A computational approach for determining rate regions and codes using entropic vector bounds

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***Abstract*—A computational technique for determining rate regions for networks and multilevel diversity coding systems based on inner and outer bounds for the region of entropic vectors is discussed. An inner bound based on binary representable matroids is discussed that has the added benefit of identifying optimal linear codes. The technique is demonstrated on a series of small examples of multilevel diversity coding systems.**

I. Introduction

The core contribution of this paper is the collection of a series of techniques from the literature that can be used to determine key characteristics, the fundamental rate region, together with the field size and linear codes to achieve it, for multilevel diversity coding systems (MDCS) [1], [2], [3] and more broadly network coded networks [4]. The techniques are enabled by an expression of the fundamental rate region for these networks in terms of the closure of the region of entropic vectors [4]. These expressions are combined with computational inner and outer bounds for the closure of the region of entropic vectors $\bar{\Gamma}^*_N$ in order to obtain inner and outer bounds on the rate regions. The inner bounds to $\bar{\Gamma}^*_N$ are based on facts known about the representability of matroids over certain finite fields as well as inequalities known among ranks of subspaces [5], [6], [7], [8], [9]. Owing to this fact, when the inner bounds match with the outer bounds, both the rate region, and the codes necessary to achieve it are determined, for the representation of the associated matroid provides the necessary code.

The computational techniques are demonstrated on a series of 2 level 3 encoder multilevel diversity coding systems. In particular, following Hau [3], we exhaustively consider all such possible systems, and show the subset of which for which simple scalar linear binary codes suffice. The results are extensible using forbidden minor characterizations of matroids that are representable over other fields, as we discuss in the conclusions.

The paper proceeds by first reviewing the definition of the region of entropic vectors in §II, discussing properties of its closure and its best known outer bounds. Next inner bounds based on matroid theory and the ranks of subspaces are discussed in §III, with a particular focus on binary representability. The relationship between the regions of entropic vectors and the rate regions for multiple multicast network coding and multilevel diversity coding systems is discussed in §IV-A. The synthesis of these bounds and the rate region description into a complete computational technique for determine the rate regions is provided in §V. Finally, the technique is demonstrated on several small multilevel diversity coding systems in §VI.

II. The Region of Entropic Vectors $\bar{\Gamma}^*_N$ and the Shannon Outer Bound Γ_N

Let $(X_1, \dots, X_N)$ be an arbitrary collection of discrete random variables. To each of the $2^N - 1$ non-empty subsets of the collection of random variables, $X_\mathcal{A} := (X_i \,|\, i \in \mathcal{A})$ with $\mathcal{A} \subseteq \{1, \dots, N\}$, there is associated a joint Shannon entropy $H(X_\mathcal{A})$. Stacking these subset entropies for different subsets into a $2^N - 1$ dimensional vector we form an entropic vector

$$\mathbf{h} = [H(X_\mathcal{A}) \,|\, \mathcal{A} \subseteq \{1, \dots, N\}, \mathcal{A} \neq \emptyset] \tag{1}$$

By virtue of having been created in this manner, the vector $\mathbf{h}$ must live in some subset of $\mathbb{R}^{2^N-1}$, and is said to be *entropic*. In particular, this entropic vector is clearly a function $\mathbf{h}(p_{\boldsymbol{X}})$ of the joint probability mass function $p_{\boldsymbol{X}}$ for the random variables. Letting $\mathcal{P}_N$ denote the set of all such possible joint probability mass functions $p_{\boldsymbol{X}}$, we define $\Gamma^*_N := \mathbf{h}(\mathcal{P}_{\boldsymbol{X}})$, the region of entropic vectors, to be the image of $\mathcal{P}_N$ under this function $\mathbf{h}(\cdot)$. It is known that the closure of the region of entropic vectors $\bar{\Gamma}^*_N$ is a convex cone [4].

