Algorithms for computing network coding rate regions via single element extension of matroids

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Outline

• Motivation
• Matroid bounds on region of entropic vectors
• Non-isomorphic matroid enumeration via matroid extension
• Characterization of matroid bounds upto isomorphism via matroid extension
• Characterization of network coding rate regions upto isomorphism via matroid extension
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The big picture slide

Yan et. al's characterization
\[ R = \Lambda \left( \text{proj}_Y \left( \text{con} \left( \Gamma^* \cap L_{123} \cap L_{45} \right) \right) \right). \]

network \[\rightarrow\] SOFTWARE

Achievable rate region

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Motivation: $\overline{\Gamma^*_N}$ has proven very difficult to characterize for $N>3$

$\overline{\Gamma^*_N}$:

- The region of entropic vectors for $N$ discrete random variables.
- Image of the set $D_N$ of all valid joint probability mass functions on $N$ discrete random variables under a vector valued function that maps each subset marginal to associated Shannon entropy.
- Its closure $\overline{\Gamma^*_N}$ is not polyhedral.
- Rate regions of multi-source multi-sink network coding (MSNC) problems depend on it! (among several other things)
Motivation: Restriction to certain families of codes yields polyhedral inner bounds on $\overline{\Gamma}_N^*$

- We want to characterize these bounds!
- We also want to compute network coding rate regions under these restrictions
Motivation: Generality of matroid extension based approach

- We know that there are networks for which linear coding does not suffice.
- There are entropic vectors that don’t correspond to any scalar or vector linear codes.
- Matroids generalize codes and there are entropic matroids that don’t arise from any code over a finite fields or reals.
Motivation: Generality of matroid extension based approach

There are entropic matroids here! (Outside the scalar/vector linear bounds)
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(Representable) Matroid Inner bound(s)

- Denoted as $\Gamma^q_N$ where $q$ is some prime power

- Conic hull of rank vectors of $\mathbb{F}_q$-representable matroids with ground set size $N$

- For a set $E$ of cardinality $N$ map $r : 2^E \setminus \emptyset \to \mathbb{Z}_{\geq 0}$ is rank function of a matroid if it is non-decreasing, submodular and bounded by cardinality of respective subsets

- $r(E)$ is called rank of matroid

- Rank vector of a matroid is formed by stacking together ranks of $2^N - 1$ non-empty subsets of $E$

- Representability of a matroid $M$ with ground set $E$ of size $N$ and rank $r(E) = m$ over $\mathbb{F}_q$:
  $\exists$ a matrix $A \in \mathbb{F}_q^{m \times N}$ such that
  $\forall B \subseteq E$, $r(B) = \text{rank}(A_{:,B})$, the matrix rank of the columns of indexed by $B$
(Representable) Matroid Inner bound(s)

- Denoted as $\Gamma_N^q$ where $q$ is some prime power
- Conic hull of rank vectors of $\mathbb{F}_q$-representable matroids with ground set size $N$
- For a set $E$ of cardinality $N$ map $r : 2^E \setminus \emptyset \to \mathbb{Z}_{\geq 0}$ is rank function of a matroid if it is non-decreasing, submodular and bounded by cardinality of respective subsets
- $r(E)$ is called rank of matroid
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  $\forall B \subseteq E, r(B) = \text{rank}(A_{:,B})$, the matrix rank of the columns of indexed by $B$
Subspace Inner Bound(s)

- Denoted as $\Gamma_{N,k,q}^{\subset space}$

- A vector $d \in \mathbb{Z}^{2^N - 1}$ is dimension vector of a $N$-subspace arrangement over $\mathbb{F}_q$ if it can be obtained from rank vector of some $\mathbb{F}_q$-representable matroid on $k > N$ elements by linear projection onto ranks of subsets of a $N$-partition of $E$

$$d(P_1) = r(\{1,2\}) \quad d(P_2) = r(\{3\}) \quad d(P_N) = r(\{k-1,k\})$$

$$d(\{P_1, P_2\}) = r(\{1,2\} \cup \{3\})$$
Subspace Inner Bound(s)

