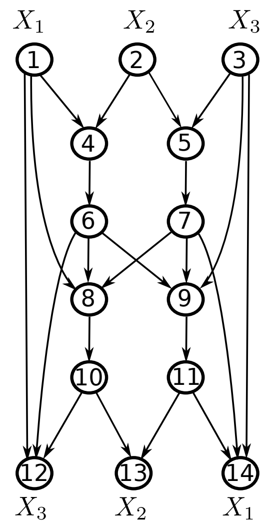

Figure 5: The 3-UNC instance symmFano

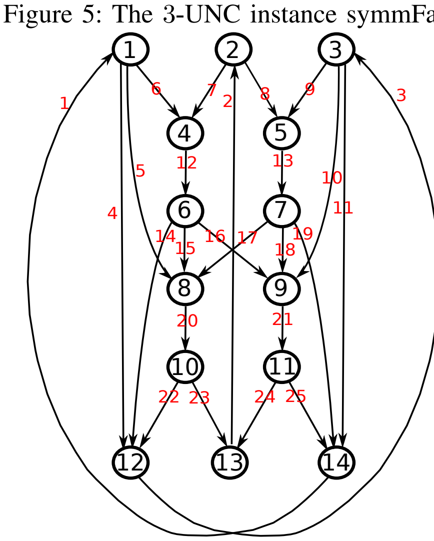

Figure 6: Circulation graph corresponding to symmFano (subscripts of variables associated with edges in red)

The network symmetry group of 3-UNC instance symmFano (a symmetric version of the Fano network) is of order 2 with the only non-trivial automorphism being

$$\begin{aligned}
g_1 =&((12,3),(14,1))((1,4),(3,5))((1,8),(3,9))\\
&((1,12),(3,14))((2,4),(2,5))((4,6),(5,7))\\
&((6,8),(7,9))((6,9),(7,8))((6,12),(7,14))\\
&((8,10),(9,11))((10,12),(11,14))((10,13),(11,13))
\end{aligned} \tag{8}$$

Appendix B
Special cases: I-DSC

The symmetry results in this section pertain to a superclass of MDCS [5], [18] that is obtained when we relax the priorities on sources and restrictions on the access structure. We call these instances *relaxed* MDCS or I-DSC.

Definition 10. *An I-DSC instance is described by the tuple* $(k, \mathcal{E}, \mathcal{D}, \mathcal{R}, \beta)$ *where* k *is the number of source random variables,* $\mathcal{E}, \mathcal{D}$ *are sets indexing the encoders and decoders,* *relation $\mathcal{R} \subseteq \mathcal{E} \times \mathcal{D}$ gives the access structure and $\beta(d) \subseteq [k], d \in \mathcal{D}$ giving decoder demands.*

We assume the sets $[k], \mathcal{E}$ and $\mathcal{D}$ to be disjoint. Now we define the circulation graph for I-DSC. Let $V \triangleq [k] \cup \{s\} \cup \mathcal{E} \cup \mathcal{D}$. Let $E_{\mathcal{S}} \triangleq \{(i,s) \mid i \in [k]\}$ be the source edge set, $E_{\mathcal{E}} \triangleq \{(s,e), e \in \mathcal{E}\}$ be the encoder edge set, $E_{\mathcal{R}} \triangleq \{(e,d) \mid (e,d) \in \mathcal{R}\}$ be the access edge set and $E_{\mathcal{F}} \triangleq \{(d,i) \mid i \in \beta(d), d \in \mathcal{D}\}$ be the feedback edge set. Let $E \triangleq E_{\mathcal{S}} \cup E_{\mathcal{E}} \cup E_{\mathcal{R}} \cup E_{\mathcal{D}}$. The circulation graph for an I-DSC instance is then $\mathcal{G}_c \triangleq (V, E)$. We associate a random variable X_i with edge $(i,s), i \in [k]$ and a random variable U_e with edge $(s,e), e \in \mathcal{E}$. Let $\mathcal{X}_n \triangleq \{X_i \mid i \in [k]\} \cup \{U_e \mid e \in \mathcal{E}\}$. The network constraints in I-DSC are defined in the same manner as the MDCS instances. We classify them into sets $\mathcal{L}_1$, which contains the source independence constraint, $\mathcal{L}_2$, which contains the encoding constraints and the decoding constraints and $\mathcal{L}_3$, which contains the rate constraints. Note that each encoding constraint is associated with an encoder and each decoding constraint is associated with a decoder. The dual circulation graph $\mathcal{G}_c^*$ of an I-DSC instance is the line graph of its circulation graph. In this setup, an automorphism of the dual circulation of graph induces a permutation on $\mathcal{X}_n$. Denote the subgroup of $\mathsf{AUT}(\mathcal{G}_c^*)$ that stabilizes $E_{\mathcal{S}}$ setwise

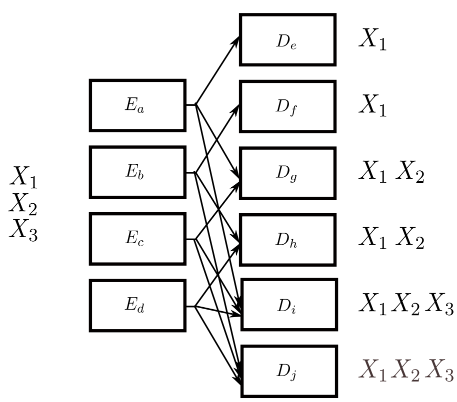

Figure 7: An I-DSC instance $\mathcal{I}$

as $G_{E_{\mathcal{S}}}$. Note that $G_{E_{\mathcal{S}}}$ stabilizes $E_{\mathcal{E}}, E_{\mathcal{R}}$ and $E_{\mathcal{F}}$ setwise. Hence $\sigma(E_{\mathcal{S}} \cup E_{\mathcal{E}}) = E_{\mathcal{S}} \cup E_{\mathcal{E}}$. This, along with the fact that members of $G_{E_{\mathcal{S}}}$ must preserve length 2 directed paths, imply following lemma.

