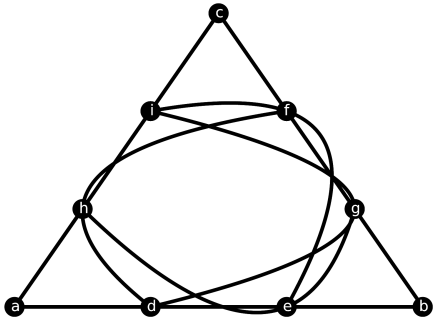
Algorithms for computing network coding rate regions via single element extension of matroids



Jayant Apte*, Congduan Li,
John MacLaren Walsh
ECE Dept. Drexel University

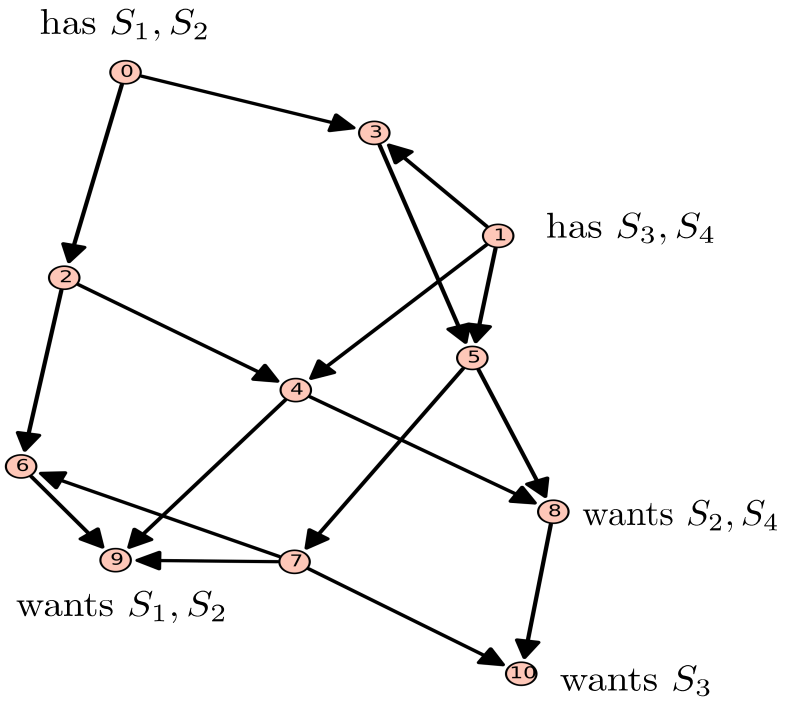
Outline

- Motivation
- Matroid bounds on region of entropic vectors
- Non-isomorphic matroid enumeration via matroid extension
- Characterization of matroid bounds upto isomorphism via matroid extension
- Characterization of network coding rate regions upto isomorphism via matroid extension

Outline

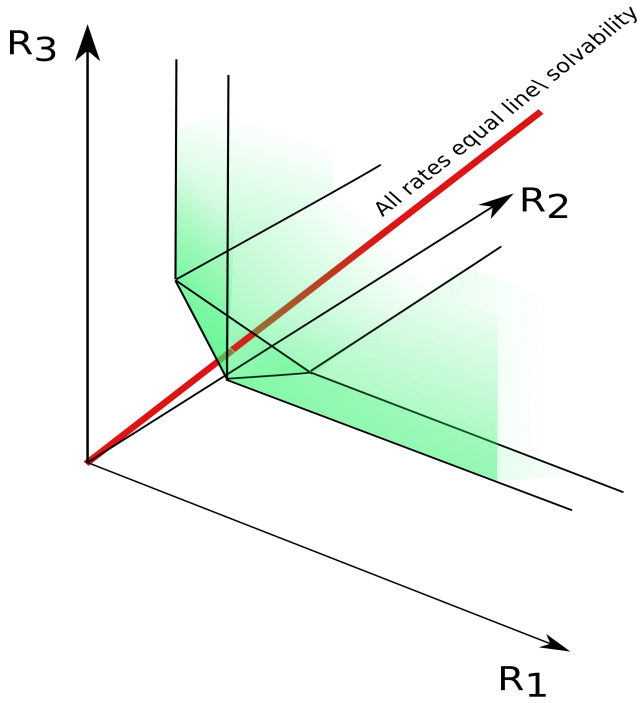
- Motivation
- Matroid bounds on region of entropic vectors
- Non-isomorphic matroid enumeration via matroid extension
- Characterization of matroid bounds upto isomorphism via matroid extension
- Characterization of network coding rate regions upto isomorphism via matroid extension

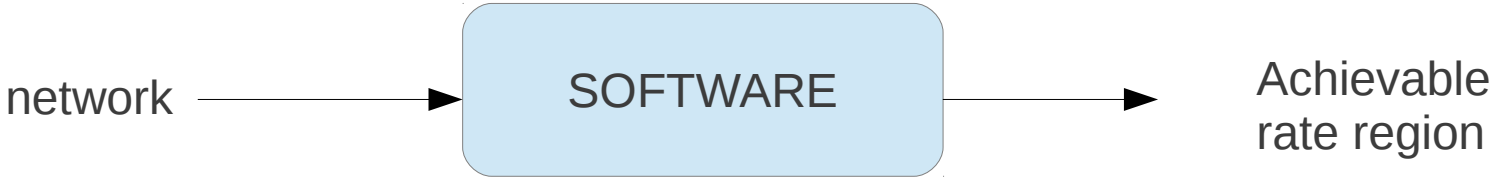
The big picture slide



Yan et. al's characterization

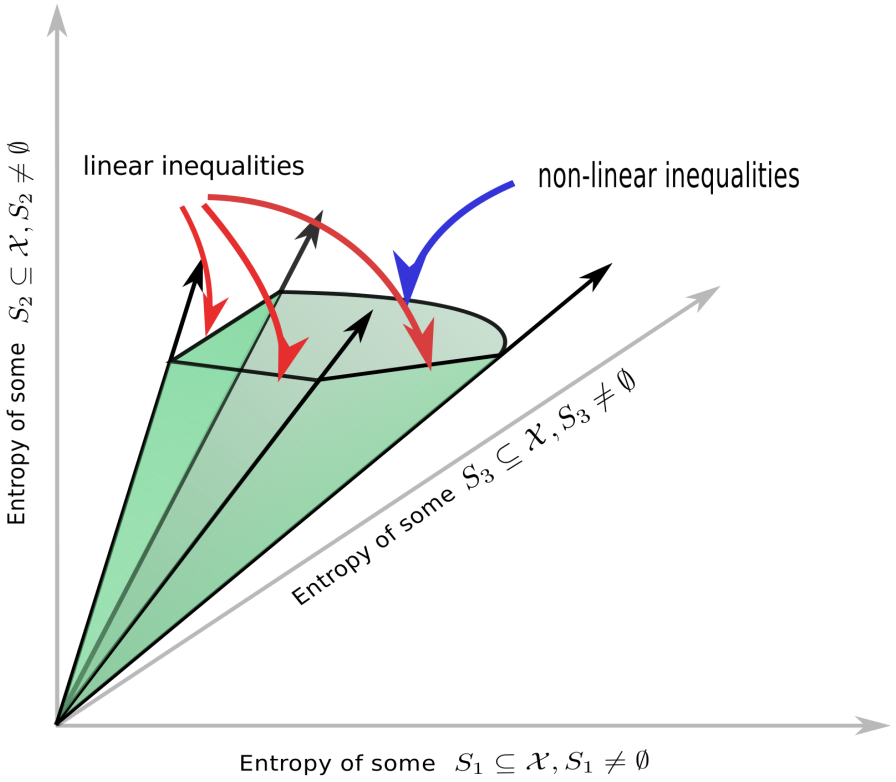
$$\mathcal{R} = \Lambda\left(\text{proj}_{Y_S}\left(\overline{\text{con}\left(\Gamma_N^* \cap \mathcal{L}_{123}\right)} \cap \mathcal{L}_{45}\right)\right).$$





Motivation: $\overline{\Gamma_N^*}$ has proven very difficult to characterize for N>3

A 3-D rendition of $\overline{\Gamma_N^*}$



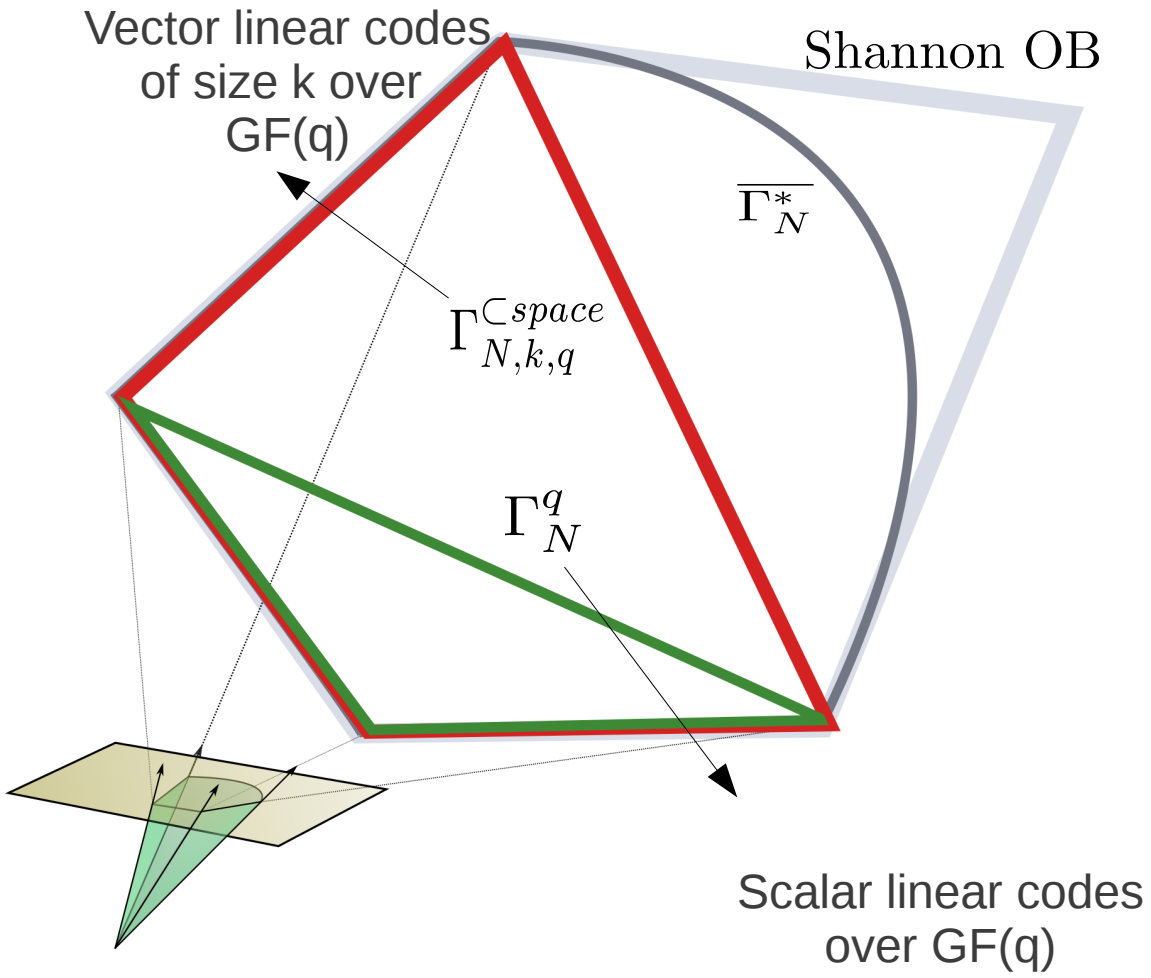
(Not yet completely characterized)

$\boldsymbol{\Gamma_N^*}$:

- The region of entropic vectors for N discrete random variables.
- Image of the set $\mathcal{D}_N$ of all valid joint probability mass functions on N discrete random variables under a vector valued function that maps each subset marginal to associated Shannon entropy
- Its closure $\overline{\Gamma_N^*}$ is not polyhedral
- Rate regions of multi-source multi-sink network coding (MSNC) problems depend on it! (among several other things)

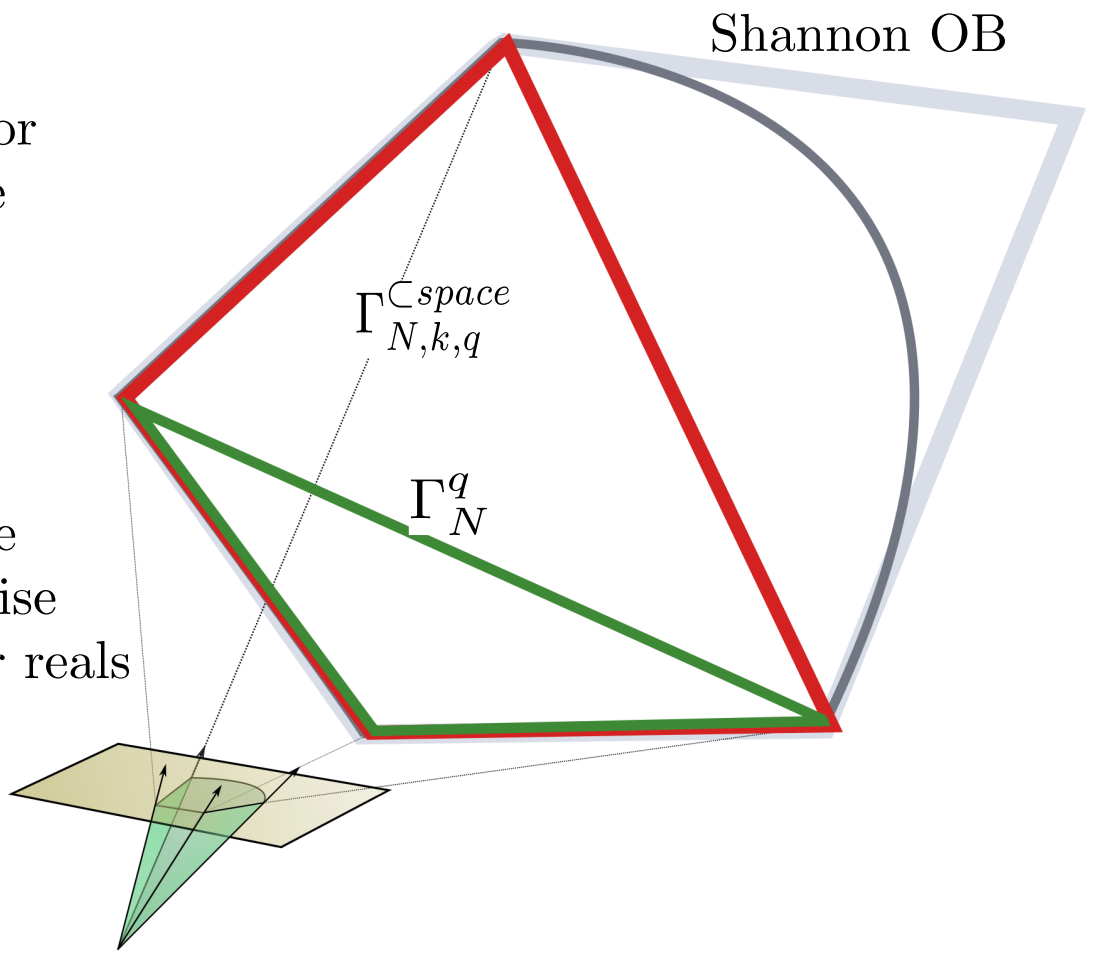
Motivation: Restriction to certain families of codes yields polyhedral inner bounds on $\overline{\Gamma_N^*}$