- Denoted as $\Gamma_{N,k,q}^{\subseteq space}$

- A vector $d \in \mathbb{Z}^{2^N-1}$ is dimension vector of a $N$-subspace arrangement over $\mathbb{F}_q$ if it can be obtained from rank vector of some $\mathbb{F}_q$-representable matroid on $k > N$ elements by linear projection onto ranks of subsets of a $N$-partition of $E$

- Extreme rays of $\Gamma_{N,k,q}^{\subseteq space}$ can be obtained from those of $\Gamma_k^q$ by linear projection
Polyhedral Inner Bounds on rate region via Yan and Yeung's characterization

\[ \mathcal{R}^{in} = \Lambda \left( \text{proj}_{Y_S} \left( \Gamma_N^{\text{poly}} \cap \mathcal{L}_{12345} \right) \right). \]
Polyhedral Inner Bounds on rate region via Yan and Yeung's characterization

Computing \( \Gamma^\text{poly}_N \cap L_{12345} \) boils down to membership checking! [4]

Arbitrary linear Constraint

Network Coding Constraint
Polyhedral Inner Bounds on rate region via Yan and Yeung's characterization

A matroid-network mapping is a surjective map $\gamma : E(M) \to \mathcal{V}$, and its inverse image $\gamma^{-1} : \mathcal{V} \to 2^{E(M)} \setminus \phi$ is a network-matroid mapping.
Polyhedral Inner Bounds on rate region via Yan and Yeung's characterization

A matroid $M$ with rank function $r$ and a matroid-network mapping $\gamma$ form a **feasible linear network code** if $r(\gamma^{-1}(\cdot))$ satisfies all network constraints $\mathcal{L}_{12345}$ imposed on the entropy $h(\cdot)$ of subsets of random variables by the network.
Number of non-isomorphic matroids in various polyhedral bounds

Without non-isomorphism, this number will be potentially multiplied by at most $N!$ to consider all isomorphic copies.
From naïve algorithms to less naïve ones

- Polymatroids
- All matroids
- Almost entropic matroids
- Entropic matroids
- GF(q) representable matroids
- Matroids over GF(q) satisfying given network constraints with some N-partition

Try and list these directly!
Single element extensions of a matroid

- If matroid $M$ is obtained from matroid $N$ by deleting a single element from $E(N)$, then $N$ is said to be single element extension of $M$.

- Deletion is matroid theory analog of shortening of a code.

- Single element extension can be thought of as adding a column to a matrix or adding an edge to graph.

- Finding all non-isomorphic extensions of a matroid on $n - 1$ elements means finding all non-isomorphic matroids on $n$ elements such that the given matroid can be obtained via deletion of a single element from each of them.
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Blackburn, Crapo and Higgs' Algorithm [8]

Non-isomorphic matroids on n-1 elements

unique single element extensions

Non-isomorphic matroids on n elements

Isomorphism Testing
Matsumoto et al.'s Algorithm [7]

Non-isomorphic (canonical) matroids on n-1 elements

Canonical matroids only

Non-isomorphic (canonical) matroids on n elements
Using canonical extensions as a workhorse

- Canonical extensions introduced by Matsumoto et al. provide a means to produce all size $n$ matroids from size $n-1$ matroids.

- We abbreviate this procedure as $\text{see}(\cdot)$ which takes a set of canonical matroids as input and produces their canonical extensions.
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(P1) List all non-isomorphic matroids forming $\Gamma_q^N$

Start with lists of size $n-1$ canonical matroids representable over finite field of size $q$

**Input**: List $\mathcal{M}_{n-1}$ of canonical $\mathbb{F}_q$-representable matroids on $n - 1$ elements

**Output**: List $\mathcal{M}_n$ of canonical $\mathbb{F}_q$-representable matroids on $n$ elements

1. $\mathcal{M}_n' \leftarrow \text{see}(\mathcal{M}_{n-1})$
2. $\mathcal{M}_n \leftarrow \emptyset$
3. **foreach** $M \in \mathcal{M}_n'$ **do**
   4. **if** $M$ has no $\mathbb{F}_q$ forbidden minors **then**
   5. $\mathcal{M}_n \leftarrow \mathcal{M}_n \cup \{M\}$
   6. **end**
7. **end**
8. **return** $\mathcal{M}_n$
(P1) List all non-isomorphic matroids forming $\Gamma^q_N$