Lemma 3. *Every $\sigma \in G_{E_{\mathcal{S}}}$ induces a permutation on V that stabilizes $\mathcal{E}, \mathcal{D} \subseteq V$ setwise.*

Theorem 5 gives characterization of network symmetry groups

Theorem 5. *The network symmetry group $G^{\mathcal{I}}$ of an I-DSC instance $\mathcal{I}$ is the group of permutations of $\mathcal{X}_n$ induced by the subgroup of $\mathsf{AUT}(\mathcal{G}_c^*)$ that stabilizes the source edge set $E_{\mathcal{S}}$ setwise.*

Proof: Let σ' be a permutation induced on $\mathcal{X}_n$ by a $\sigma \in G_{E_{\mathcal{S}}}$. Set of source random variables is stabilized setwise since $E_{\mathcal{S}}$ is stabilized setwise, fixing the source independence constraint. The set of encoding constraints is stabilized setwise since $E_{\mathcal{E}}$ is stabilized setwise, i.e. an encoding constraint for $U_{e_1}, e_1 \in \mathcal{E}$ maps to encoding constraint for $U_{e_2}, e_2 \in \mathcal{E}$, since σ stabilizes $E_{\mathcal{E}}$ and $E_{\mathcal{S}}$ setwise. A decoding constraint associated with a decoder $d \in \mathcal{D}$ can be written in general form $h_{X \cup Y} = h_X$ where $X \subseteq \mathcal{S}$ and $Y \subseteq \mathcal{E}$ where X is the set of encoders accessed by d and Y is the set of sources demanded by d. Let $X^{\sigma'} = \sigma'(X)$ and $Y^{\sigma'} = \sigma'(Y)$. From lemma 3, $X^{\sigma'}$ is the set of encoders accessed by some decoder $d' \in \mathcal{D}$ while $Y^{\sigma'}$ is the set of sources demanded by d'. Hence $\mathcal{L}_2$ is stabilized setwise by σ'. $\mathcal{L}_3$ is also stabilized setwise by σ' since $E_{\mathcal{S}} \cup E_{\mathcal{E}}$ is stabilized setwise by σ.

Conversely, consider a $\sigma \in G^{\mathcal{I}}$. It induces a permutation on $E_{\mathcal{S}} \cup E_{\mathcal{E}}$ via the association of random variables with edges we described, which we also refer to as σ. Since σ stabilizes $\mathcal{L}_1$ setwise, $\sigma(E_{\mathcal{S}}) = E_{\mathcal{S}}$ and $\sigma(E_{\mathcal{E}}) = E_{\mathcal{E}}$. Furthermore, a permutation on $V_{\mathcal{S}}, V_{\mathcal{E}}$ and $V_{\mathcal{D}}$ is induced via the permutation of sources and the permutation of encoding and decoding constraints respectively that is brought about by σ, which we refer to as σ as well. We explicitly construct an automorphism σ' of $\mathcal{G}_c^*$ that stabilizes $E_{\mathcal{S}}$ setwise and induces σ as follows:

$$\sigma'(u,v) = \begin{cases} \sigma(u,v) & \text{if } (u,v) \in E_S \cup E_{\mathcal{E}} \\ (\sigma(u), \sigma(v)) & \text{if } (u,v) \in E_{\mathcal{R}} \\ (\sigma(u), \sigma(v)) & \text{if } (u,v) \in E_{\mathcal{D}}. \end{cases} \tag{9}$$

First note that $\sigma'(u,v) = (\sigma(u), \sigma(v)) \in E_{\mathcal{R}}$ for $(u,v) \in E_{\mathcal{R}}$, since σ stabilizes the set of encoding constraints setwise i.e. decoder $\sigma(v)$ has access to encoder $\sigma(u)$. Similarly, $\sigma'(u,v) \in E_{\mathcal{D}}$ for $(u,v) \in E_{\mathcal{D}}$. Hence, σ' is indeed a permutation of the set of edges E of $\mathcal{G}_c$. If $\sigma'(d) = d'$ for some $d, d' \in \mathcal{D}$, the length 2 directed paths of type $(s,e,d), e \in \mathcal{E}$ is preserved as $\sigma'(e) = e'$ s.t. $e' \in \beta(d')$ in order to preserve decoding constraint associated with d. Similarly, directed paths of type $(e,d,i), e \in \mathcal{E}, d \in \mathcal{D}, i \in [k]$ are preserved due to preservation of $\mathcal{L}_3$ by σ'. Finally, directed paths of type $(i,s,e), i \in [k], e \in \mathcal{E}$ are preserved by construction. ∎

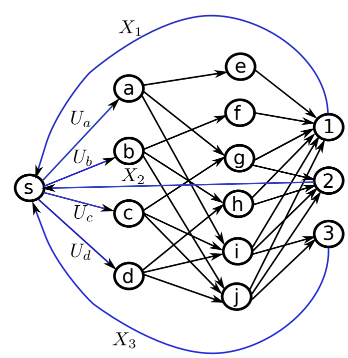

Figure 8: The circulation graph for I-DSC $\mathcal{I}$

Corollary 1. *The network symmetry group $G^{\mathcal{I}}$ of a MDCS instance $\mathcal{I}$ is the group of permutations of $\mathcal{X}_n$ induced by the subgroup of $\mathsf{AUT}(\mathcal{G}_c^*)$ that stabilizes the source edge set $E_{\mathcal{S}}$ pointwise.*

Appendix C
Special cases: regenerating I-DSC

In this section we consider a generalization I-DSC which is a superclass containing several problems in literature concerning regeneration in distributed storage systems such as exact repair (considered by Shah et al. and Tian), multi-source exact repair (Considered by Apte et al), MDC-R (considered by Tian and Liu). Regenerating I-DSC allows arbitrary decoder access and repair access with or without priorities enforced on the sources.