- We want to characterize these bounds!
- We also want to compute network coding rate regions under these restrictions



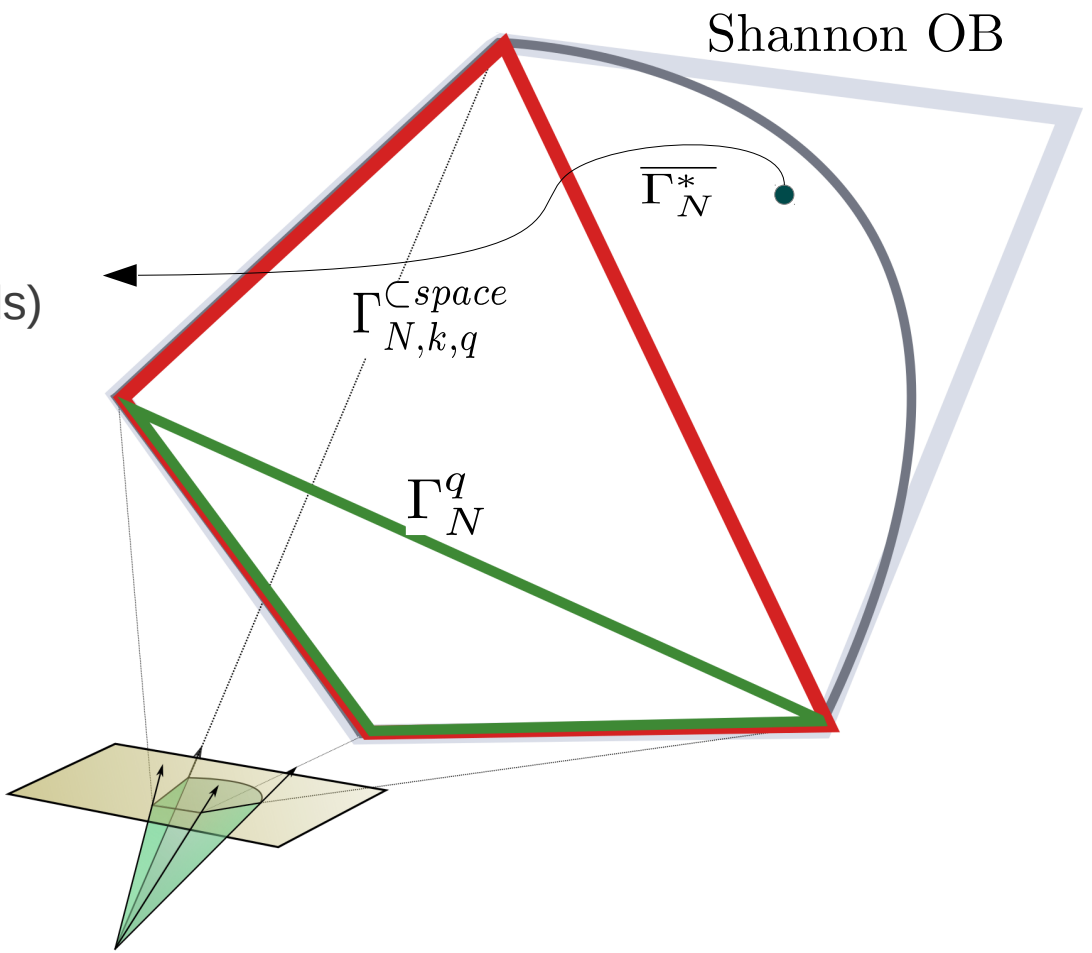
Motivation: Generality of matroid extension based approach

- We know that there are networks for which linear coding does not suffice
- There are entropic vectors that don't correspomd to any scalar or vector linear codes
- Matroids generalize codes and there are entropic matroids that don't arise from any code over a finite fields or reals



Motivation: Generality of matroid extension based approach

There are entropic matroids here!
(Outside the scalar/vector linear bounds)

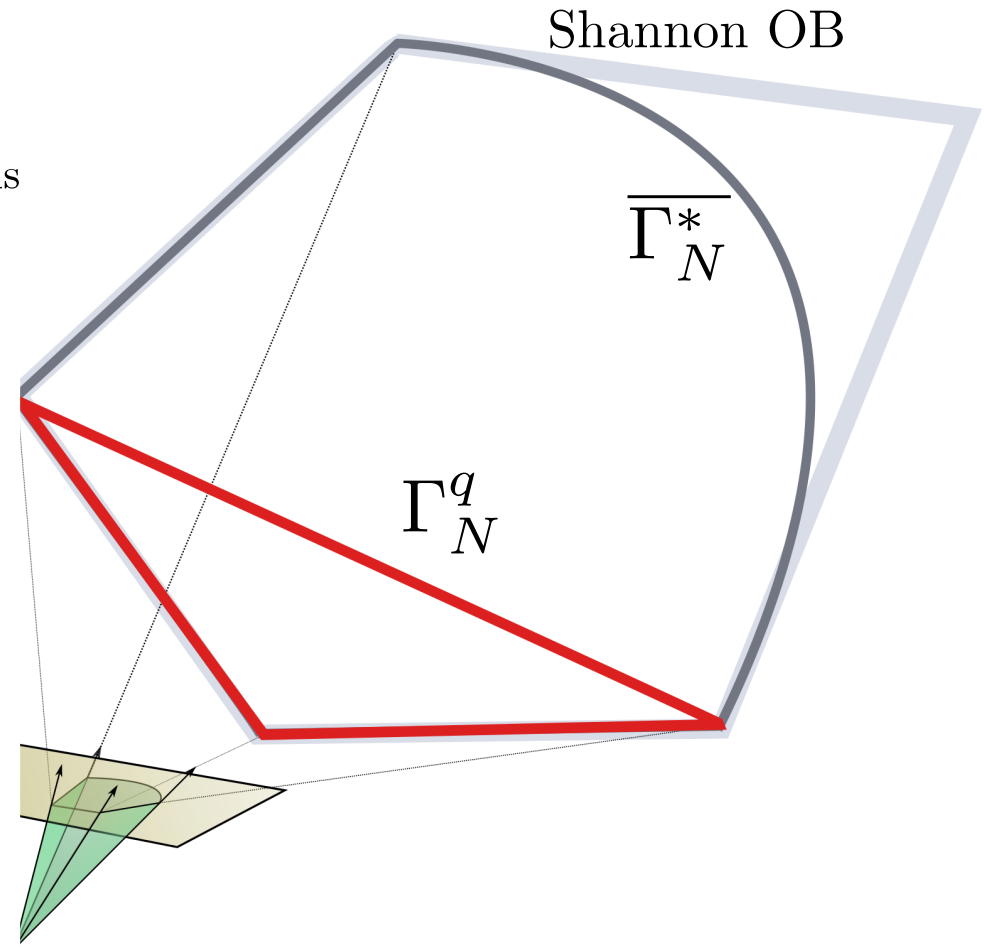


Outline

- Motivation
- Matroid bounds on region of entropic vectors
- Non-isomorphic matroid enumeration via matroid extension
- Characterization of matroid bounds upto isomorphism via matroid extension
- Characterization of network coding rate regions upto isomorphism via matroid extension

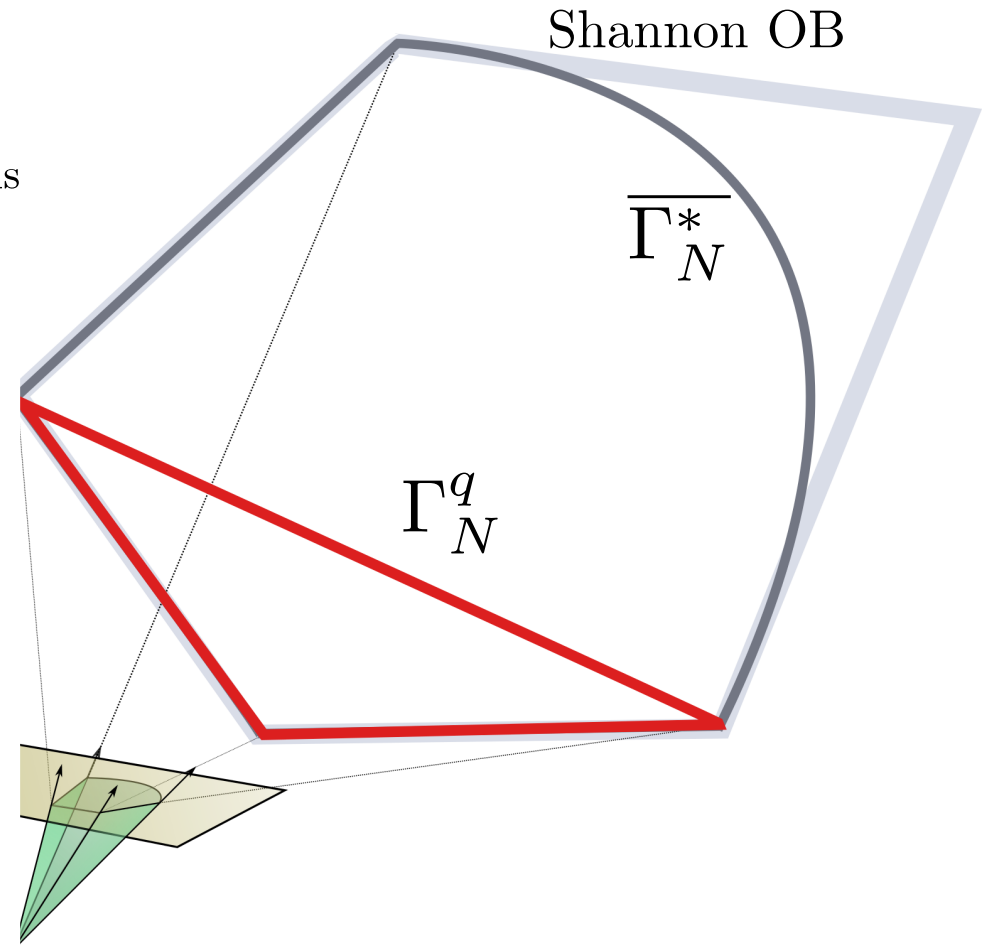
(Representable) Matroid Inner bound(s)

- Denoted as Γ_N^q where q is some prime power
- Conic hull of *rank vectors* of $\mathbb{F}_q$-*representable* matroids with ground set size N
- For a set E of cardinality N map $r : 2^E \setminus \emptyset \to \mathbb{Z}_{\geq 0}$ is *rank function* of a matroid if it is non-decreasing, submodular and bounded by cardinality of respective subsets
- $r(E)$ is called rank of matroid
- *Rank vector* of a matroid is formed by stacking together ranks of $2^N - 1$ non-empty subsets of E
- Representability of a matroid M with ground set E of size N and rank $r(E) = m$ over $\mathbb{F}_q$:
  $\exists$ a matrix $\mathbf{A} \in \mathbb{F}_q^{m \times N}$ such that
  $\forall B \subseteq E$, $r(B) = \text{rank}(\mathbf{A}_{:,B})$, the matrix rank of the columns of indexed by B



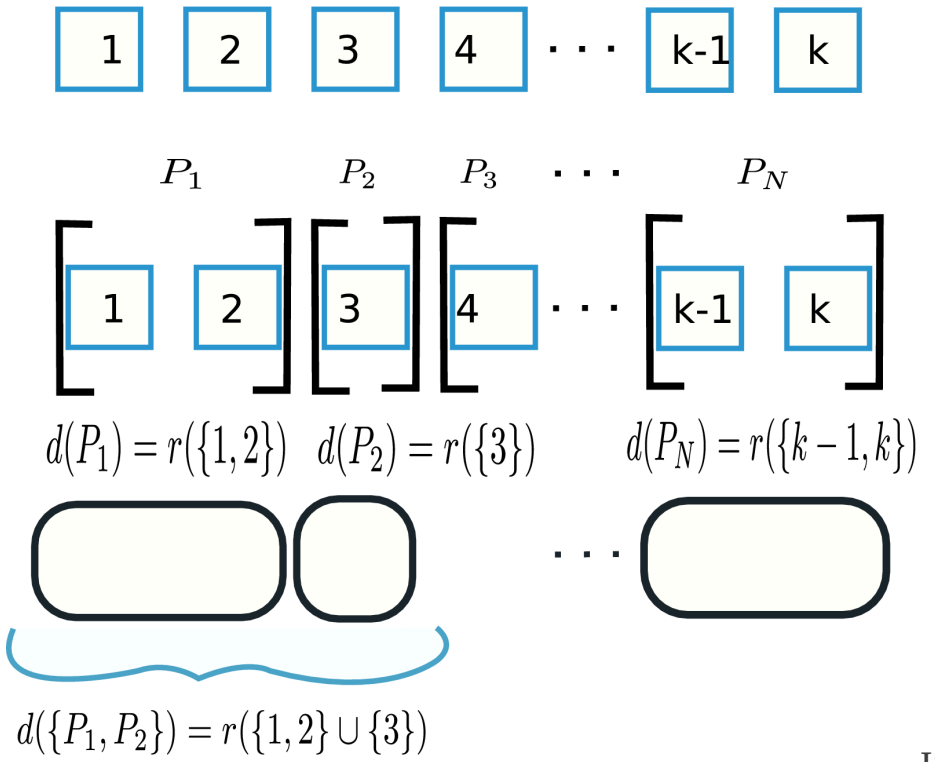
(Representable) Matroid Inner bound(s)