Start with lists of size $n-1$ canonical matroids representable over finite field of size $q$

Obtain canonical single element extensions

Input: List $\mathcal{M}_{n-1}$ of canonical $\mathbb{F}_q$-representable matroids on $n - 1$ elements

Output: List $\mathcal{M}_n$ of canonical $\mathbb{F}_q$-representable matroids on $n$ elements

```python
1 $\mathcal{M}_n' \leftarrow \text{see}(\mathcal{M}_{n-1})$
2 $\mathcal{M}_n \leftarrow \emptyset$
3 foreach $M \in \mathcal{M}_n'$ do
4   if $M$ has no $\mathbb{F}_q$ forbidden minors then
5     $\mathcal{M}_n \leftarrow \mathcal{M}_n \cup \{M\}$
6   end
7 end
8 return $\mathcal{M}_n$
```
(P1) List all non-isomorphic matroids forming $\Gamma^q_N$

Start with lists of size $n-1$ canonical matroids representable over finite field of size $q$

Obtain canonical single element extensions

Reject the matroids having $\mathbb{F}_q$ forbidden minors ($q=2,3,4$)

**Input:** List $\mathcal{M}_{n-1}$ of canonical $\mathbb{F}_q$-representable matroids on $n-1$ elements

**Output:** List $\mathcal{M}_n$ of canonical $\mathbb{F}_q$-representable matroids on $n$ elements

1. $\mathcal{M}_n' \leftarrow \text{see}(\mathcal{M}_{n-1})$
2. $\mathcal{M}_n \leftarrow \phi$
3. foreach $M \in \mathcal{M}_n'$ do
   4. if $M$ has no $\mathbb{F}_q$ forbidden minors then
   5. \hspace{1em} $\mathcal{M}_n \leftarrow \mathcal{M}_n \cup \{M\}$
   6. end
4. end
7. return $\mathcal{M}_n$
(P1) List all non-isomorphic matroids forming $\Gamma^q_N$

Start with lists of size $n-1$ canonical matroids representable over finite field of size $q$

Obtain canonical single element extensions

Reject the matroids having $\mathbb{F}_q$ forbidden minors ($q=2,3,4$)

**Why does this work?**

If a matroid $M$ has a minor $K$ then all its extensions have Minor $K$.

Hence, if matroid $M$ is not representable over finite field of size $q$, so are its extensions

**Input:** List $\mathcal{M}_{n-1}$ of canonical $\mathbb{F}_q$-representable matroids on $n - 1$ elements

**Output:** List $\mathcal{M}_n$ of canonical $\mathbb{F}_q$-representable matroids on $n$ elements

1. $\mathcal{M}'_n \leftarrow \text{see}(\mathcal{M}_{n-1})$
2. $\mathcal{M}_n \leftarrow \phi$
3. foreach $M \in \mathcal{M}'_n$ do
   4. if $M$ has no $\mathbb{F}_q$ forbidden minors then
   5. $\mathcal{M}_n \leftarrow \mathcal{M}_n \cup \{M\}$
   6. end
4. end
7. return $\mathcal{M}_n$
(P2) List all non-isomorphic extreme rays of $\Gamma_{N}^{q}$
(P2) List all non-isomorphic extreme rays of $\Gamma^q_N$

**Theorem** (Li et al. [3]) Extreme rays of $\Gamma^q_N$ are rank vectors of connected matroids
(P2) List all non-isomorphic extreme rays of $\Gamma^q_N$

**Theorem** (Li et al.) Extreme rays of $\Gamma^q_N$ are rank vectors of connected matroids

- Connectivity of matroids generalizes 2-connectivity of graphs

- **Definition:** A matroid $M$ on ground set $E$ is said to be connected if for every $U \subset E$,

$$r_M(U) + r_M(E \setminus U) > r_M(E)$$
(P2) List all non-isomorphic extreme rays of $\Gamma^q_N$