Definition 11. *A regenerating I-DSC instance $\mathcal{I}$ is described by the tuple $(k, \mathcal{E}, \mathcal{D}, \mathcal{D}', \mathcal{R}, \mathcal{R}', \beta, \gamma)$ where k is the number of source random variables, $\mathcal{E}, \mathcal{D}$ and $\mathcal{D}'$ are sets indexing the encoders, decoders and repair decoders, relations $\mathcal{R} \subseteq \mathcal{E} \times \mathcal{D}$ and $\mathcal{R}' \subseteq \mathcal{E} \times \mathcal{D}'$ give the access and repair access structures respectively and $\beta : \mathcal{D} \to 2^{[k]} \setminus \emptyset$ and $\gamma : \mathcal{D}' \to \mathcal{E} \setminus \emptyset$ giving decoder and repair decoder demands.*

Let $\mathcal{R}$ and $\mathcal{R}'$ be available through two functions $f : \mathcal{D} \to 2^{\mathcal{E}} \setminus \emptyset$ and $f' : \mathcal{D}' \to 2^{\mathcal{E}}$. Note that if $\gamma(d') = e \in \mathcal{E}$ then $e \notin f'(d')$ for all $d' \in \mathcal{D}'$. We now define the circulation graph for regenerating I-DSC instance $\mathcal{I}$. Let $V \triangleq [k] \cup \{s\} \cup \mathcal{E} \cup \mathcal{E}' \cup \mathcal{D} \cup \mathcal{D}'$ where $\mathcal{E}' \triangleq \{e' \mid e \in \mathcal{E}\}$. Let $E_{\mathcal{S}} \triangleq \{(i,s) \mid i \in [k]\}$ be the source edge set, $E_{\mathcal{E}'} \triangleq \{(s,e'), e' \in \mathcal{E}'\}$ and $E_{\mathcal{E}} \triangleq \{(e',e) \mid e \in \mathcal{E}\}$ be the two encoder edge sets, $E_{\mathcal{R}} \triangleq \{(e,d) \mid (e,d) \in \mathcal{R}\}$ be the access edge set, $E_{\mathcal{R}'} \triangleq \{(e,d') \mid (e,d') \in \mathcal{R}'\}$ be the repair access edge set, $E_{\mathcal{F}} \triangleq \{(d,i) \mid i \in \beta(d), d \in \mathcal{D}\}$ be the feedback edge set and finally, $E_{\mathcal{F}'} \triangleq \{(d',e') \mid e \in \gamma(d'), d' \in \mathcal{D}'\}$. Let $E \triangleq E_{\mathcal{S}} \cup E_{\mathcal{E}} \cup E_{\mathcal{E}'} \cup E_{\mathcal{R}} \cup E_{\mathcal{R}'} \cup E_{\mathcal{F}} \cup E_{\mathcal{F}'}$. The circulation graph for a regenerating I-DSC instance $\mathcal{I}$ is then $\mathcal{G}_c \triangleq (V, E)$. In a regenerating I-DSC instance, there are k RVs corresponding to the sources, $|\mathcal{E}|$ RVs corresponding to the encoded messages and $|\mathcal{R}'|$ random variables corresponding to repair encodings giving a total of $n = k + |\mathcal{E}| + |\mathcal{R}'|$. The constraints associated with regenerating I-DSC are very similar to those of I-DSC. The set $\mathcal{L}_1$ contains the source independence constraint. The set $\mathcal{L}_2 = \bigcup_{i \in [4]} \mathcal{L}_2^i$, where $\mathcal{L}_2^1$ and $\mathcal{L}_2^2$ contain the encoding and decoding constraints respectively, while sets $\mathcal{L}_2^3$ and $\mathcal{L}_2^4$ contain the repair encoding and repair decoding constraints. Moreover, $|\mathcal{L}_2^1| = |\mathcal{E}|, |\mathcal{L}_2^2| = |\mathcal{D}|, |\mathcal{L}_2^3| = |\mathcal{R}'|$ and $|\mathcal{L}_2^4| = |\mathcal{D}'|$. Finally, the set $\mathcal{L}_3$ contains one rate constraint per random variable. In order to obtain an automorphism group characterization of symmetries, we associate random variables with edges of $\mathcal{G}_c$.
With each edge $(e, d') \in \mathcal{R}'$, we associate a random variable $U_{e,d'}$, with each edge $(i, s') \in E_{\mathcal{S}}$, we associate a random variable X_i and with each edge $(s, e') \in E_{\mathcal{E}}$, we associate a random variable U_e. Let $\mathcal{X}_n$ be set of all random variables associated with the instance. Denote by $G_{E_{\mathcal{S}}}$ The network symmetry group $G^{\mathcal{I}}$ for a regenerating I-DSC instance $\mathcal{I}$ is characterized as follows. A $\sigma \in G_{E_{\mathcal{S}}}$ also stabilizes sets $E_{\mathcal{X}}, \mathcal{X} \in \{\mathcal{E}, \mathcal{R}', \mathcal{E}', \mathcal{F}, \mathcal{R}, \mathcal{F}'\}$ setwise. In this setting, every automorphism of $\mathcal{G}_c^*$ induces a permutation on $\mathcal{X}_n$.