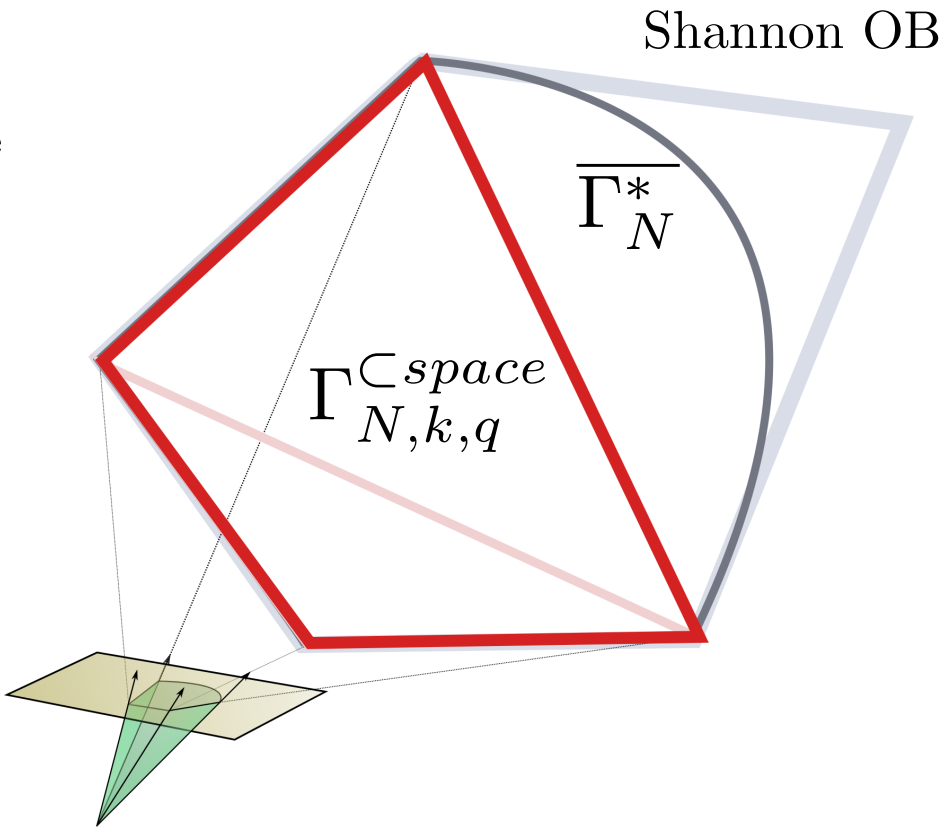
- Denoted as Γ_N^q where q is some prime power
- Conic hull of *rank vectors* of $\mathbb{F}_q$-*representable* matroids with ground set size N
- For a set E of cardinality N map $r : 2^E \setminus \emptyset \to \mathbb{Z}_{\geq 0}$ is *rank function* of a matroid if it is non-decreasing, submodular and bounded by cardinality of respective subsets
- $r(E)$ is called rank of matroid
- *Rank vector* of a matroid is formed by stacking together ranks of $2^N - 1$ non-empty subsets of E
- Representability of a matroid M with ground set E of size N and rank $r(E) = m$ over $\mathbb{F}_q$:
  $\exists$ a matrix $\mathbf{A} \in \mathbb{F}_q^{m \times N}$ such that
  $\forall B \subseteq E$, $r(B) = \text{rank}(\mathbf{A}_{:,B})$, the matrix rank of the columns of indexed by B



Subspace Inner Bound(s)

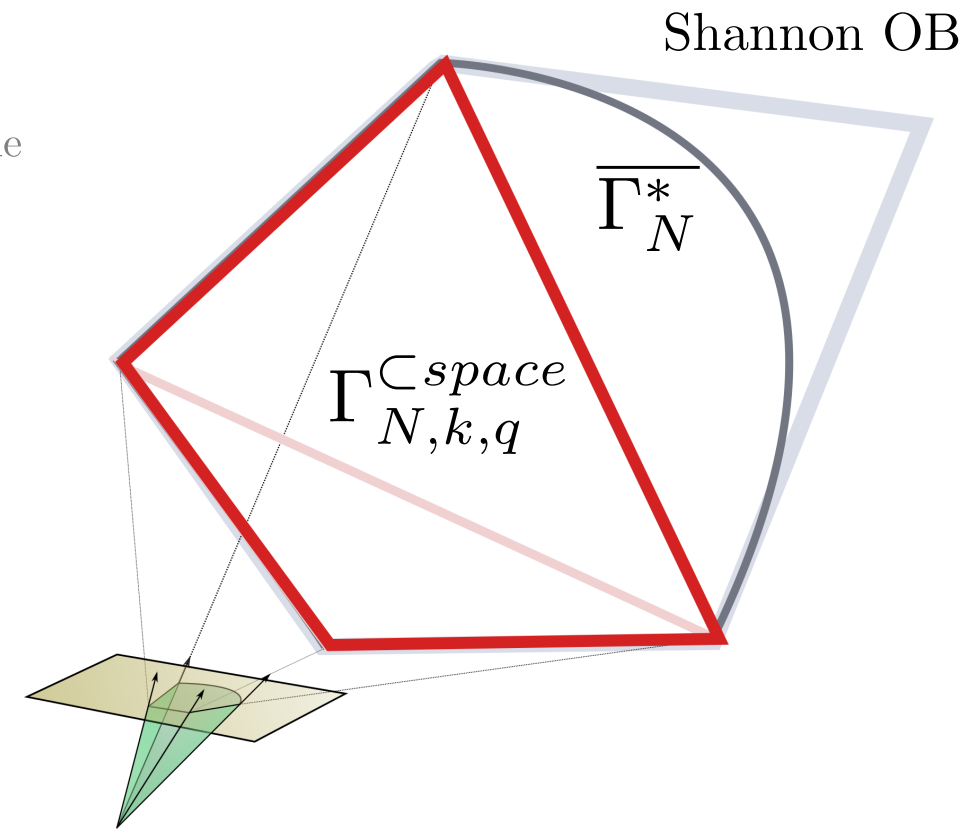
- Denoted as $\Gamma^{\subset space}_{N,k,q}$
- A vector $d \in \mathbb{Z}^{2^N-1}$ is dimension vector of a N-subspace arrangement over $\mathbb{F}_q$ if it can be obtained from rank vector of some $\mathbb{F}_q$-representable matroid on $k > N$ elements by linear projection onto ranks of subsets of a N-partition of E





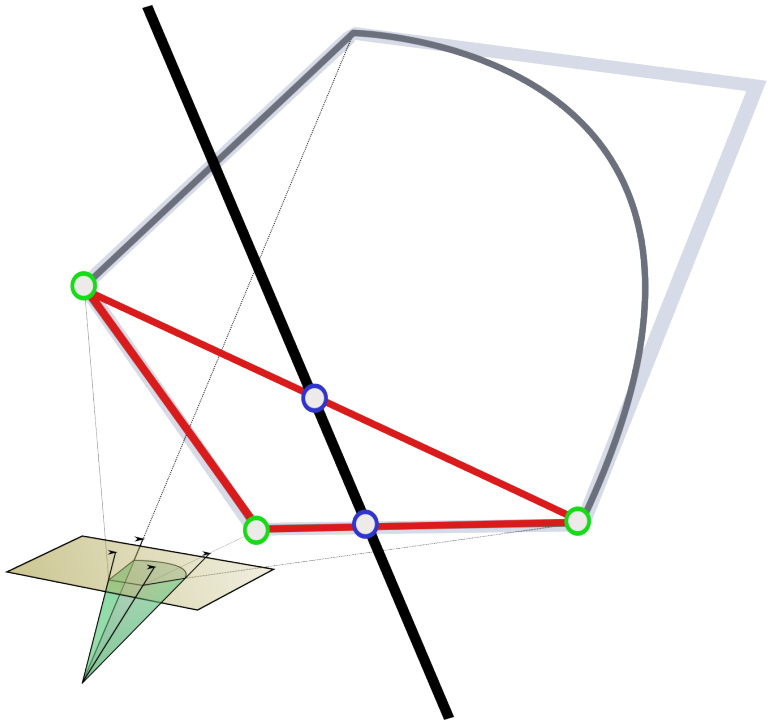
Subspace Inner Bound(s)

- Denoted as $\Gamma^{\subset space}_{N,k,q}$
- A vector $d \in \mathbb{Z}^{2^N-1}$ is dimension vector of a N-subspace arrangement over $\mathbb{F}_q$ if it can be obtained from rank vector of some $\mathbb{F}_q$-representable matroid on $k > N$ elements by linear projection onto ranks of subsets of a N-partition of E
- Extreme rays of $\Gamma^{\subset space}_{N,k,q}$ can be obtained from those of Γ^q_k by linear projection

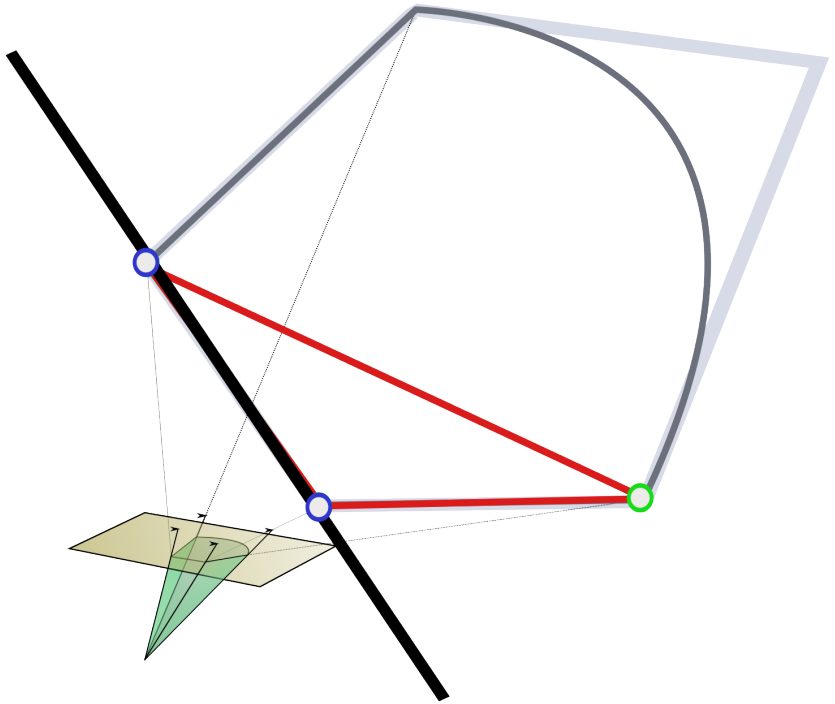


Polyhedral Inner Bounds on rate region via Yan and Yeung's characterization

$$\mathcal{R}^{in} = \Lambda\left(\mathrm{proj}_{Y_{\mathcal{S}}}\left(\Gamma_N^{poly} \cap \mathcal{L}_{12345}\right)\right).$$



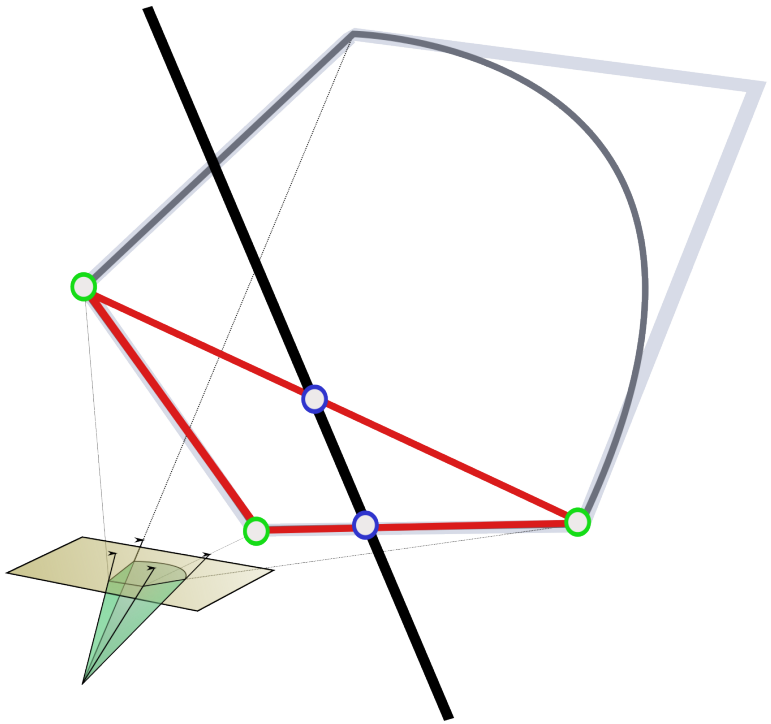
Arbitrary linear Constraint



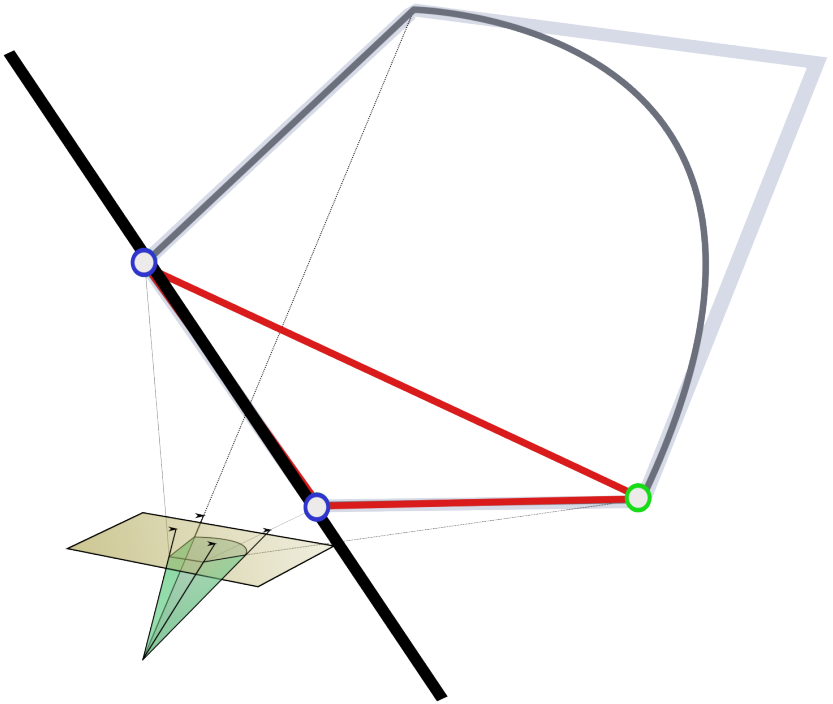
Network Coding Constraint

Polyhedral Inner Bounds on rate region via Yan and Yeung's characterization

Computing $\left(\Gamma_N^{poly} \cap \mathcal{L}_{12345}\right)$ boils down to membership checking! [4]