Theorem (Li et al.) Extreme rays of $\Gamma^q_N$ are rank vectors of connected matroids

- Connectivity of matroids generalizes 2-connectivity of graphs

- Definition: A matroid $M$ on ground set $E$ is said to be connected if for every $U \subset E$,

$$r_M(U) + r_M(E \setminus U) > r_M(E)$$

- Lemma: Single element extensions of connected matroids are also connected
(P2) List all non-isomorphic extreme rays of $\Gamma_{N}^q$
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(P2) List all non-isomorphic extreme rays of $\Gamma_N^q$

Ground set size $n-1$  
Ground set size $n$

Connected matroid whose every single element deletion is not connected
(P2) List all non-isomorphic extreme rays of $\Gamma^q_N$

Example: Uniform matroid $U_{n-1}^n$ i.e. a matroid on ground set $E$ of size $n$ such that $r(S) = |S|$, $\forall S \subseteq E$ and $r(E) = n - 1$
(P2) List all non-isomorphic extreme rays of $\Gamma_N^q$.
(P2) List all non-isomorphic extreme rays of $\Gamma_{N}^q$

- **Theorem:** A matroid is connected iff its dual is connected.

- **Theorem:** If $M$ is a connected matroid then, for every $e \in E(M)$ at least one of $M \setminus e$ or $M/e$ are connected.
(P2) List all non-isomorphic extreme rays of $\Gamma^q_N$ 

- **Theorem:** A matroid is connected iff its dual is connected.

- **Theorem:** If $M$ is a connected matroid then, for every $e \in E(M)$ at least one of $M \setminus e$ or $M/e$ are connected.

Ground set size $n-1$  
Ground set size $n$ 

Original  

Dual  

Single element shortening  
Single element puncturing
(P2) List all non-isomorphic extreme rays of $\Gamma^q_N$

- **Theorem:** A matroid is connected iff its dual is connected.

- **Theorem:** If $M$ is a connected matroid then, for every $e \in E(M)$ at least one of $M \setminus e$ or $M/e$ are connected.

**Theorem:** Dual of rank $r$ unreachable connected matroid can be obtained via single element extension of some rank $n - r$ matroid or rank $n - r - 1$ matroid on $n - 1$ elements.
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(P3) List all non-isomorphic matroids forming bounds \( \Gamma^q_N \cap \mathcal{L}_{12345} \) and \( \Gamma^{\subseteq \text{space}}_{N,k,q} \cap \mathcal{L}_{12345} \)

- **Definition**: Partial matroid-network mapping \( \xi \) from the ground set of matroid \( M \) to set of network variables \( \mathcal{V} \) is a not necessarily surjective map \( \xi : E(M) \rightarrow \mathcal{V} \cup \nu_{\phi} \)
(P3) List all non-isomorphic matroids forming bounds $\Gamma_N^q \cap \mathcal{L}_{12345}$ and $\Gamma_{N,k,q}^{\subset space} \cap \mathcal{L}_{12345}$

- **Definition:** Partial matroid-network mapping $\xi$ from the ground set of matroid $M$ to set of network variables $\mathcal{V}$ is a not necessarily surjective map $\xi : E(M) \rightarrow \mathcal{V} \cup v_\phi$

- $v_\phi$ is the empty variable

- Call partial matroid-network mapping as $p$-map.

- Call a variable $v \in \mathcal{V}$ defined wrt a $p$-map $\xi$ if $\exists e \text{ s.t. } \xi(e) = v$
(P3) List all non-isomorphic matroids forming bounds 
\( \Gamma_N^q \cap \mathcal{L}_{12345} \) and \( \Gamma_{N,k,q}^{\subseteq \text{space}} \cap \mathcal{L}_{12345} \)

- **Definition:** Partial matroid-network mapping \( \xi \) from the ground set of matroid \( M \) to set of network variables \( \mathcal{V} \) is a not necessarily surjective map \( \xi : E(M) \to \mathcal{V} \cup \nu_\phi \)

- \( \nu_\phi \) is the empty variable

- Call partial matroid-network mapping as \( p \)-map.