Lemma 4. *Every $\sigma \in G_{E_{\mathcal{S}}}$ induces a permutation on V that stabilizes $\mathcal{E}, \mathcal{E}', \mathcal{D}, \mathcal{D}' \subseteq V$ setwise.*

Theorem 6. *The network symmetry group $G^{\mathcal{I}}$ of a regenerating I-DSC instance $\mathcal{I}$ is the group of permutations of $\mathcal{X}_n$ induced by the subgroup of $\mathsf{AUT}(\mathcal{G}_c^*)$ that stabilizes source edge set $E_{\mathcal{S}}$.*

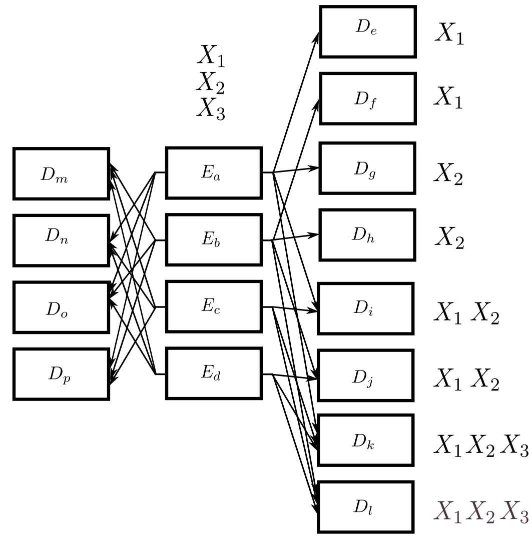

Figure 9: An I-DSC-R instance $\mathcal{I}$

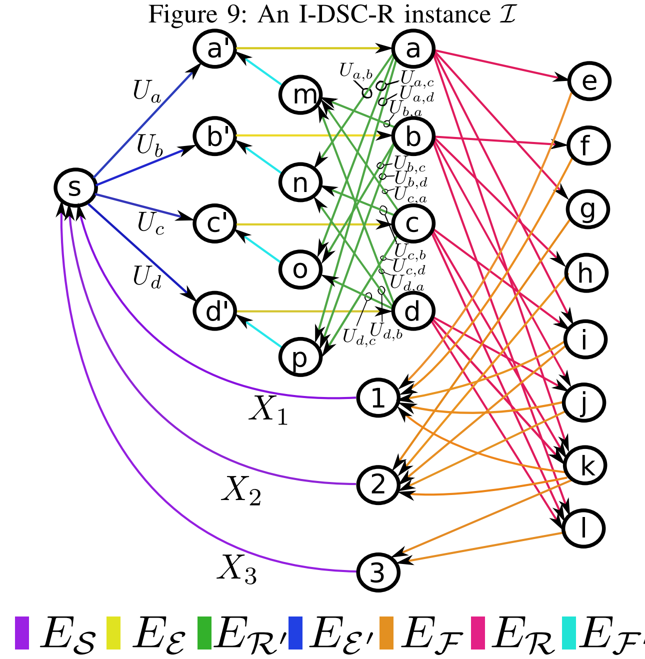

Figure 10: Circulation graph $\mathcal{G}_c$ for I-DSC-R instance $\mathcal{I}$

Proof: Let σ' be the permutation induced on $\mathcal{X}_n$ by $\sigma' \in G_{E_{\mathcal{S}}}$. Set of source random variables is stabilized setwise by σ', since $E_{\mathcal{S}}$ is stabilized setwise, fixing the source independence constraint. The set $\mathcal{L}_2^1$ of encoding constraints is stabilized setwise since sets $E_{\mathcal{S}}$ and $E_{\mathcal{E}}$ are stabilized setwise, i.e. an encoding constraint for $U_{e_1}, e_1 \in \mathcal{E}$ maps to encoding constraint for some $U_{e_2}, e_2 \in \mathcal{E}$. Set $\mathcal{L}_2^2$ of decoding constraints is preserved by σ' since from lemma 4, σ obeys a permutation on decoders $\mathcal{D}$ and if $\sigma(d_1) = d_2$ for $d_1, d_2 \in \mathcal{D}$, then $\sigma'(\mathsf{In}(d_1)) = \mathsf{In}(d_2)$ and $\sigma'(\mathsf{Out}(d_1)) = \mathsf{Out}(d_2)$. Set $\mathcal{L}_2^3$ of repair encoding constraints contains one constraint per RV $U_{x,y}, (x,y) \in \mathcal{R}'$ and is stabilized setwise by σ' since σ preserves length 2 directed paths of type (s, e_1', e_1) and (e_1', e_1, d') where $e_1' \in \mathcal{E}', e_1 \in \mathcal{E}$ and $d' \in \mathcal{D}'$. Set $\mathcal{L}_2^4$ of repair decoding constraints contains one constraint per encoder in $\gamma(D') \subseteq \mathcal{E}$. Note that permutation on $\mathcal{E}$ induced by σ according to lemma 4 also stabilizes $\gamma(\mathcal{D}')$ setwise implying that σ' stabilizes $\mathcal{L}_2^4$ setwise. Finally, $\mathcal{L}_3$ is stabilized setwise as $E_{\mathcal{S}}$, $E_{\mathcal{E}'}$ and $E_{\mathcal{R}'}$ are stabilized setwise by σ'.

Conversely, let $\sigma \in G^{\mathcal{I}}$. Such a σ induces a permutation on $E_{\mathcal{E}}, E_{\mathcal{S}}$ and $E_{\mathcal{R}'}$ based on aforementioned association of random variables with edges $E_{\mathcal{E}}$, $E_{\mathcal{E}'}$ and $E_{\mathcal{S}}$. It also induces a permutation on $\mathcal{E}$ and $\mathcal{E}'$ based on permutation of $\mathcal{L}_2^1$, on $\mathcal{D}, \mathcal{D}'$ based on permutation of $\mathcal{L}_2^2$ and $\mathcal{L}_2^4$ respectively and on $[k]$ based on permutation of source random variables. We refer to these permutations as σ as well. We now construct a $\sigma' \in \mathsf{AUT}(\mathcal{G}_c^*)$ s.t. σ is induced by σ'.