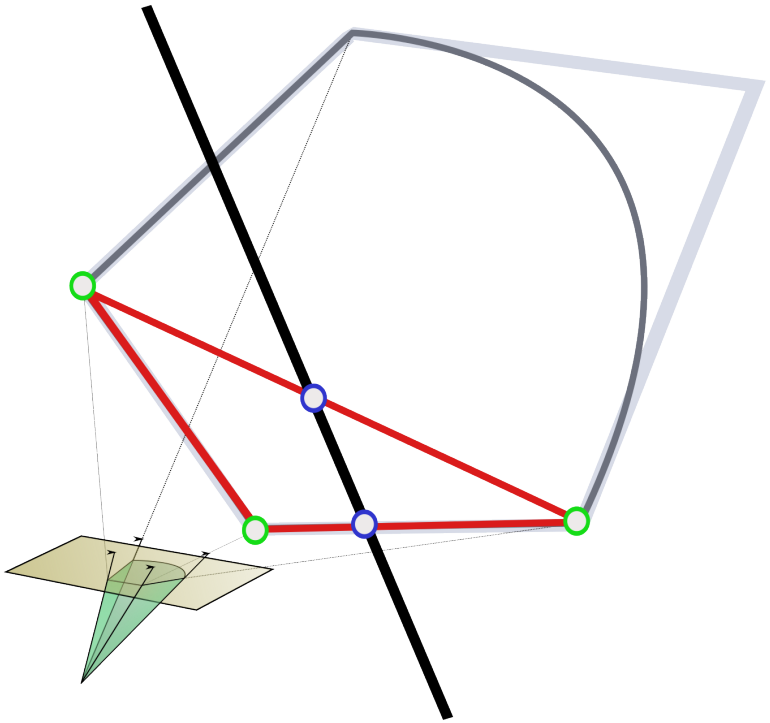
Arbitrary linear Constraint



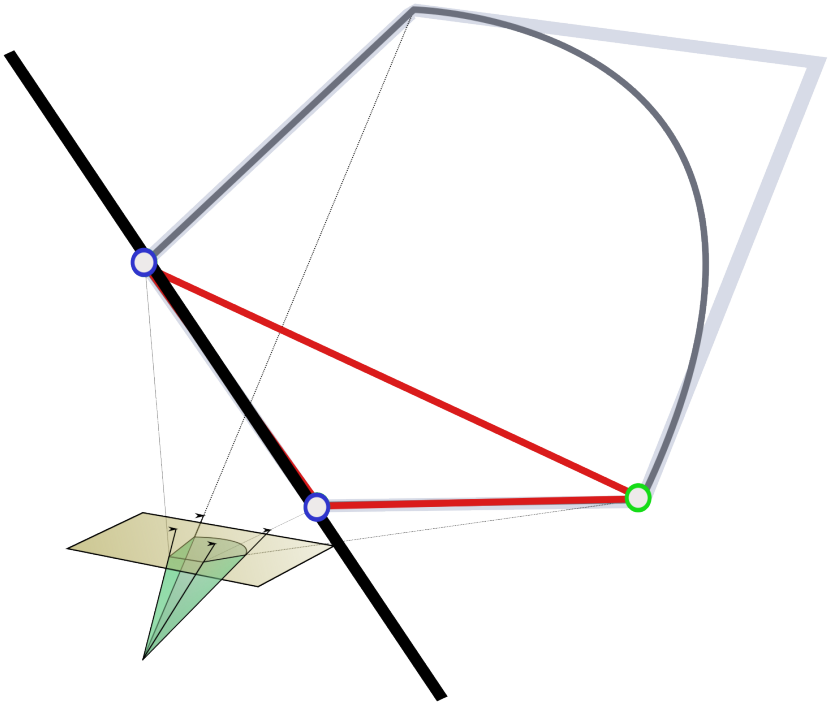
Network Coding Constraint

Polyhedral Inner Bounds on rate region via Yan and Yeung's characterization

A *matroid-network mapping* is a surjective map $\gamma : E(M) \to \mathcal{V}$, and its inverse image $\gamma^{-1} : \mathcal{V} \to 2^{E(M)} \setminus \phi$ is a *network-matroid mapping*.



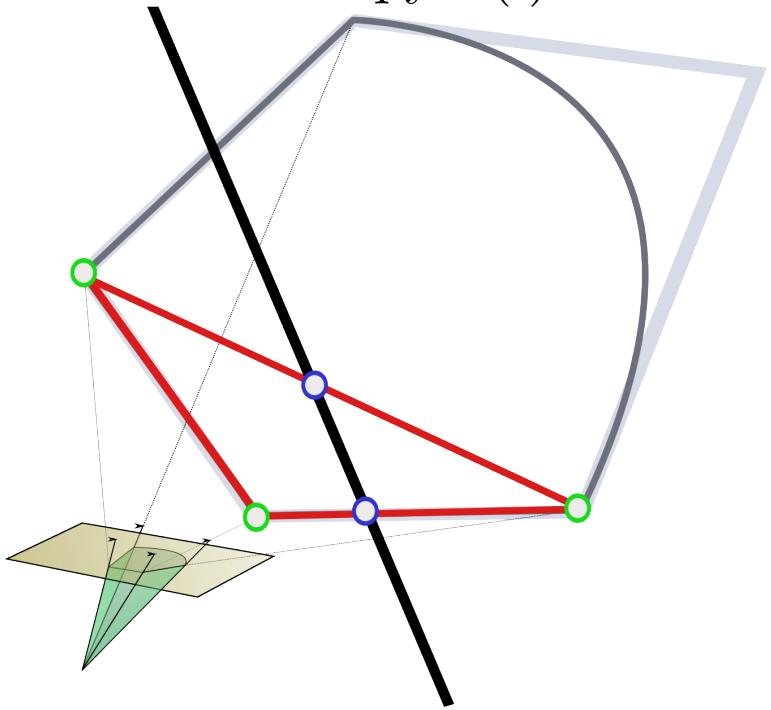
Arbitrary linear Constraint



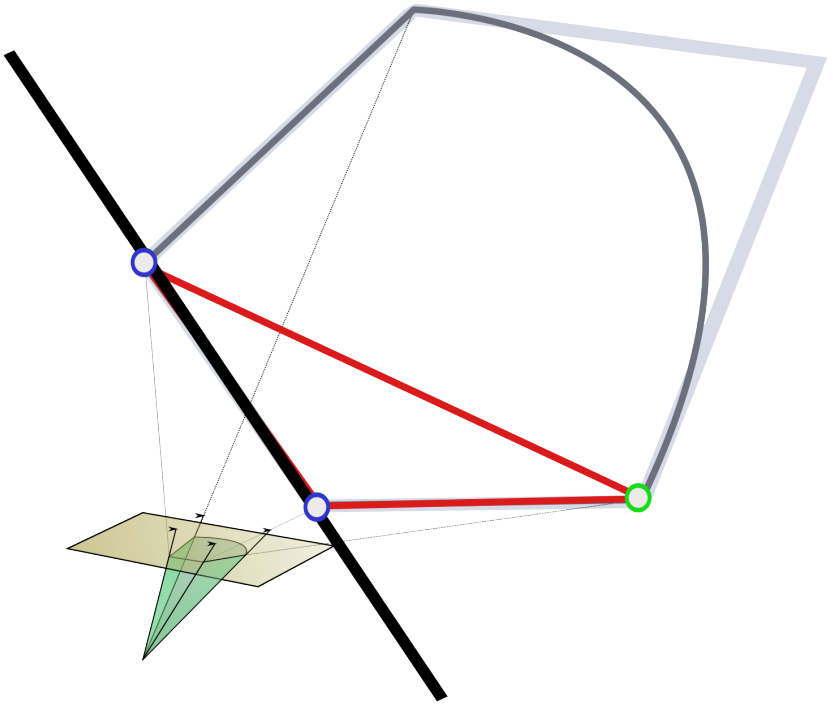
Network Coding Constraint

Polyhedral Inner Bounds on rate region via Yan and Yeung's characterization

A matroid M with rank function r and a matroid-network mapping γ form a *feasible linear network code* if $r(\gamma^{-1}(\cdot))$ satisfies all network constraints $\mathcal{L}_{12345}$ imposed on the entropy $h(\cdot)$ of subsets of random variables by the network



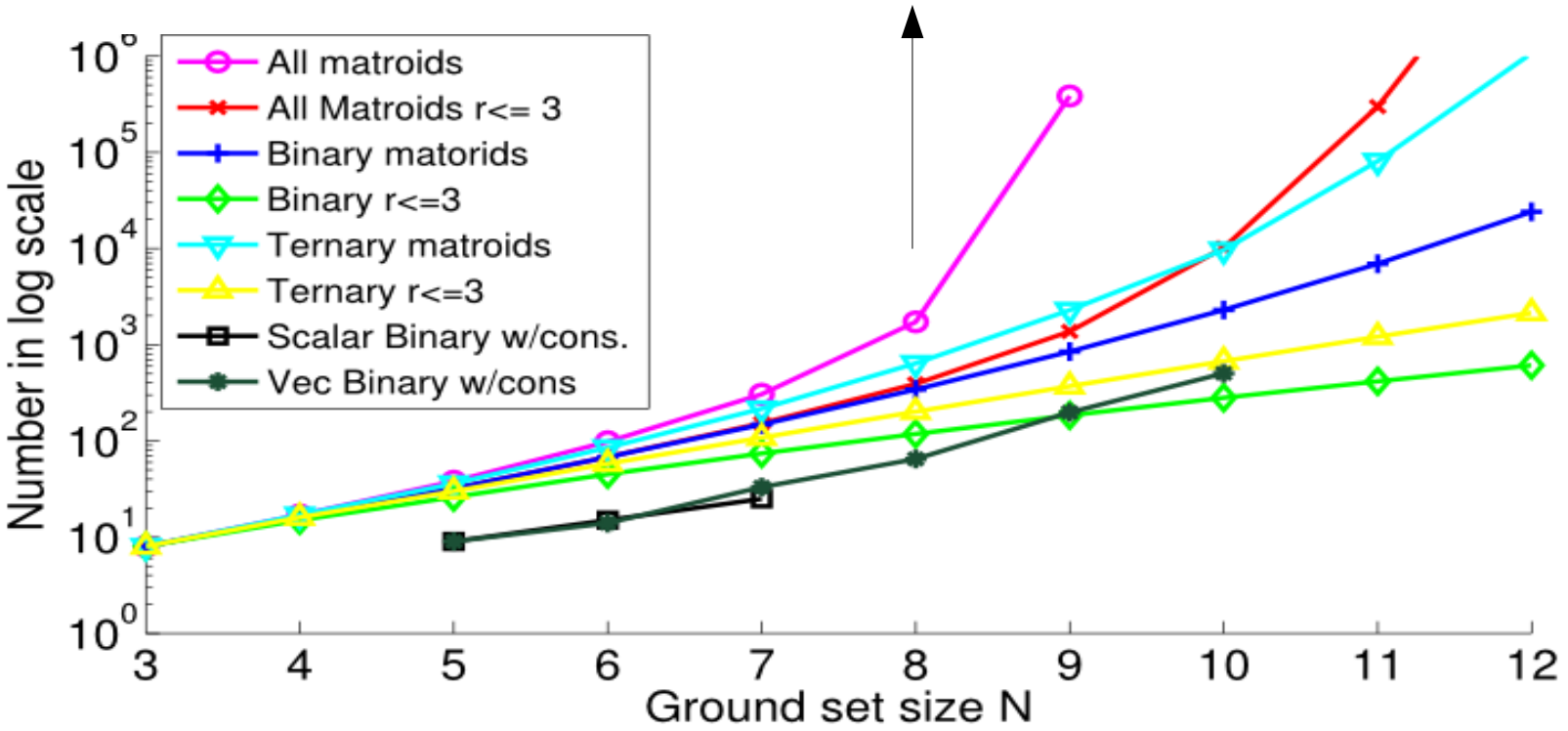
Arbitrary linear Constraint



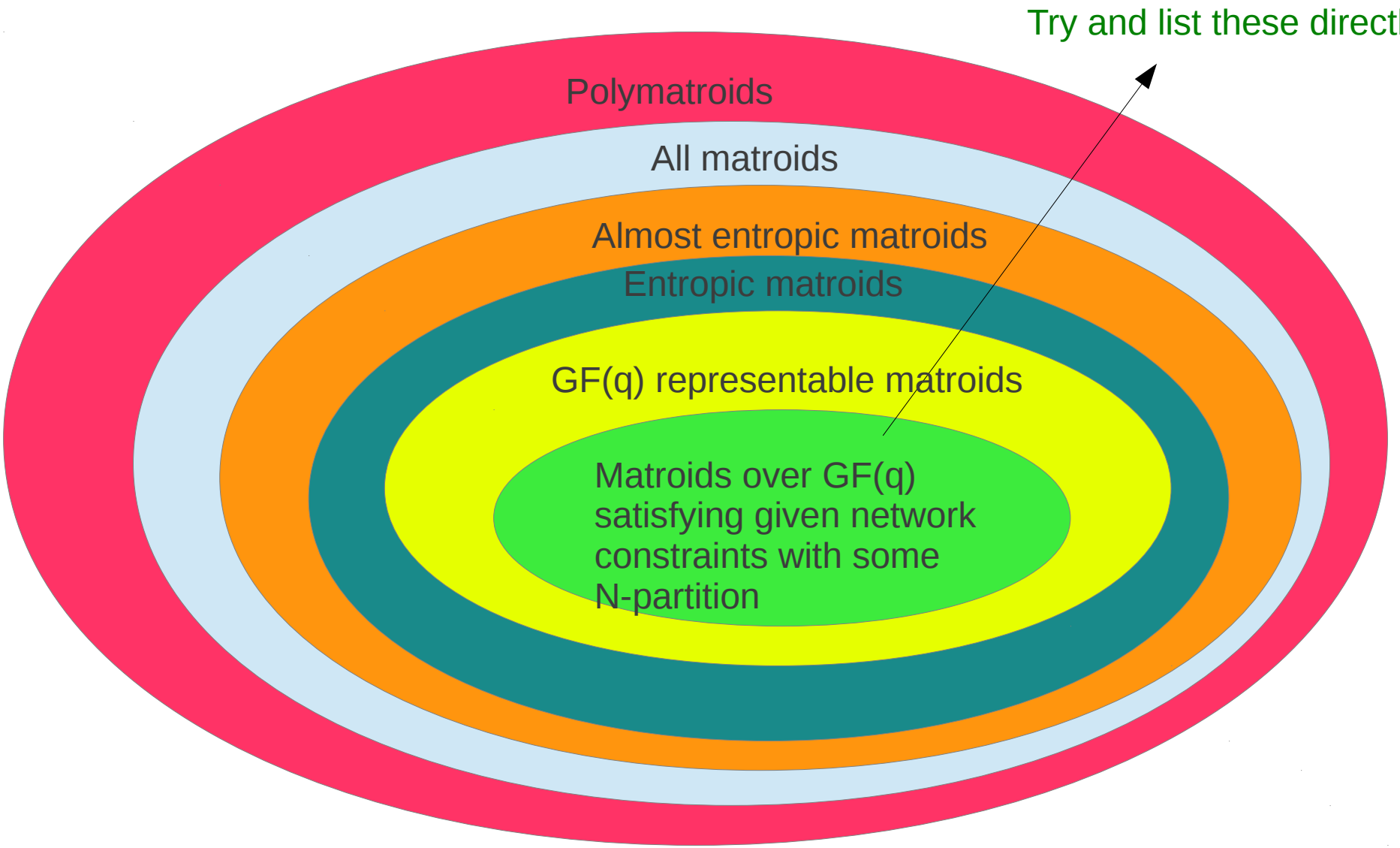
Network Coding Constraint

Number of non-isomorphic matroids in various polyhedral bounds

Without non-isomprphism, this number will be potentially multiplied by at most N! to consider all isomorphic copies