- Call a variable \( \nu \in \mathcal{V} \) defined wrt a \( p \)-map \( \xi \) if \( \exists e \text{ s.t. } \xi(e) = \nu \)

- **Definition:** A matroid \( M \) with a \( p \)-map \( \xi \) is a Partially Feasible Linear Network Code (\( p \)-code) if \( r_M(\xi^{-1}(\cdot)) \) satisfies all network constraints imposed on the entropy \( h(\cdot) \) of subsets of defined random variables by the network.
(P3) List all non-isomorphic matroids forming bounds \( \Gamma^q_N \cap \mathcal{L}_{12345} \) and \( \mathcal{C}^{\text{space}}_{N,k,q} \cap \mathcal{L}_{12345} \)

The goal is to produce all canonical matroids of ground set size up to \( N^* \) that together with some surjective matroid network map form feasible linear network codes for a given n/w.

**Input:** A \( N \)-variable MSNC problem and maximum ground set size \( N^* \)

**Output:** All pairs \((M, \xi)\) such that \( M \) is a non-isomorphic (canonical) matroid with \( N \leq E(M) \leq N^* \) and \((M, \xi)\) is a feasible linear network code

1. \( \mathcal{M}^0 \leftarrow \{\text{matroid } M^\emptyset \text{ on empty set}\} \)
2. \( Q^M \leftarrow \{\text{empty } p\text{-map}\} \)
3. \( \text{for } i \in \{1, \ldots, N^*\} \text{ do} \)
4. \( \mathcal{M}^i \leftarrow \text{see} (\mathcal{M}^{i-1}, q) \)
5. \( \text{foreach } M \in \mathcal{M}^i \text{ do} \)
6. \( M' \leftarrow \text{parent matroid of } M \)
7. \( Q^M \leftarrow \text{extendmaps}(M, M', Q^{M'}) \)
8. \( \text{if } Q^M \text{ is empty then} \)
9. \( \quad \mathcal{M}^i \leftarrow \mathcal{M}^i \setminus M \)
10. \( \text{end} \)
11. \( \text{foreach } \xi \in Q^M \text{ do} \)
12. \( \quad \text{if } (M, \xi) \text{ is a feasible linear network code then} \)
13. \( \quad \quad \text{Output } (M, \xi) \)
14. \( \quad \text{end} \)
15. \( \text{end} \)
16. \( \text{end} \)
17. \( \text{return} \)

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(P3) List all non-isomorphic matroids forming bounds 
\[ \Gamma_N^q \cap \mathcal{L}_{12345} \text{ and } \Gamma_{N,k,q}^{\subseteq \text{space}} \cap \mathcal{L}_{12345} \]

The goal is to produce all canonical matroids of ground set size up to \( N^* \) that together with some surjective matroid network map form feasible linear network codes for a given n/w

Start with matroid on empty ground Set and an empty p-map

**Input:** A \( N \)-variable MSNC problem and maximum ground set size \( N^* \)

**Output:** All pairs \( (M, \xi) \) such that \( M \) is a non-isomorphic (canonical) matroid with \( N \leq E(M) \leq N^* \) and \( (M, \xi) \) is a feasible linear network code

```plaintext
1. \( M^0 \leftarrow \{ \text{matroid } M^\emptyset \text{ on empty set} \} \)
2. \( Q^M \leftarrow \{ \text{empty p-map} \} \)
3. for \( i \in \{1, \ldots, N^*\} \) do
   4. \( M^i \leftarrow \text{see}(M^{i-1}, q) \)
   5. foreach \( M \in M^i \) do
      6. \( M' \leftarrow \text{parent matroid of } M \)
      7. \( Q^M \leftarrow \text{extendpmaps}(M, M', Q^{M'}) \)
      8. if \( Q^M \) is empty then
         9. \( M^i \leftarrow M^i \setminus M \)
      end
   endforeach
   11. foreach \( \xi \in Q^M \) do
      12. if \( (M, \xi) \) is a feasible linear network code then
          13. Output \( (M, \xi) \)
      end
   end
   15. end
17. return
```