$$\sigma'(u,v) = \begin{cases} \sigma(u,v) & \text{if } (u,v) \in E_S \cup E_{\mathcal{E}'} \cup E_{\mathcal{R}'} \\ (\sigma(u), \sigma(v)) & \text{if } (u,v) \in E_{\mathcal{E}} \cup E_{\mathcal{R}} \cup E_{\mathcal{F}} \cup E_{\mathcal{F}'} \end{cases} \tag{10}$$

Note that for $(u, u') \in E_{\mathcal{E}}$, $\sigma'(u, u') = (\sigma(u), \sigma(u')) = (\sigma(u), \sigma(u)') \in E_{\mathcal{R}}$. For $(u, v) \in E_{\mathcal{R}}$ $(E_{\mathcal{F}}, E_{\mathcal{F}'})$, $\sigma'(u, v) = (\sigma(u), \sigma(v))$ and since σ stabilizes $\mathcal{L}_2^2$ $(\mathcal{L}_2^2, \mathcal{L}_2^4)$ setwise, $(\sigma(e), \sigma(d)) \in \mathcal{R}$ i.e. $(\sigma(e), \sigma(d)) \in E_{\mathcal{R}}$ $(E_{\mathcal{F}}, E_{\mathcal{F}'})$ as well. Hence, σ' is indeed a permutation of E. Let $e_1, e_2 \in \mathcal{E}, e_1', e_2' \in \mathcal{E}', d_1, d_2 \in \mathcal{D}, d_1', d_2' \in \mathcal{D}'$ and $i_1, i_2 \in [k]$. The directed paths of type (s, e_1', e_1) are preserved by construction of σ'. Directed paths of type (e', e, d) are preserved since if $\sigma'(d_1) = d_2$ then $\sigma'(e_1) \in \beta(d_2)$ in order for σ to preserve $\mathcal{L}_2^2$ while $\sigma'(e_1', e_1) \in E_{\mathcal{E}'}$ by construction of σ'. Similarly, directed paths of types (e_1, d_1, i_1) and (d_1, i_1, s) are preserved under σ' as σ preserves $\mathcal{L}_2^2$ whereas directed paths of types (e_1', e_1, d_1') and (e_1, d_1', e_2') are preserved under σ' as σ preserves repair encoding constraint set $\mathcal{L}_2^3$ repair decoding constraint set $\mathcal{L}_2^4$ respectively. Directed paths of type (d_1, e_1', e_1) are preserved by construction. ■

APPENDIX D
GENERALIZED p-SYMMETRICAL ENTROPY FUNCTIONS

In this section we consider several generalizations of p-symmetrical entropy functions of Chen and Yeung [6] in increasing order of generality. Chen and Yeung considered action of special subgroups of S_n which we refer to as p-stabilizer groups on $\mathcal{H}_n$. Roughly speaking, a p-stabilizer group setwise stabilizes each block of partition p. A precise definition can be stated as follows:

Definition 12. *Given a partition p of $\mathcal{X}_n$ with blocks $\mathcal{X}_{n_1}, \ldots, \mathcal{X}_{|p|}$ the p-stabilizer group Σ_p is defined as the subgroup of S_n that contains permutations σ such that if $X_i \in \mathcal{X}_j$ then, $\sigma(X_i) \in \mathcal{X}_j$, for all $i \in [n], j \in [|p|]$*

In fact, Chen and Yeung [6] mention the generalization to action of arbitrary subgroups of S_n on $\mathcal{H}_n$ as a future direction. The following example shows that a network symmetry group need not be a p-stabilizer group correnponding to its orbits p in the set of variables. Hence, such a generalization is in fact warranted.

The orbits of symmetry group $G^{\mathcal{I}}$ of butterfly network considered in section IV form a partition of $\mathcal{X}_n$, namely $\{\{X_1, X_2\}, \{X_3\}, \{X_4, X_5\}, \{X_6, X_7\}, \{X_8, X_9\}\}$. Let's denote this partition as p. Let Σ_p be the associated p-stabilizer group. This group has order 16. Hence, one can see that automorphism groups of network coding instances are more general than p-stabilizer groups. We can associate with each $G^{\mathcal{I}}$ a p-stabilizer group Σ_p. Both $G^{\mathcal{I}}$ and Σ_p act on $\mathcal{H}_n$ by permuting $2^{\mathcal{X}_n} \setminus \emptyset$. With slight abuse of notation, henceforth, we will refer to $2^{\mathcal{X}_n} \setminus \emptyset$ as simply $2^{\mathcal{X}_n}$. The sequence of induced actions of $G^{\mathcal{I}}$ can be written as:

$$\mathsf{Act}_{G^{\mathcal{I}}}(\mathcal{X}_n) \xrightarrow[\text{action}]{\text{induced}} \mathsf{Act}_{G^{\mathcal{I}}}(2^{\mathcal{X}_n}) \xrightarrow[\text{action}]{\text{induced}} \mathsf{Act}_{G^{\mathcal{I}}}(\mathcal{H}_n) \tag{11}$$

and similarly for Σ_p:

$$\mathsf{Act}_{\Sigma_p}(\mathcal{X}_n) \xrightarrow[\text{action}]{\text{induced}} \mathsf{Act}_{\Sigma_p}(2^{\mathcal{X}_n}) \xrightarrow[\text{action}]{\text{induced}} \mathsf{Act}_{\Sigma_p}(\mathcal{H}_n) \tag{12}$$

Let the sets of orbits under their action on $2^{\mathcal{X}_n}$ be denoted as $\mathcal{P}^{\mathcal{I}}$ and $\mathcal{P}$ which also also partitions of $2^{\mathcal{X}_n}$. Number of blocks in $\mathcal{P}$ is denoted as n_p (as defined in [6]) while number of blocks in $\mathcal{P}^{\mathcal{I}}$ is denoted as N_p. One can see that $N_p \geq n_p$ since $\mathcal{P}^{\mathcal{I}}$ is essentially a refinement of $\mathcal{P}$. We denote different blocks of $\mathcal{P}$ and $\mathcal{P}^{\mathcal{I}}$ as $\mathcal{P}_i, i \in [n_p]$ and $\mathcal{P}_i^{\mathcal{I}}, i \in [N_p]$ respectively. Following Chen and Yeung [6] the set of points in $\mathcal{H}_n$ fixed under action of Σ_p is defined as follows.