From naïve algorithms to less naïve ones



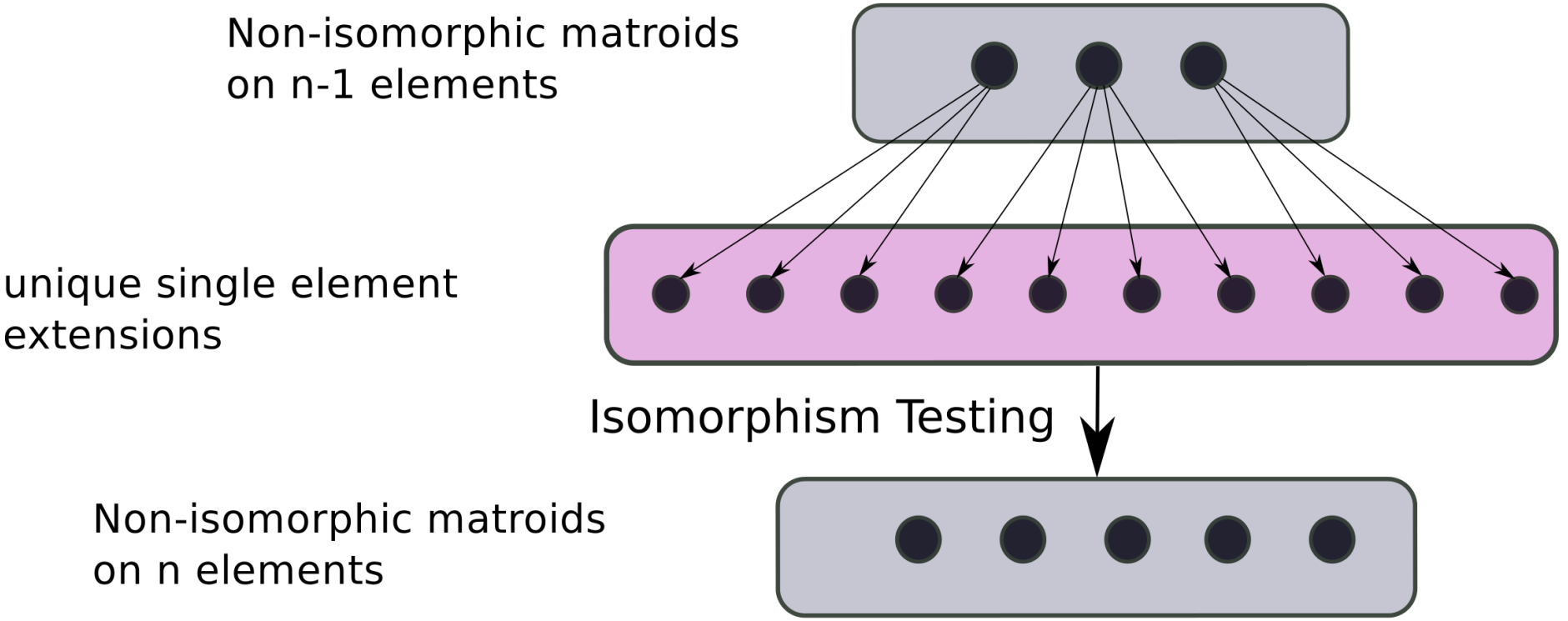
Single element extensions of a matroid

- If matroid M is obtained from matroid N by deleting a single element from $E(N)$, then N is said to be single element extension of M
- Deletion is matroid theory analog of shortening of a code
- Single element extenion can be thought of as adding a column to a matrix or adding an edge to graph
- finding all non-isomorphic extensions of a matroid on $n-1$ elements means finding all non-isomorphic matroids on n elements such that the given matroid can be obtained via deletion of a single element from each of them.

Outline

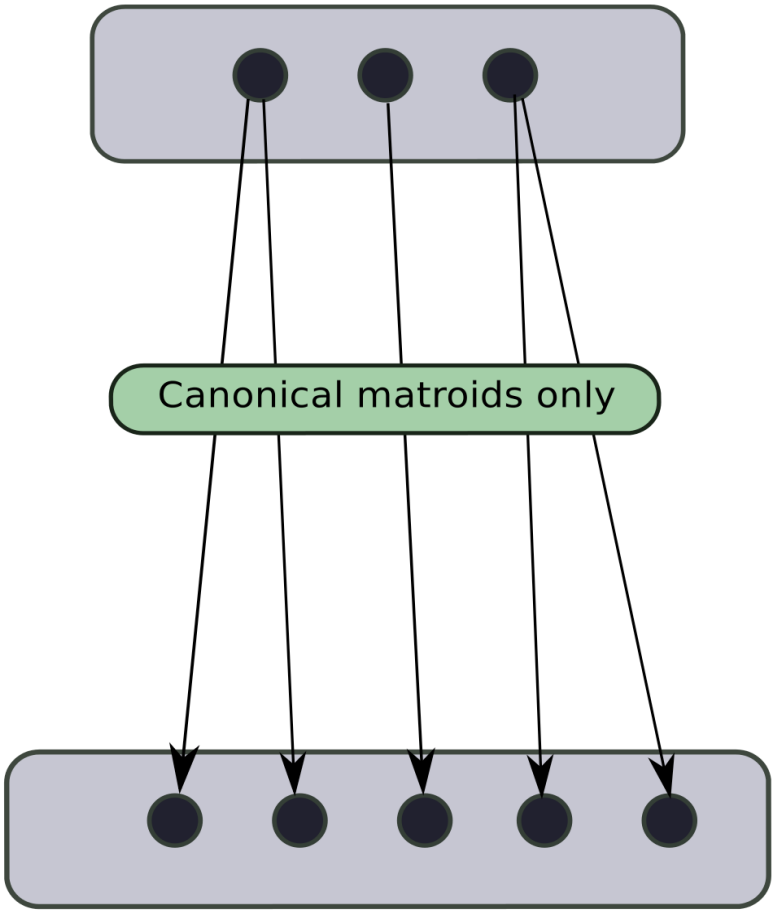
- Motivation
- Matroid bounds on region of entropic vectors
- Non-isomorphic matroid enumeration via matroid extension
- Characterization of matroid bounds upto isomorphism via matroid extension
- Characterization of network coding rate regions upto isomorphism via matroid extension

Blackburn, Crapo and Higgs' Algorithm [8]



Matsumoto et al.'s Algorithm [7]

Non-isomorphic (canonical) matroids on n-1 elements



Non-isomorphic (canonical) matroids on n elements

Using canonical extensions as a workhorse

- Canonical extensions inroduced by Matsumoto et al. provide a means to produce all size *n* matroids from size *n-1* matroids
- We abbreviate this procedure as $see(\cdot)$ which takes a set of canonical matroids as input and produces their canonical extensions

Outline

- Motivation
- Matroid bounds on region of entropic vectors
- Non-isomorphic matroid enumeration via matroid extension
- Characterization of matroid bounds upto isomorphism via matroid extension
- Characterization of network coding rate regions upto isomorphism via matroid extension

(P1) List all non-isomorphic matroids forming Γ_N^q

Start with lists of size *n-1 canonical* matroids representable over finite field of size *q*

Input: List $\mathscr{M}_{n-1}$ of canonical $\mathbb{F}_q$-representable matroids on $n-1$ elements
Output: List $\mathscr{M}_n$ of canonical $\mathbb{F}_q$-representable matroids on n elements

```
1 M'_n ← see(M_{n-1})
2 M_n ← φ
3 foreach M ∈ M'_n do
4 |   if M has no F_q forbidden minors then
5 |   |   M_n ← M_n ∪ {M}
6 |   end
7 end
8 return M_n
```

(P1) List all non-isomorphic matroids forming Γ_N^q

Start with lists of size *n-1 canonical* matroids representable over finite field of size *q*

Obtain canonical single element extensions

Input: List $\mathscr{M}_{n-1}$ of canonical $\mathbb{F}_q$-representable matroids on $n-1$ elements
Output: List $\mathscr{M}_n$ of canonical $\mathbb{F}_q$-representable matroids on n elements

1 $\mathscr{M}'_n \leftarrow see(\mathscr{M}_{n-1})$
2 $\mathscr{M}_n \leftarrow \phi$
3 **foreach** $M \in \mathscr{M}'_n$ **do**
4 | **if** M *has no* $\mathbb{F}_q$ *forbidden minors* **then**
5 | | $\mathscr{M}_n \leftarrow \mathscr{M}_n \cup \{M\}$
6 | **end**
7 **end**
8 **return** $\mathscr{M}_n$

(P1) List all non-isomorphic matroids forming Γ_N^q

Start with lists of size *n-1 canonical* matroids representable over finite field of size *q*

Obtain canonical single element extensions

Reject the matroids having $\mathbb{F}_q$ forbidden minors *(q=2,3,4)*

Input: List $\mathscr{M}_{n-1}$ of canonical $\mathbb{F}_q$-representable matroids on $n-1$ elements
Output: List $\mathscr{M}_n$ of canonical $\mathbb{F}_q$-representable matroids on n elements

1 $\mathscr{M}'_n \leftarrow see(\mathscr{M}_{n-1})$
2 $\mathscr{M}_n \leftarrow \phi$
3 **foreach** $M \in \mathscr{M}'_n$ **do**
4 | **if** M *has no* $\mathbb{F}_q$ *forbidden minors* **then**
5 | | $\mathscr{M}_n \leftarrow \mathscr{M}_n \cup \{M\}$
6 | **end**
7 **end**
8 **return** $\mathscr{M}_n$

(P1) List all non-isomorphic matroids forming Γ_N^q

Start with lists of size *n-1 canonical* matroids representable over finite field of size *q*

Obtain canonical single element extensions

Reject the matroids having $\mathbb{F}_q$ forbidden minors *(q=2,3,4)*

Why does this work?

If a matroid *M* has a minor *K* then all its extensions have Minor *K.*

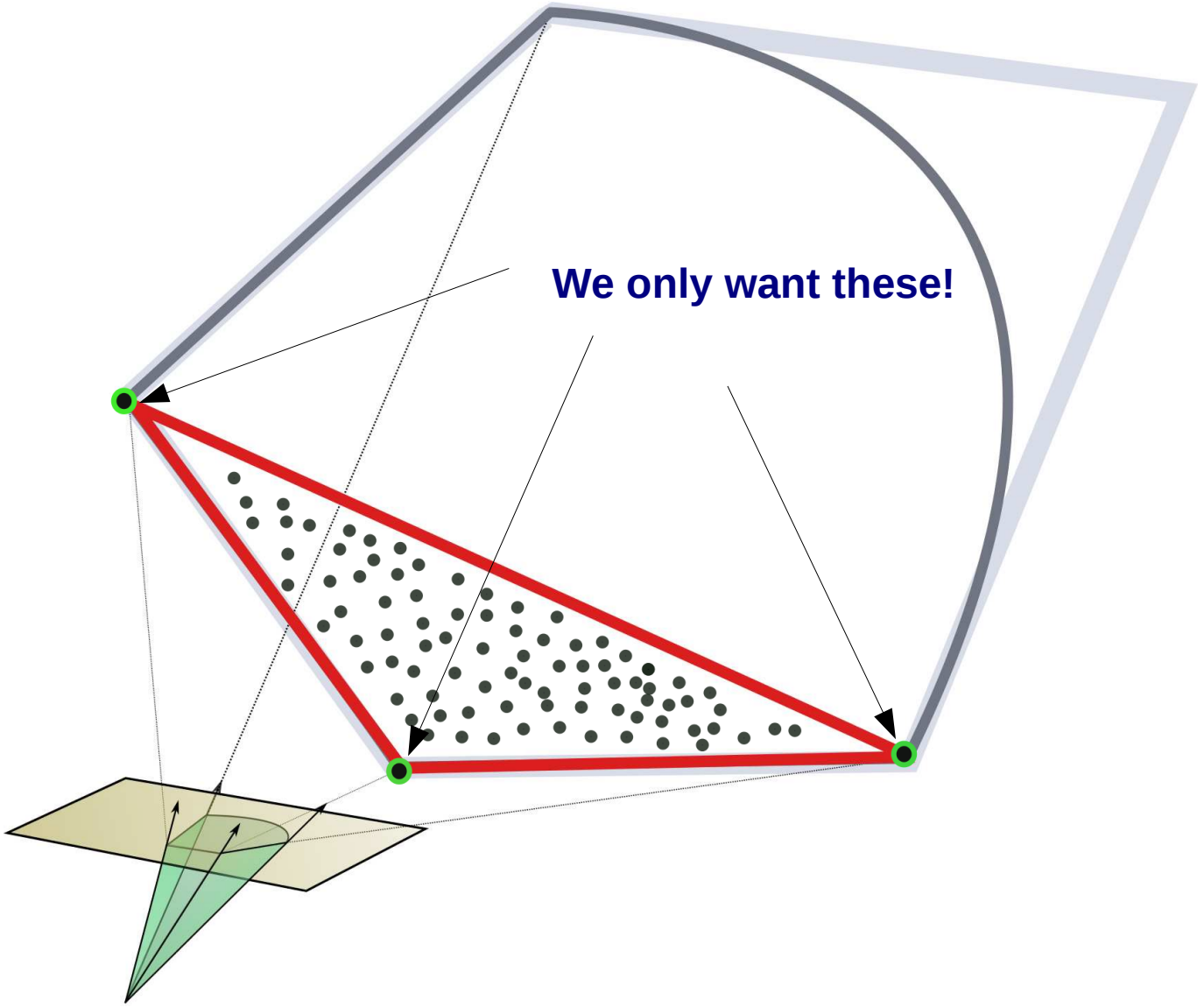
Hence, if matroid M is not representable over finite field of size q, so are its extensions