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(P3) List all non-isomorphic matroids forming bounds $\Gamma_{N}^{q} \cap \mathcal{L}_{12345}$ and $\Gamma_{N,k,q}^{\subseteq space} \cap \mathcal{L}_{12345}$

The goal is to produce all canonical matroids of ground set size up to $N^*$ that together with some surjective matroid network map form feasible linear network codes for a given n/w

Start with matroid on empty ground
Set and an empty p-map

Extend matroids ensuring representability

\begin{verbatim}
Input: A $N$-variable MSNC problem and maximum ground set size $N^*$
Output: All pairs $(M, \xi)$ such that $M$ is a non-isomorphic (canonical) matroid with $N \leq E(M) \leq N^*$ and $(M, \xi)$ is a feasible linear network code

1. $\mathcal{M}^0 \leftarrow \{\text{matroid } M^\emptyset \text{ on empty set}\}$
2. $Q^M \leftarrow \{\text{ empty p-map}\}$
3. for $i \in \{1, \ldots, N^*\}$ do
   4. $\mathcal{M}^i \leftarrow \text{see}(\mathcal{M}^{i-1}, q)$
   5. foreach $M \in \mathcal{M}^i$ do
      6. $M' \leftarrow \text{parent matroid of } M$
      7. $Q^M \leftarrow \text{extendpmaps}(M, M', Q^{M'})$
      8. if $Q^M$ is empty then
         9. $\mathcal{M}^i \leftarrow \mathcal{M}^i \setminus M$
     end
   10. foreach $\xi \in Q^M$ do
      11. if $(M, \xi)$ is a feasible linear network code then
          12. Output $(M, \xi)$
      end
     end
   end
end
return
\end{verbatim}

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(P3) List all non-isomorphic matroids forming bounds \( \Gamma^q_N \cap \mathcal{L}_{12345} \) and \( \Gamma^{\subset space}_{N,k,q} \cap \mathcal{L}_{12345} \)

The goal is to produce all canonical matroids of ground set size up to \( N^* \) that together with some surjective matroid network map form feasible linear network codes for a given n/w

Start with matroid on empty ground
Set and an empty p-map

Extend matroids ensuring representability

Obtain p-codes for extension from p-codes of parent

```
Input: A \( N \)-variable MSNC problem and maximum ground set size \( N^* \)
Output: All pairs \((M, \xi)\) such that \( M \) is a non-isomorphic (canonical) matroid with \( N \leq E(M) \leq N^* \) and \((M, \xi)\) is a feasible linear network code

1. \( \mathcal{M}^0 \leftarrow \{ \text{matroid } M^\emptyset \text{ on empty set} \} \)
2. \( Q^M \leftarrow \{ \text{empty p-map} \} \)
3. for \( i \in \{1, \ldots, N^*\} \) do
   4. \( \mathcal{M}^i \leftarrow \text{see}(\mathcal{M}^{i-1}, q) \)
   5. foreach \( M \in \mathcal{M}^i \) do
      6. \( M' \leftarrow \text{parent matroid of } M \)
      7. \( Q^M \leftarrow \text{extendpmaps}(M, M', Q^{M'}) \)
      8. if \( Q^M \text{ is empty} \) then
         9. \( \mathcal{M}^i \leftarrow \mathcal{M}^i \setminus M \)
      end
   endforeach
   11. foreach \( \xi \in Q^M \) do
      12. if \((M, \xi)\) is a feasible linear network code then
         13. Output \((M, \xi)\)
      end
   end
   16. end
17. return
```
(P3) List all non-isomorphic matroids forming bounds
\[ \Gamma^q_N \cap \mathcal{L}_{12345} \text{ and } \Gamma^\subset_{N,k,q} \cap \mathcal{L}_{12345} \]

The goal is to produce all canonical matroids of ground set size up to \( N^* \) that together with some surjective matroid network map form feasible linear network codes for a given n/w