Next observe that viewed as a function of the subscript indices $\mathcal{A}$, $H(X_\mathcal{A})$ is a non-decreasing submodular function, so that $\forall \mathcal{A} \subseteq \mathcal{B} \subseteq [N]$ and $\forall \mathcal{C}, \mathcal{D} \subseteq [N]$

$$\begin{aligned} H(X_\mathcal{A}) &\leq H(X_\mathcal{B}) & (2)\\ H(X_{\mathcal{C}\cup\mathcal{D}}) + H(X_{\mathcal{C}\cap\mathcal{D}}) &\leq H(X_\mathcal{C}) + H(X_\mathcal{D}) & (3)\end{aligned}$$

Because they are true for any collection of subspace entropies, these linear inequalities (2,3) can be viewed as supporting halfspaces for Γ^*_N. Thus, the intersection of all such inequalities form a polyhedral outer bound Γ_N for Γ^*_N and $\bar{\Gamma}^*_N$, where

$$\Gamma_N := \left\{ \mathbf{h} \in \mathbb{R}^{2^N-1} \,\middle|\, \begin{array}{c} h_\mathcal{A} \leq h_\mathcal{B} \quad \forall \mathcal{A} \subseteq \mathcal{B} \\ h_{\mathcal{C}\cup\mathcal{D}} + h_{\mathcal{C}\cap\mathcal{D}} \leq h_\mathcal{C} + h_\mathcal{D} \ \forall \mathcal{C}, \mathcal{D} \end{array} \right\}$$

This outer bound Γ_N is known as the *Shannon outer bound*, as it can be thought of as the set of all inequalities resulting from the positivity of Shannon's information measures among the random variables.

While $\Gamma_2 = \Gamma_2^*$ and $\Gamma_3 = \bar{\Gamma}_3^*$, $\bar{\Gamma}_N^* \subsetneq \Gamma_N$ for all $N \geq 4$ [4], and indeed it is known [10] that $\bar{\Gamma}_N^*$ is not even polyhedral for $N \geq 4$. Through the remainder of the paper we will use the Shannon outer bound as our outer bound for $\bar{\Gamma}_N^*$. However, several authors [11], [12], [10], [13] have proven new non-Shannon inequalities forming tighter polyhedral outer bounds, and these could alternatively be substituted in the technique we will describe.

III. Binary Representable Matroid Inner Bound for the Region of Entropic Vectors

We now proceed to describe the inner bound on the closure of the region of entropic vectors, $\Gamma_N^{\text{bin}} \subseteq \bar{\Gamma}_N^*$, where Γ_N^{bin} is formed by taking the convex hull of the extreme rays of the Shannon outer bound Γ_N that equal (up to a scalar multiple) the rank function of a binary representable matroid. To fully explain and motivate these ideas, we first give a brief background on relevant portions of matroid theory in §III-A and rank functions for matroids, polymatroids, and subspaces §III-B. Then in §III-C we formally define the inner bound, prove a theorem by Hassibi important in its efficient computation, and apply the theorem to compute the inner bound.

A. Matroid theory

Matroid theory (c.f., [5]) is an abstract generalization of the familiar notion of independence in the context of linear algebra to the more general setting of set systems, i.e., collections of subsets of a ground set obeying certain axioms. The ground set of size N is without loss of generality $S = \{1, \ldots, N\} \equiv [N]$, and in our context each element of the ground set will correspond to a random variable as discussed in the previous section. There are numerous equivalent definitions of matroids, we present just one.

Definition 1. [5] A matroid M is an ordered pair $(S, \mathcal{I})$ consisting of a finite set S (the ground set) and a collection $\mathcal{I}$ (called independent sets) of subsets of S obeying:

1) Normalization: $\emptyset \in \mathcal{I}$;
2) Heredity: If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$;
3) Independence augmentation: If $I_1 \in \mathcal{I}$ and $I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there is an element $e \in I_2 - I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

Representable matroids are an important class of matroids which connect the independent sets to the conventional notion of independence in a vector space. See §III-B for r_M in the definition below.

Definition 2. A matroid M with ground set S of size $|S| = N$ and rank $r_M = r$ is representable over a field $\mathbb{F}$ if there exists a matrix $A \in \mathbb{F}^{r \times N}$ such that for each independent set $I \in \mathcal{I}$ the corresponding columns in A, viewed as vectors in $\mathbb{F}^r$, are linearly independent.