Input: List $\mathscr{M}_{n-1}$ of canonical $\mathbb{F}_q$-representable matroids on $n-1$ elements

Output: List $\mathscr{M}_n$ of canonical $\mathbb{F}_q$-representable matroids on n elements

1 $\mathscr{M}'_n \leftarrow see(\mathscr{M}_{n-1})$
2 $\mathscr{M}_n \leftarrow \phi$
3 **foreach** $M \in \mathscr{M}'_n$ **do**
4 | **if** M *has no* $\mathbb{F}_q$ *forbidden minors* **then**
5 | | $\mathscr{M}_n \leftarrow \mathscr{M}_n \cup \{M\}$
6 | **end**
7 **end**
8 **return** $\mathscr{M}_n$

(P2) List all non-isomorphic extreme rays of Γ_N^q



(P2) List all non-isomorphic extreme rays of Γ_N^q

Theorem(Li et al. [3]) Extreme rays of Γ_N^q are rank vectors of connected matroids

(P2) List all non-isomorphic extreme rays of Γ_N^q

Theorem(Li et al.) Extreme rays of Γ_N^q are rank vectors of connected matroids

- Connectivity of matroids generalizes 2-connectivity of graphs
- **Definition**: A matroid M on ground set E is said to be connected if for every $U \subset E$,

$$r_M(U) + r_M(E \setminus U) > r_M(E)$$

(P2) List all non-isomorphic extreme rays of Γ_N^q

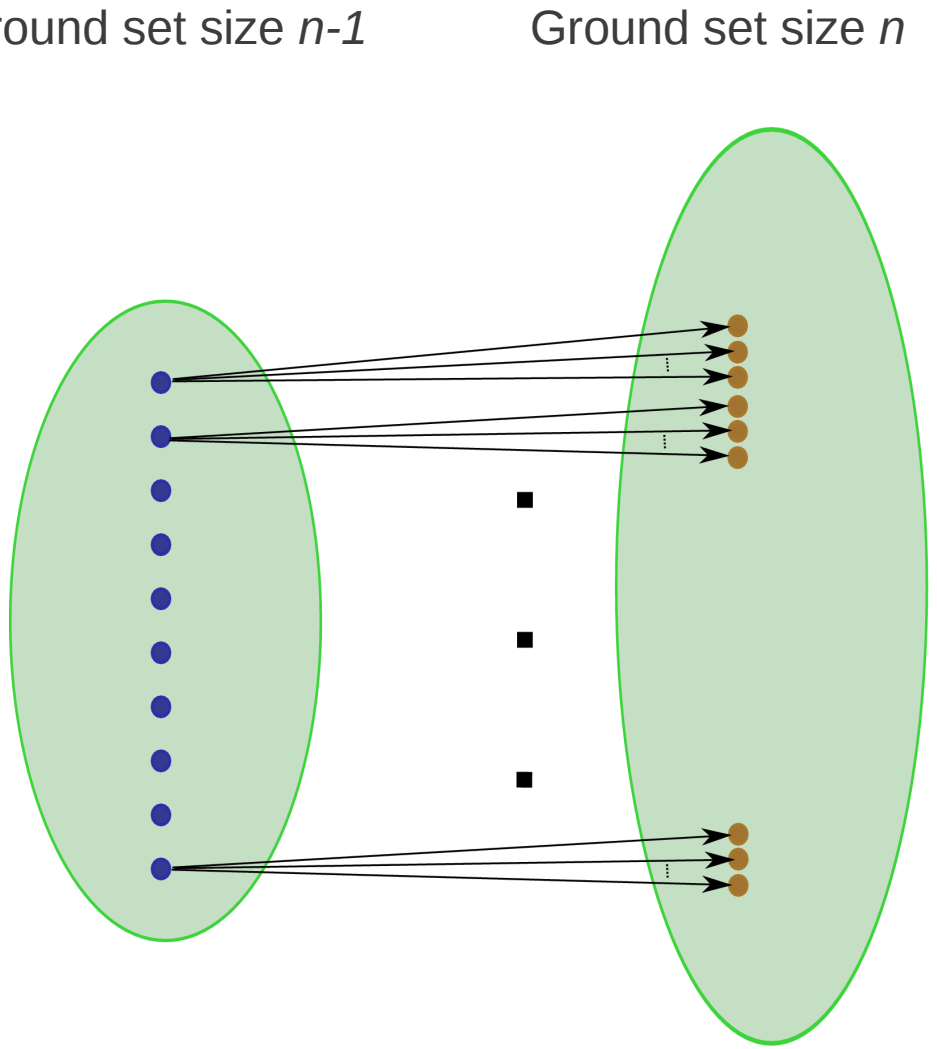
Theorem(Li et al.) Extreme rays of Γ_N^q are rank vectors of connected matroids

- Connectivity of matroids generalizes 2-connectivity of graphs
- **Definition**: A matroid M on ground set E is said to be connected if for every $U \subset E$,

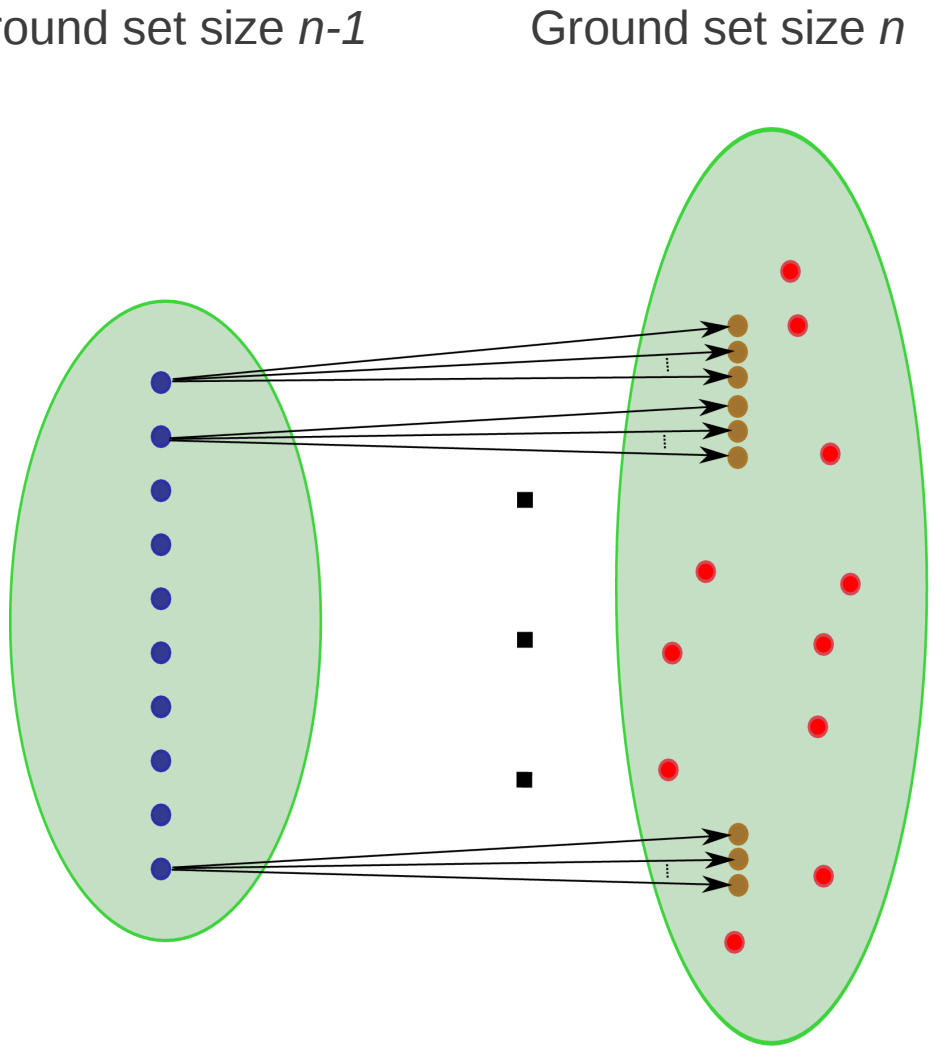
$$r_M(U) + r_M(E \setminus U) > r_M(E)$$

- **Lemma**: Single element extensions of connected matroids are also connected

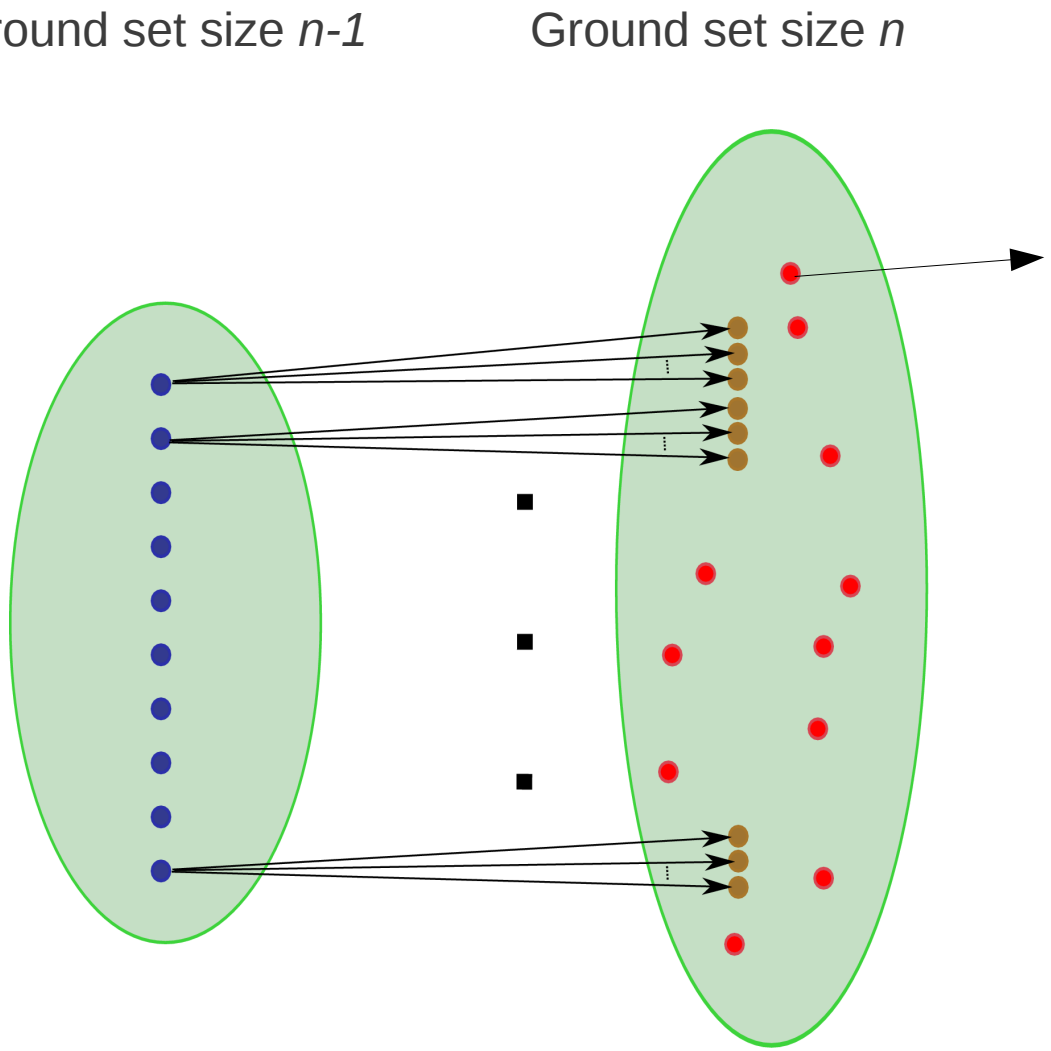
(P2) List all non-isomorphic extreme rays of Γ_N^q