$$\mathsf{Fix}_{\Sigma_p}(\mathcal{H}_n) = \left\{ \mathbf{h} \in \mathcal{H}_n \,\middle|\, \begin{array}{l} \mathbf{h}(\mathcal{A}) = \mathbf{h}(\mathcal{B}) \text{ if } \mathcal{A}, \mathcal{B} \in \mathcal{P}_i \\ \text{for some } i \in [n_p] \end{array} \right\} \tag{13}$$

Similarly we define set of points in $\mathcal{H}_n$ fixed under action of $G^{\mathcal{I}}$ as follows.

$$\mathsf{Fix}_{G^{\mathcal{I}}}(\mathcal{H}_n) = \left\{ \mathbf{h} \in \mathcal{H}_n \,\middle|\, \begin{array}{l} \mathbf{h}(\mathcal{A}) = \mathbf{h}(\mathcal{B}) \text{ if } \mathcal{A}, \mathcal{B} \in \mathcal{P}_i^{\mathcal{I}} \\ \text{for some } i \in [N_p] \end{array} \right\} \tag{14}$$

$\mathsf{Fix}_{\Sigma_p}(\mathcal{H}_n)$ and $\mathsf{Fix}_{G^{\mathcal{I}}}(\mathcal{H}_n)$ are subspaces of $\mathcal{H}_n$ of dimension n_p and N_p respectively. For automorphism group of butterfly network, we can compute N_p using Burnside Lemma. Let a group G act on set S. Denote by S^g the set of elements in S that are fixed under the action of a specific $g \in G$.

Lemma 5. ***(Burnside)*** *No. of orbits* $= \frac{1}{|G|} \sum_{g \in G} |S^g|$

Considering action of $G^{\mathcal{I}}$ on $2^{\mathcal{X}_n}$ we get,

$$N_p = \frac{1}{2}(511 + 31) = 271 \tag{15}$$

where number of elements in $2^{\mathcal{X}_n}$ fixed under identity permutation is $|2^{\mathcal{X}_9}| = 511$, while $2^5 - 1 = 31$ is the number of elemenst of $2^{\mathcal{X}_n}$ fixed under action of $G^{\mathcal{I}}$ (to see this, note that the only non-empty subsets of $\mathcal{X}_n$ fixed under $(1,2)(3)(4,5)(6,7)(8,9)$ are the ones that can be obtained as union of some collection of blocks in the partition p).

On the other hand, the formula given by Chen and Yeung yields

$$n_p = 3^4 \times 2 = 162 \tag{16}$$

Following lemma establishes containment relationship between subspaces of points in $\mathcal{H}_n$ fixed under network symmetry group $G^{\mathcal{I}}$ and corresponding p-stabilizer group Σ_p

Lemma 6. $\mathsf{Fix}_{\Sigma_p}(\mathcal{H}_n) \subseteq \mathsf{Fix}_{G^{\mathcal{I}}}(\mathcal{H}_n)$

Naturally, corresponding to a network coding instance $\mathcal{I}$, we can define a *$\mathcal{I}$-symmetrical entropy function region* as:

$$\Psi^*_{\mathcal{I}} \triangleq \mathsf{Fix}_{G_{\mathcal{I}}}(\mathcal{H}_n) \cap \Gamma^*_n \tag{17}$$

and a *$\mathcal{I}$-symmetrical polymatroidal region* as:

$$\Psi_{\mathcal{I}} \triangleq \mathsf{Fix}_{G_{\mathcal{I}}}(\mathcal{H}_n) \cap \Gamma_n \tag{18}$$

Using lemma 6 we conclude the following:

Theorem 7. *For any network coding instance $\mathcal{I}$ with symmetry group $G^{\mathcal{I}}$ and associated p-stabilizer group Σ_p,*

1) $\Psi^*_p \subseteq \Psi^*_{\mathcal{I}}$
2) $\Psi_p \subseteq \Psi_{\mathcal{I}}$

A further generalization of p-symmetrical entropy functions can be obtained by considering action of a subgroup of $GL(\mathbb{R}, 2^n - 1)$ on $\mathcal{H}_n$ which is essentially a group of automorphisms of $\mathcal{H}_n$ (i.e. bijective linear transformations $\sigma : H_n \to H_n$). We call symmetries captured by such a group the *geometric symmetries*. Simplest geometric symmetries one can consider are the groups $G \leq O(\mathbb{R}, 2^n - 1) \leq GL(\mathbb{R}, 2^n - 1)$ where $O(\mathbb{R}, 2^n - 1)$ is the group of orthogonal linear transformations of $\mathcal{H}_n$. Elements of G are $2^n - 1 \times 2^n - 1$ permutation matrices. Such a group G acts naturally on $\mathcal{H}_n$ by permuting $2^{\mathcal{X}_n}$ (equivalently, the set of standard basis vectors of $\mathcal{H}_n$). Let $\hat{\mathcal{P}}$ be the set of orbits in $2^{\mathcal{X}_n}$ under action of $G \leq O(\mathbb{R}, 2^n - 1)$ and let $N = |\hat{\mathcal{P}}|$. Similar to equations 13,14 we can define the set of points in $\mathcal{H}_n$ fixed under the action of G as:

$$\mathsf{Fix}_G(\mathcal{H}_n) = \left\{ \mathbf{h} \in \mathcal{H}_n \,\middle|\, \begin{array}{l} \mathbf{h}(\mathcal{A}) = \mathbf{h}(\mathcal{B}) \text{ if } \mathcal{A}, \mathcal{B} \in \hat{\mathcal{P}}_i \\ \text{for some } i \in [N] \end{array} \right\} \tag{19}$$

We can now define the *G-symmetrical entropy function region* as:

$$\Psi^*_G \triangleq \mathsf{Fix}_G(\mathcal{H}_n) \cap \Gamma^*_n \tag{20}$$

and the *G-symmetrical polymatroidal region* as:

$$\Psi_G \triangleq \mathsf{Fix}_G(\mathcal{H}_n) \cap \Gamma_n \tag{21}$$

$G^{\mathcal{I}}$ corresponding to a network coding instance $\mathcal{I}$ and Σ_p can be realized as subgroups of $O(\mathbb{R}, 2^n - 1)$ with the generators being permutation matrices corresponding to permutations of $2^{\mathcal{X}_n}$ they induce.