Start with matroid on empty ground
Set and an empty p-map
Extend matroids ensuring representability
Obtain p-codes for extension from p-codes of parent
Remove matroids without any feasible p-codes

\begin{verbatim}
\textbf{Input:} A \( N \)-variable MSNC problem and maximum ground set size \( N^* \)
\textbf{Output:} All pairs \((M, \xi)\) such that \( M \) is a non-isomorphic (canonical) matroid with \( N \leq E(M) \leq N^* \) and \((M, \xi)\) is a feasible linear network code

1. \( \mathcal{M}^0 \leftarrow \{ \text{matroid } M^\emptyset \text{ on empty set} \} \)
2. \( Q^M \leftarrow \{ \text{ empty p-map} \} \)
3. for \( i \in \{1, \ldots, N^*\} \) do
   4. \( \mathcal{M}^i \leftarrow \text{see}(\mathcal{M}^{i-1}, q) \)
   5. foreach \( M \in \mathcal{M}^i \) do
      6. \( M' \leftarrow \text{parent matroid of } M \)
      7. \( Q^M \leftarrow \text{extendmaps}(M, M', Q^{M'}) \)
      8. if \( Q^M \) is empty then
         9. \( \mathcal{M}^i \leftarrow \mathcal{M}^i \setminus M \)
   end
   10. foreach \( \xi \in Q^M \) do
      11. if \((M, \xi)\) is a feasible linear network code then
          12. Output \((M, \xi)\)
      end
   end
end
return
\end{verbatim}
(P3) List all non-isomorphic matroids forming bounds
\[ \Gamma^q_N \cap \mathcal{L}_{12345} \] and \[ \Gamma^{\text{space}}_{N,k,q} \cap \mathcal{L}_{12345} \]

The goal is to produce all canonical matroids of ground set size up to \( N^* \) that together with some surjective matroid network map form feasible linear network codes for a given n/w

Start with matroid on empty ground
Set and an empty p-map

Extend matroids ensuring representability

Obtain p-codes for extension from p-codes of parent

Remove matroids without any feasible p-codes

Output p-codes that map surjectively to \( \mathcal{V} \)

**Input:** A \( N \)-variable MSNC problem and maximum ground set size \( N^* \)

**Output:** All pairs \((M, \xi)\) such that \( M \) is a non-isomorphic (canonical) matroid with \( N \leq E(M) \leq N^* \) and \((M, \xi)\) is a feasible linear network code

1. \( \mathcal{M}^0 \leftarrow \{\text{matroid } M^\emptyset \text{ on empty set}\} \)
2. \( Q^M \leftarrow \{ \text{empty p-map} \} \)
3. for \( i \in \{1, \ldots, N^*\} \) do
   - \( \mathcal{M}^i \leftarrow \text{see}(\mathcal{M}^{i-1}, q) \)
   - foreach \( M \in \mathcal{M}^i \) do
     - \( M' \leftarrow \text{parent matroid of } M \)
     - \( Q^M \leftarrow \text{extendpmaps}(M, M', Q^{M'}) \)
     - if \( Q^M \) is empty then
       - \( \mathcal{M}^i \leftarrow \mathcal{M}^i \setminus M \)
     - end
   - foreach \( \xi \in Q^M \) do
     - if \((M, \xi)\) is a feasible linear network code then
       - Output \((M, \xi)\)
     - end
   - end
- end
- return
Open Problems/Future work

- Enumerate only the extreme rays of $\Gamma^q_N \cap \mathcal{L}_{12345}$ and $\Gamma^{space}_{N,k,q} \cap \mathcal{L}_{12345}$
- Find all almost entropic algebraic matroids satisfying network constraints of a given network
- Are there other minor closed families of almost entropic matroids?
- Nontrivial bound on number of non-isomorphic extensions of a matroid?*
- Complexity of computing all non-isomorphic extensions of a matroid*
- Complexity of computing all non-isomorphic extensions of a GF(q) representable matroid that are also representable over GF(q)*

*Thanks to Dr. Rudi Pendavingh (TU\(e\)) for letting me know that these problems are in fact open
Thanks for listening!

(or for coming here just in case I put you to sleep)
References