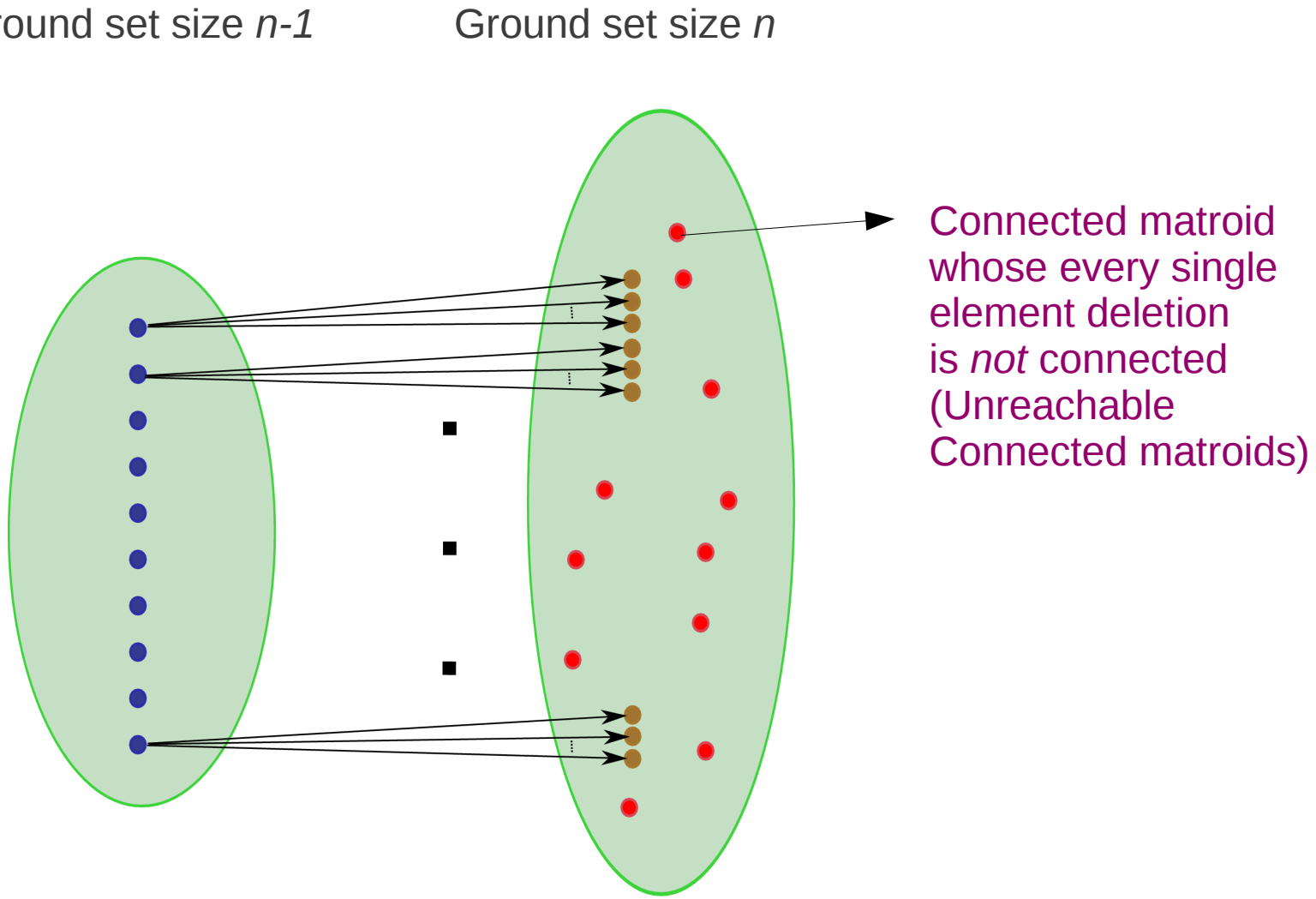
(P2) List all non-isomorphic extreme rays of Γ_N^q



(P2) List all non-isomorphic extreme rays of Γ_N^q

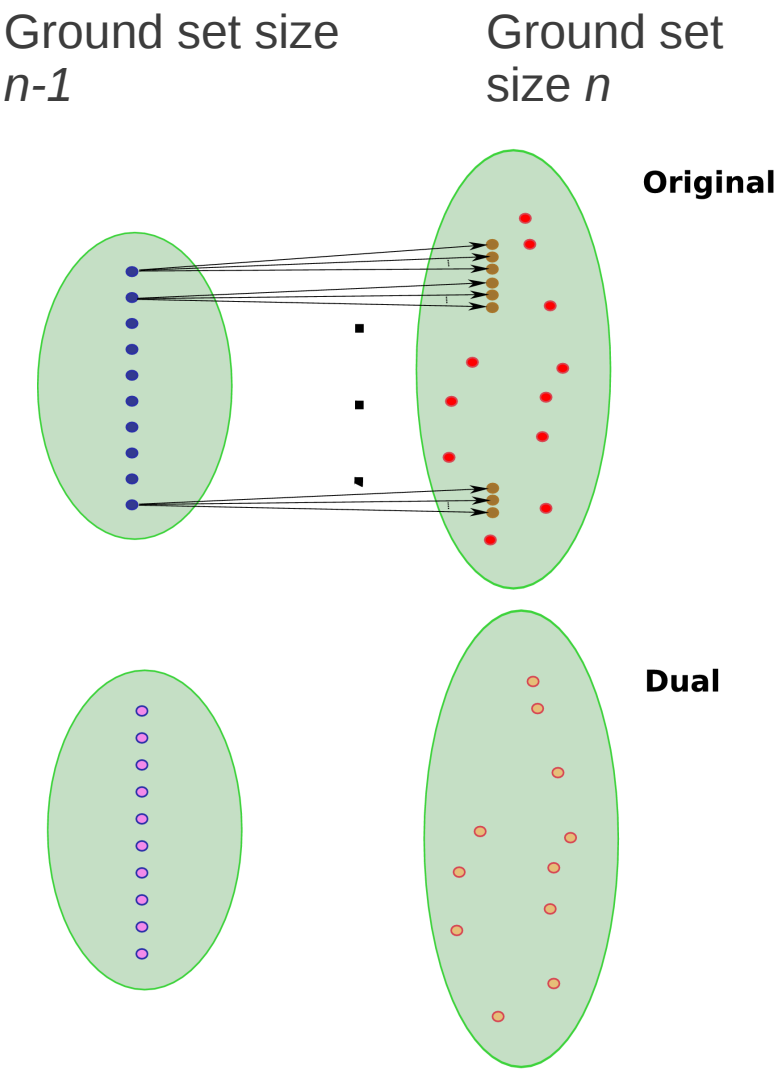


(P2) List all non-isomorphic extreme rays of Γ_N^q

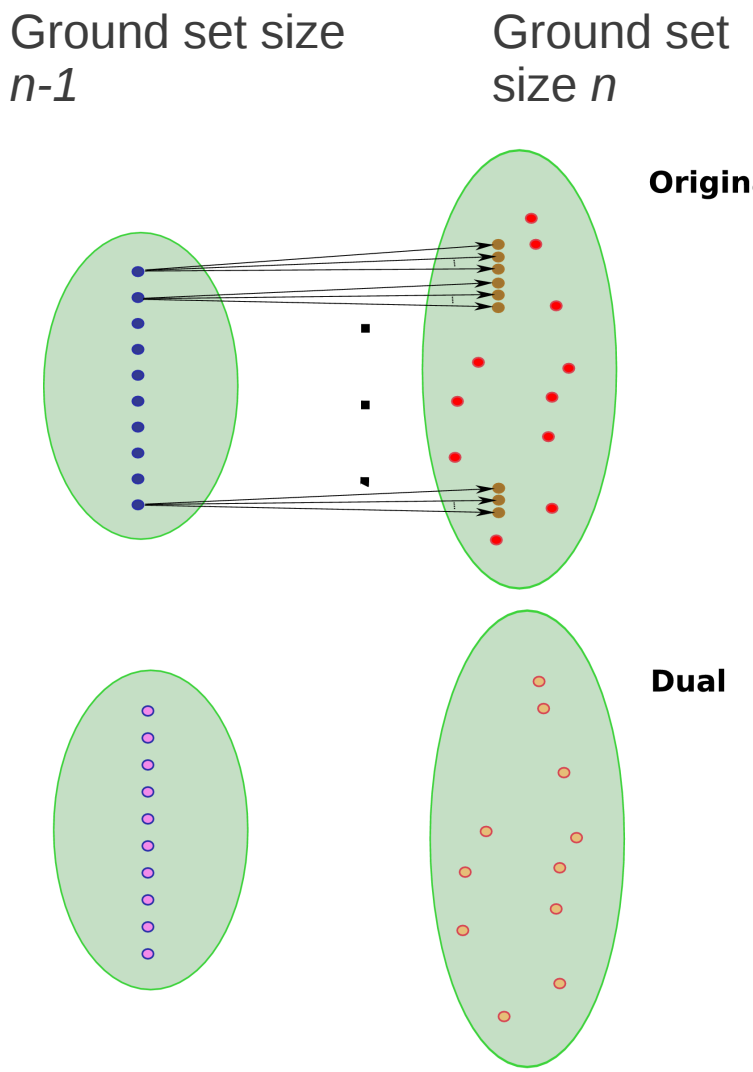


Example: Uniform matroid U_n^{n-1} i.e. a matroid on ground set E of size n such that $r(S) = |S|, \ \forall S \subset E$ and $r(E) = n - 1$

(P2) List all non-isomorphic extreme rays of Γ_N^q

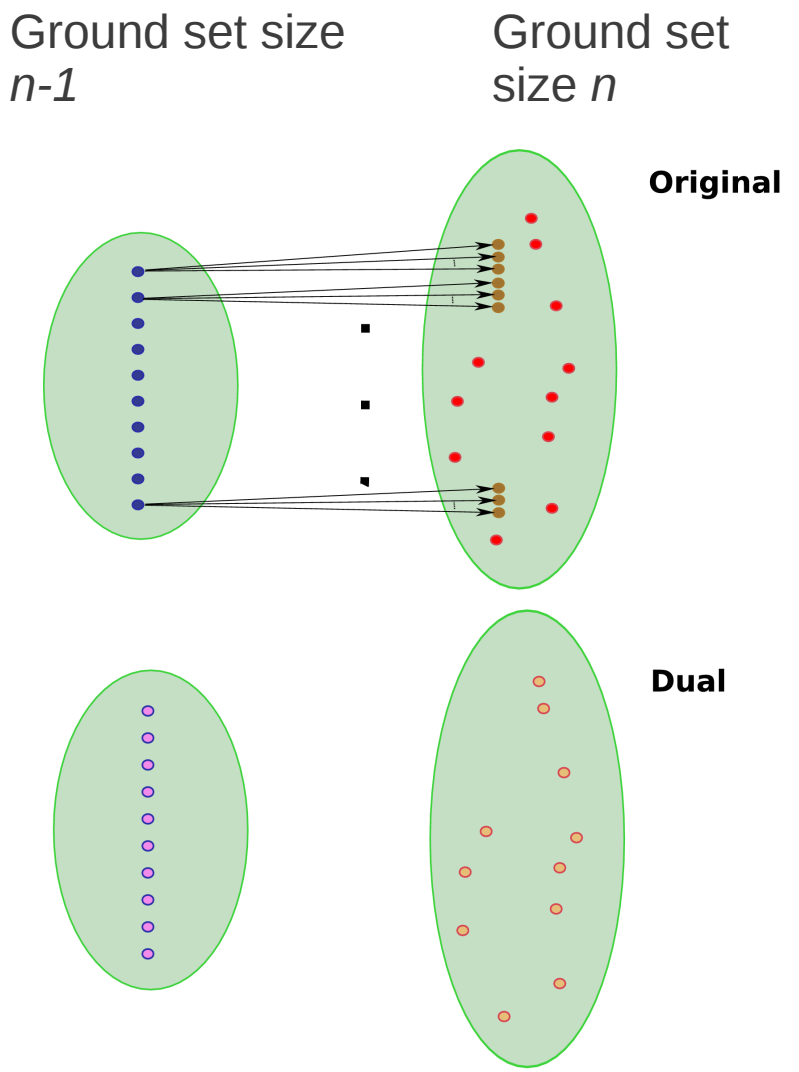


(P2) List all non-isomorphic extreme rays of Γ_N^q



- **Theorem**: A matroid is connected iff its dual is connected
- **Theorem**: If M is a connected matroid then, for every $e \in E(M)$ at least one of $M \setminus e$ or M/e are connected.

(P2) List all non-isomorphic extreme rays of Γ_N^q

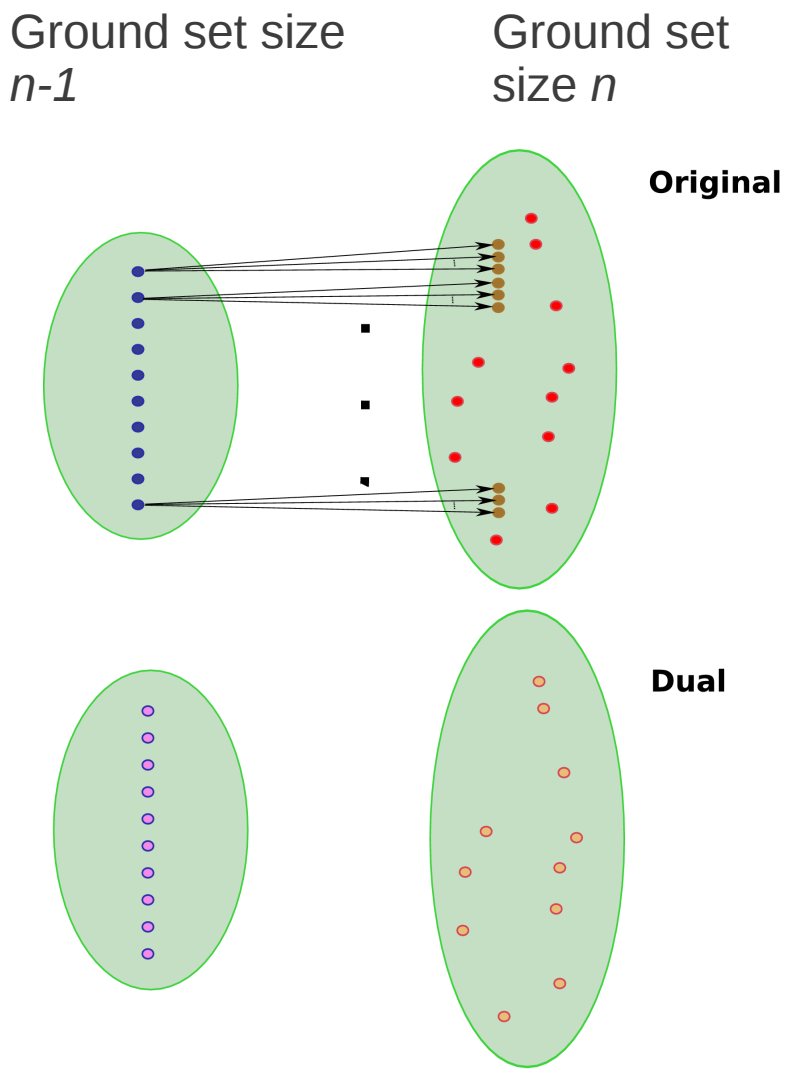


- **Theorem**: A matroid is connected iff its dual is connected
- **Theorem**: If M is a connected matroid then, for every $e \in E(M)$ at least one of $M \setminus e$ or M/e are connected.

Single element shortening

Single element puncturing

(P2) List all non-isomorphic extreme rays of Γ_N^q



- **Theorem**: A matroid is connected iff its dual is connected
- **Theorem**: If M is a connected matroid then, for every $e \in E(M)$ at least one of $M \setminus e$ or M/e are connected.