The last notion of symmetry we consider is specific to polyhedral bounds on $\overline{\Gamma^*_n}$ such as LP outer bound (Γ_n), $\mathbb{F}_q$-representable matroid inner bound (Γ^q_n) and subspace inner bounds ($\Gamma^{\subset space}_{N,k,q}$). These are called the *combinatorial symmetries*. Our exposition of combinatorial symmetries follows Rehn [19]. First we define isomorphism of the face lattice.

Definition 13. *f is a face lattice isomorphism between two face lattices $L(P)$ and $L(Q)$ if f is a bijection of the faces of P to the faces of Q such that for all faces F, G of P, it holds that*

$$F \subseteq G \iff f(G) \subseteq f(G) \tag{22}$$

A *face lattice automorphism* of polyhedron P is a face lattice isomorphism betwen $L(P)$ and itself.

Definition 14. *A combinatorial symmetry of a polyhedron P is an automorphism f of its face lattice $L(P)$*

Set of all combinatorial symmetries forms a group called *combinatorial symmetry group*. Given the inequalities (H-representation) and extreme rays (V-representation) describing a polyhedron P, one can compute its combinatorial symmetry group from incidence relationships between facets and extreme rays [20]. In case of polyhedral bounds on $\overline{\Gamma^*_n}$, we are usually presented with either inequalities or the extreme rays describing polyhedral bound, eg. Shannon outer bound has a redily available inequality description in form of elemental inequalities, see [**?**], while matroid and subspace inner bounds are readily available in extreme ray description from enumeration of connected matroids [21]. The alternative descriptions of both of these bounds (V-representation for Γ_n and H-representation for Γ^q_n) tend to be probhibitively large. e.g. Γ_5 has $117,983$ extreme rays and and only 85 inequalities. Hence one would like to compute at least a subgroup of combinatorial automorphism groups of Γ_n and Γ^q_n while avoiding representation conversion. This purpose is served by the restricted symmetries(see [15], [19]). We first define *restricted isomorphism*. The following terminology is defined for polyhedral cones described in terms of its extreme rays (V-representation). Note that these definitions can be extended to polyhedral cones described in terms of inequalities (H-representation) by considering the associated polar polyhedral cone [22] and to arbitrary polyhedra using homogenization [23]. Denote a polyhedral cone generated by a set of vectors $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}, \mathbf{v}_i \in \mathbb{R}^d, \ \forall i \in [n]$ as $P(V)$,

Definition 15. *A **Restricted Isomorphism** of between two polyhedral cones $P(V)$ and $P(V')$, with $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $V' = \{\mathbf{v}'_1, \ldots, \mathbf{v}'_n\}$ in $\mathbb{R}^d$ is given by a matrix $\mathbf{A} \in GL_d(\mathbb{R})$ sucha that there exists a permutation σ satisfying $\mathbf{A}\mathbf{v}_i = \mathbf{v}'_{\sigma(i)}$ for $i \in [n]$*

A *restricted automorphism* of a polyhedral cone $P(V)$ is a restricted isomorphism between $P(V)$ and itself.

Definition 16. *A restricted symmetry of a polyhedral cone P is a restricted automorphism of P*

All restricted symmetries of a polyhedral cone P form a group with matrix multiplication as operation, called *restricted symmetry group*. Restricted symmetries can be computed by obtaining automorphism group of an appropriately constructed edge-colored graph [15] using software such as SymPol [24]. Using SymPol, we computed restricted symmetry group of Γ_4. It is a group of order 1152 with 8 generators.

Placeholder: Symmetries of Γ^2_n

Placeholder: Inclusion relationships between notions of symmetry

Appendix E
Exploiting Symmetry in Linear Programming

This section briefly discusses symmetries of linear programs and related terminology, as defined by [13], [14] and how to use them to reduce the complexity of solving the linear program. We consider linear programs in following standard form

$$(\Lambda) \qquad \begin{aligned} &max\ \mathbf{c}^t\mathbf{x} \\ &s.t.\ \mathbf{A}\mathbf{x} \leq \mathbf{b},\ x \in \mathbb{R}^d \end{aligned} \tag{23}$$

where $\mathbf{A} \in \mathbb{R}^{m\times d}, \mathbf{b} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ $\mathbf{x} \in \mathbb{R}^d$ is *feasible* if it satisfies all constraints of the LP. LP is *feasible* if it has atleast one feasible point. Set of feasible solutions of an LP is $X \triangleq \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$, which is a polyhedron. A *solution* of an LP is an element $\mathbf{x}^* \in \mathbb{R}^d$ that is feasible and maximizes the cost function. We will restrict our attention to subgroups of $O(d,\mathbb{R})$ whose elements are permutation matrices. A *permutation matrix* is a matrix whose rows are a permutation of rows of the indentity matrix. Denote the group of all $d \times d$ permutation matrices by $\mathsf{Perm}(d)$. A subgroup of $\mathsf{Perm}(d)$ acts naturally on $\mathbb{R}^d$ via matrix vector multiplication. e.g.