Single element shortening

Single element puncturing

Theorem: Dual of rank r unreachable connected matroid can be obtained via single element extension of some rank $n-r$ matroid or rank $n-r-1$ matroid on $n-1$ elements

Outline

- Motivation
- Matroid bounds on region of entropic vectors
- Non-isomorphic matroid enumeration via matroid extension
- Characterization of matroid bounds upto isomorphism via matroid extension
- Characterization of network coding rate regions upto isomorphism via matroid extension

(P3) List all non-isomorphic matroids forming bounds $\Gamma_N^q \cap \mathcal{L}_{12345}$ and $\Gamma_{N,k,q}^{\subset space} \cap \mathcal{L}_{12345}$

- **Definition**: **Partial matroid-network mapping** ξ from the ground set of matrioid M to set of network variables $\mathcal{V}$ is a not necessarily surjective map $\xi : E(M) \to \mathcal{V} \cup v_\phi$

(P3) List all non-isomorphic matroids forming bounds $\Gamma_N^q \cap \mathcal{L}_{12345}$ and $\Gamma_{N,k,q}^{\subset space} \cap \mathcal{L}_{12345}$

- **Definition**: **Partial matroid-network mapping** ξ from the ground set of matrioid M to set of network variables $\mathcal{V}$ is a not necessarily surjective map $\xi : E(M) \to \mathcal{V} \cup v_\phi$
- v_ϕ is the empty variable
- Call partial matroid-network mapping as p-map.
- Call a variable $v \in \mathcal{V}$ *defined* wrt a p-map ξ if $\exists e$ s.t. $\xi(e) = v$

(P3) List all non-isomorphic matroids forming bounds $\Gamma_N^q \cap \mathcal{L}_{12345}$ and $\Gamma_{N,k,q}^{\subset space} \cap \mathcal{L}_{12345}$

- **Definition**: **Partial matroid-network mapping** ξ from the ground set of matrioid M to set of network variables $\mathcal{V}$ is a not necessarily surjective map $\xi : E(M) \to \mathcal{V} \cup v_\phi$
- v_ϕ is the empty variable
- Call partial matroid-network mapping as p-map.
- Call a variable $v \in \mathcal{V}$ *defined* wrt a p-map ξ if $\exists e$ s.t. $\xi(e) = v$
- **Definition**: A matroid M with a p-map ξ is a **Partially Feasible Linear Network Code** (p-code) if $r_M(\xi^{-1}(\cdot))$ satisfies all network constraints imposed on the entropy $h(\cdot)$ of subsets of *defined* random variables by the network.

(P3) List all non-isomorphic matroids forming bounds $\Gamma_N^q \cap \mathcal{L}_{12345}$ and $\Gamma_{N,k,q}^{\subset space} \cap \mathcal{L}_{12345}$

The goal is to produce all canonical matroids of ground set size up to N* that together with some surjective matroid network map form feasible linear network codes for a given n/w

Input: A N-variable MSNC problem and maximum ground set size N^*
Output: All pairs (M, ξ) such that M is a non-isomorphic (canonical) matroid with $N \leq E(M) \leq N^*$ and (M, ξ) is a feasible linear network code

$\mathscr{M}^0 \leftarrow$ {matroid M^ϕ on empty set}
$\mathcal{Q}^M \leftarrow$ { empty p-map}
for $i \in \{1, \ldots, N^*\}$ **do**
  $\mathscr{M}^i \leftarrow \mathsf{see}(\mathscr{M}^{i-1}, q)$
  foreach $M \in \mathscr{M}^i$ **do**
    $M' \leftarrow$ parent matroid of M
    $\mathcal{Q}^M \leftarrow \mathsf{extendpmaps}(M, M', \mathcal{Q}^{M'})$
    if $\mathcal{Q}^M$ *is empty* **then**
      $\mathscr{M}^i \leftarrow \mathscr{M}^i \setminus M$
    end
    foreach $\xi \in \mathcal{Q}^M$ **do**
      if (M, ξ) *is a feasible linear network code* **then**
        Output (M, ξ)
      end
    end
  end
end
return

(P3) List all non-isomorphic matroids forming bounds $\Gamma_N^q \cap \mathcal{L}_{12345}$ and $\Gamma_{N,k,q}^{\subset space} \cap \mathcal{L}_{12345}$

The goal is to produce all canonical matroids of ground set size up to N* that together with some surjective matroid network map form feasible linear network codes for a given n/w

Start with matroid on empty ground Set and an empty p-map

Input: A N-variable MSNC problem and maximum ground set size N^*
Output: All pairs (M, ξ) such that M is a non-isomorphic (canonical) matroid with $N \leq E(M) \leq N^*$ and (M, ξ) is a feasible linear network code

$\mathscr{M}^0 \leftarrow$ {matroid M^ϕ on empty set}
$\mathcal{Q}^M \leftarrow$ { empty p-map}
for $i \in \{1, \ldots, N^*\}$ **do**
  $\mathscr{M}^i \leftarrow \mathsf{see}(\mathscr{M}^{i-1}, q)$
  foreach $M \in \mathscr{M}^i$ **do**
    $M' \leftarrow$ parent matroid of M
    $\mathcal{Q}^M \leftarrow \mathsf{extendpmaps}(M, M', \mathcal{Q}^{M'})$
    if $\mathcal{Q}^M$ *is empty* **then**
      $\mathscr{M}^i \leftarrow \mathscr{M}^i \setminus M$
    end
    foreach $\xi \in \mathcal{Q}^M$ **do**
      if (M, ξ) *is a feasible linear network code* **then**
        Output (M, ξ)
      end
    end
  end
end
return

(P3) List all non-isomorphic matroids forming bounds $\Gamma_N^q \cap \mathcal{L}_{12345}$ and $\Gamma_{N,k,q}^{\subset space} \cap \mathcal{L}_{12345}$

The goal is to produce all canonical matroids of ground set size up to N* that together with some surjective matroid network map form feasible linear network codes for a given n/w

Start with matroid on empty ground Set and an empty p-map

Extend matroids ensuring representability

Input: A N-variable MSNC problem and maximum ground set size N^*
Output: All pairs (M, ξ) such that M is a non-isomorphic (canonical) matroid with $N \leq E(M) \leq N^*$ and (M, ξ) is a feasible linear network code

```
1  M^0 ← {matroid M^φ on empty set}
2  Q^M ← { empty p-map}
3  for i ∈ {1, ..., N*} do
4  |   M^i ← see(M^{i-1}, q)
5  |   foreach M ∈ M^i do
6  |   |   M' ← parent matroid of M
7  |   |   Q^M ← extendpmaps(M, M', Q^{M'})
8  |   |   if Q^M is empty then
9  |   |   |   M^i ← M^i \ M
10 |   |   end
11 |   |   foreach ξ ∈ Q^M do
12 |   |   |   if (M, ξ) is a feasible linear network code
   |   |   |   then
13 |   |   |   |   Output (M, ξ)
14 |   |   |   end
15 |   |   end
16 |   end
17 end
18 return
```

(P3) List all non-isomorphic matroids forming bounds $\Gamma_N^q \cap \mathcal{L}_{12345}$ and $\Gamma_{N,k,q}^{\subset space} \cap \mathcal{L}_{12345}$

The goal is to produce all canonical matroids of ground set size up to N* that together with some surjective matroid network map form feasible linear network codes for a given n/w

Start with matroid on empty ground Set and an empty p-map

Extend matroids ensuring representability

Obtain p-codes for extension from p-codes of parent

Input: A N-variable MSNC problem and maximum ground set size N^*
Output: All pairs (M, ξ) such that M is a non-isomorphic (canonical) matroid with $N \leq E(M) \leq N^*$ and (M, ξ) is a feasible linear network code

$\mathscr{M}^0 \leftarrow \{$matroid M^ϕ on empty set$\}$
$\mathcal{Q}^M \leftarrow \{$ empty p-map$\}$
for $i \in \{1, \ldots, N^*\}$ **do**
  $\mathscr{M}^i \leftarrow$ see$(\mathscr{M}^{i-1}, q)$
  foreach $M \in \mathscr{M}^i$ **do**
    $M' \leftarrow$ parent matroid of M
    $\mathcal{Q}^M \leftarrow$ extendpmaps$(M, M', \mathcal{Q}^{M'})$
    if $\mathcal{Q}^M$ *is empty* **then**
      $\mathscr{M}^i \leftarrow \mathscr{M}^i \setminus M$
    end
    foreach $\xi \in \mathcal{Q}^M$ **do**
      if (M, ξ) *is a feasible linear network code* **then**
        Output (M, ξ)
      end
    end
  end
end
return

(P3) List all non-isomorphic matroids forming bounds $\Gamma_N^q \cap \mathcal{L}_{12345}$ and $\Gamma_{N,k,q}^{\subset space} \cap \mathcal{L}_{12345}$

The goal is to produce all canonical matroids of ground set size up to N* that together with some surjective matroid network map form feasible linear network codes for a given n/w

Start with matroid on empty ground Set and an empty p-map

Extend matroids ensuring representability

Obtain p-codes for extension from p-codes of parent

Remove matroids without any feasible p-codes

Input: A N-variable MSNC problem and maximum ground set size N^*
Output: All pairs (M, ξ) such that M is a non-isomorphic (canonical) matroid with $N \leq E(M) \leq N^*$ and (M, ξ) is a feasible linear network code

$\mathscr{M}^0 \leftarrow$ {matroid M^ϕ on empty set}
$\mathcal{Q}^M \leftarrow$ { empty p-map}
for $i \in \{1, \ldots, N^*\}$ **do**
  $\mathscr{M}^i \leftarrow$ see$(\mathscr{M}^{i-1}, q)$
  foreach $M \in \mathscr{M}^i$ **do**
    $M' \leftarrow$ parent matroid of M
    $\mathcal{Q}^M \leftarrow$ extendpmaps$(M, M', \mathcal{Q}^{M'})$
    if $\mathcal{Q}^M$ *is empty* **then**
      $\mathscr{M}^i \leftarrow \mathscr{M}^i \setminus M$
    end
    foreach $\xi \in \mathcal{Q}^M$ **do**
      if (M, ξ) *is a feasible linear network code* **then**
        Output (M, ξ)
      end
    end
  end
end
return

(P3) List all non-isomorphic matroids forming bounds $\Gamma_N^q \cap \mathcal{L}_{12345}$ and $\Gamma_{N,k,q}^{\subset space} \cap \mathcal{L}_{12345}$

The goal is to produce all canonical matroids of ground set size up to N* that together with some surjective matroid network map form feasible linear network codes for a given n/w

Start with matroid on empty ground Set and an empty p-map

Extend matroids ensuring representability

Obtain p-codes for extension from p-codes of parent

Remove matroids without any feasible p-codes

Output p-codes that map surjectively to $\mathcal{V}$

Input: A N-variable MSNC problem and maximum ground set size N^*
Output: All pairs (M, ξ) such that M is a non-isomorphic (canonical) matroid with $N \leq E(M) \leq N^*$ and (M, ξ) is a feasible linear network code

$\mathscr{M}^0 \leftarrow \{$matroid M^ϕ on empty set$\}$
$\mathcal{Q}^M \leftarrow \{$ empty p-map$\}$
for $i \in \{1, \ldots, N^*\}$ **do**
  $\mathscr{M}^i \leftarrow$ see$(\mathscr{M}^{i-1}, q)$
  foreach $M \in \mathscr{M}^i$ **do**
    $M' \leftarrow$ parent matroid of M
    $\mathcal{Q}^M \leftarrow$ extendpmaps$(M, M', \mathcal{Q}^{M'})$
    if $\mathcal{Q}^M$ *is empty* **then**
      $\mathscr{M}^i \leftarrow \mathscr{M}^i \setminus M$
    end
    foreach $\xi \in \mathcal{Q}^M$ **do**
      if (M, ξ) *is a feasible linear network code* **then**
        Output (M, ξ)
      end
    end
  end
end
return

Open Problems/Future work

- Enumerate only the extreme rays of $\Gamma_N^q \cap \mathcal{L}_{12345}$ and $\Gamma_{N,k,q}^{\subset space} \cap \mathcal{L}_{12345}$
- Find all almost entropic algebraic matroids satisfying network constraints of a given network
- Are there other minor closed families of almost entropic matroids?
- Nontrivial bound on number of non-isomorphic extensions of a matroid?*
- Complexity of computing all non-isomorphic extensions of a matroid*
- Complexity of computing all non-isomporphic extensions of a GF(q) representable matroid that are also representable over GF(q)*

**Thanks to Dr. Rudi Pendavingh (TU\e) for letting me know that these problems are in fact open*

Thanks for listening!

(or for coming here just in case I put you to sleep)

References

- [1] X. Yan, R.W. Yeung, and Zhen Zhang. An implicit characterization of the achievable rate region for acyclic multisource multisink network coding. IEEE Trans. on Inform. Theory, 58(9):5625–5639, 2012.
- [2] J. M. Walsh and S. Weber. Relationships among bounds for the region of entropic vectors in four variables. In 2010 Allerton Conference on Communication, Control, and Computing, September 2010.
- [3] Congduan Li, John MacLaren Walsh, Steven Weber. Matroid bounds on the region of entropic vectors. In 51th Annual Allerton Conference on Communication, Control and Computing, October 2013.
- [4] Congduan Li, J. Apte, J.M. Walsh, and S. Weber. A new computational approach for determining rate regions and optimal codes for coded networks. In Network Coding (NetCod), 2013 International Symposium on, pages 1–6, 2013.
- [5] J. G. Oxley. Matroid Theory. Oxford University, 2011.
- [6] Dillon Mayhew and Gordon F. Royle. Matroids with nine elements. Journal of Combinatorial Theory, Series B, 98(2):415 – 431, 2008.
- [7] Yoshitake Matsumoto, Sonoko Moriyama, Hiroshi Imai, and David Bremner. Matroid enumeration for incidence geometry. Discrete Comput. Geom., 47(1):17–43, January 2012.
- [8] John E. Blackburn, Henry H. Crapo, and Denis A. Higgs. A catalogue of combinatorial geometries. Mathematics of Computation, 27(121):pp. 155–166, 1973.
- [9] Rota's Conjecture: Researcher solves 40-year-old math problem. phys.org article posted Aug 15, 2013.
- [10] Gian-Carlo Rota. Combinatorial theory, old and new. In Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 3, pages 229–233. Gauthier-Villars, Paris, 1971.
- [11] Alexander Schrijver. Combinatorial optimization: polyhedra and efficiency, volume 24. Springer, 2003.
- [12] WT Tutte. Connectivity in matroids. Canad. J. Math, 18:1301–1324, 1966.