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

shows action of a member of $\mathsf{Perm}(3)$ on an element of $\mathbb{R}^3$. Roughly speaking, symmetry of an LP is a group action that preserves both the cost vector and the inequality system (i.e. the underlying polyhedron).

Definition 17. *A symmetry of a matrix $\mathbf{A} \in \mathbb{R}^{m\times d}$ is a matrix $\mathbf{P} \in \mathsf{Perm}(d)$ such that there exists a $\mathbf{P}' \in \mathsf{Perm}(d)$ with $\mathbf{P}'$ with $\mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{A}$*

Definition 18. *A symmetry of an inequality system $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ is a matrix $\mathbf{P} \in \mathsf{Perm}(d)$ that is a symmetry of matrix $\mathbf{A}$ i.e. $\mathbf{P}'\mathbf{A}\mathbf{P} = A$ for some $\mathbf{P}' \in \mathsf{Perm}(m)$ and $\mathbf{P}'\mathbf{b} = \mathbf{b}$*

Symmetry of an inequality system preserves the inequality system i.e. the set of inequalities in the system is fixed setwise. Now we can define symmetry of a linear program.

Definition 19. *A symmetry of a LP Λ is a matrix $\mathbf{P} \in \mathsf{Perm}(d)$ that is a symmetry of its inequality system $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ that preserves the cost vector $\mathbf{c}^t\mathbf{P} = \mathbf{c}^t$*

Set of all symmetries of a linear program forms a subgroup of $\mathsf{Perm}(d)$.

Definition 20. *The full symmetry group of a LP is*

$$G^\Lambda \triangleq \left\{ \mathbf{P} \in \mathsf{Perm}(d) \;\middle|\; \begin{aligned} &\mathbf{c}^t\mathbf{P} = \mathbf{c} \text{ and } \exists \mathbf{P}' \in \mathsf{Perm}(m) \\ &s.t.\ (\mathbf{P}'\mathbf{b} = \mathbf{b} \wedge \mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{A}) \end{aligned} \right\} \tag{24}$$

Let $\mathsf{Fix}_{G^\Lambda}(\mathbb{R}^d)$ be the subset (in fact a subspace) of $\mathbb{R}^d$ of points in $\mathbb{R}^d$ fixed under the action of G^Λ on $\mathbb{R}^d$. The dimension of $\mathsf{Fix}_{G^\Lambda}(\mathbb{R}^d)$ is equal to the number of orbits under the action of G^Λ on the set of standard basis vectors. Let $k \triangleq \dim(\mathsf{FIx}_{G^\Lambda}(\mathbb{R}^d))$. The main result of [14] can be stated as follows:

Theorem 8. ***(Bödi and Herr)*** *For any d-dimensional linear program with full symmetry group G^Λ there exists a solution of Λ in $\mathsf{Fix}_{G^\Lambda}(\mathbb{R}^d)$*

Given the full symmetry group of LP Λ, one can construct a k-dimensional linear program Λ' such that a solution to Λ can be obtained by solving Λ'. Λ' is defined as follows:

$$(\Lambda') \qquad \begin{aligned} &max\ \mathbf{c}^t\tilde{\mathbf{P}}\mathbf{M}_r\mathbf{y} \\ &s.t.\ \mathbf{A}\tilde{\mathbf{P}}\mathbf{M}_r\mathbf{y} \leq \mathbf{b},\ \mathbf{y} \in \mathbb{R}^k_{\geq 0} \end{aligned} \tag{25}$$

$\tilde{\mathbf{P}}$ is a projection matrix, with associated map $f_{\tilde{\mathbf{P}}} : \mathbb{R}^d \to \mathsf{Fix}_{G^\Lambda}(\mathbb{R}^d) : \mathbf{x} \mapsto \tilde{\mathbf{P}}\mathbf{x}$. We will now see how $\tilde{\mathbf{P}}$ is constructed. Let $x_i, \ldots, x_d$ be the variables associated with linear program Λ in equation 23. $\tilde{\mathbf{P}}$ is constructed using orbits under action of G^Λ on set $\{x_1, \ldots, x_d\}$. Let $\mathsf{orb}(i) \triangleq \{j \mid j \in [d] \wedge x_i \text{ and } x_j \text{ are in same orbit}\}$ for any $i \in [d]$. Let $\mathsf{rep}(i)$ be the smallest index in $\mathsf{orb}(i)$. Let R be the set of all representatives. (Note that $|R| = k$)

$$\tilde{p}_{ij} = \begin{cases} 1 & \text{if } j \in R \text{ and } i \in \mathsf{orb}(j) \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

The matrix M_r is constructed as follows:

$$M_r = [\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_k}],\ \mathbf{v}_{i_p} = \sum_{j \in \mathsf{orb}(i_p)} \mathbf{e}_j,\ i_p \in R \tag{27}$$

Following theorem provides a way of obtaining optimal solution $\mathbf{x}^*_{\mathsf{fix}} \in \mathsf{Fix}(\mathbb{R}^d)$ given a solution $\mathbf{y}^*$ of Λ'

Theorem 9. ***(Bödi and Herr)*** *If $\mathbf{y}^*$ is solution of Λ' then $\mathbf{M}_r\mathbf{y}^*$ is solution of Λ*

Another result in [14] pertains to symmetry group of intersection of two inequality systems:

Theorem 10. *Given a symmetry group $G \leq \mathsf{Perm}(d)$ of two inequality systems $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ and $\mathbf{A}\mathbf{x} \leq \mathbf{b}'$, where $\mathbf{A} \in \mathbb{R}^{m\times d}, \mathbf{A}' \in \mathbb{R}^{m'\times d}, \mathbf{b} \in \mathbb{R}^m$, and $\mathbf{b}' \in \mathbb{R}^{m'}$, the group G is also a symmetry group of the inequality system*

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{A}' \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} \mathbf{b} \\ \mathbf{b}' \end{bmatrix} \tag{28}$$