There has been significant effort towards characterizing the set of matroids that are representable over various field sizes, with a complete answer only available for fields of sizes two, three, and four. These characterizations are given in terms of lists of excluded minors, where a minor is a matroid that is obtained from another matroid by a pair of operations called deletion and contraction. We refer the interested reader to [5] for the definitions of these operations. A central result in matroid theory is the characterization of binary representable matroids due to Tutte.

Theorem 3. *(Tutte) A matroid M is binary representable (representable over a binary field) iff it does not have the matroid $U_{2,4}$ as a minor.*

Here, $U_{k,N}$ is the *uniform* matroid on the ground set $S = [N]$ with independent sets $\mathcal{I}$ equal to all subsets of $[N]$ of size at most k. For example, $U_{2,4}$ has as its independent sets

$$\mathcal{I} = \{\emptyset, 1, 2, 3, 4, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}. \tag{4}$$

In the following section we define the rank function of a matroid, and relate it to the rank function for a polymatroid and a collection of subspaces.

B. Rank functions for matroids, polymatroids, and subspaces

We clarify the relationships between the rank function for matroids, the rank function for polymatroids, and the dimension function for arrangements of subspaces

1) Matroid rank function: For a matroid $M = (S, \mathcal{I})$ with $|S| = N$ the rank function $r : 2^S \to \{0, \ldots, N\}$ is defined as the size of the largest independent set contained in each subset of S, i.e., $r(A) = \max_{B \subseteq A}\{|B| : B \in \mathcal{I}\}$. The rank of a matroid, r_M, is the rank of the ground set, $r_M = r(S)$. The rank function of a matroid can be shown to obey the following properties. In fact these properties may instead be viewed as an alternate definition of a matroid in that any vector obeying these axioms is the rank function of a matroid.

Definition 4. A set function $r : 2^S \to \{0, \ldots, N\}$ is a rank function of a matroid if it obeys the following axioms:

1) Cardinality: $r(A) \leq |A|$;
2) Monotonicity: if $A \subseteq B \subseteq S$ then $r(A) \leq r(B)$;
3) Submodularity: if $A, B \subseteq S$ then $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$.

The operations of deletion and contraction mentioned in the previous subsection yield new matroids with new rank functions. Specifically, let M/T denote the matroid obtained by contraction of M on $T \subset S$, and let $M \setminus T$ denote the matroid obtained by deletion from M of $T \subset S$. Then, by [5] (3.1.5,7)

$$\begin{array}{rcl} r_{M/T}(X) & = & r_M(X \cup T) - r_M(T) \\ r_{M \setminus T}(X) & = & r_M(X) \end{array}, \ \forall X \subseteq S - T. \tag{5}$$

2) Polymatroid rank function: A polymatroid (S, ρ) consists of a ground set $|S| = N$ and a rank function $\rho : 2^S \to \mathbb{R}_+$ obeying the following axioms

Definition 5. A set function $\rho : 2^S \to \mathbb{R}_+$ is a rank function of a polymatroid if it obeys the following axioms:

1) Normalization: $\rho(\emptyset) = 0$;

2) Monotonicity: if $A \subseteq B \subseteq S$ then $\rho(A) \leq \rho(B)$;
3) Submodularity: if $A, B \subseteq S$ then $\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B)$.

Observe the set of all possible matroid rank functions $r \in \mathbb{N}^{2^N}$ is countably infinite, and is a subset of the uncountably infinite set of all possible polymatroid rank functions. A polymatroid (S, ρ) determines a corresponding polyhedron in $\mathbb{R}_+^N$ (sometimes itself referred to as a polymatroid):

$$\mathcal{P}(S, \rho) = \left\{ x \in \mathbb{R}_+^N : \sum_{i \in A} x_i \leq \rho(A),\ \forall A \subseteq S \right\}, \quad (6)$$

i.e., the polyhedron formed by the intersection of all 2^N halfspaces defined by the rank function inequalities. Fujishige observed in 1978 [14] that the entropy function for a collection of random variables $(X_i, i \in [N])$ viewed as a set function is a polymatroid rank function.

Theorem 6. *(Fujishige) The Shannon outer bound Γ_N is the same as the region of all polymatroid rank functions on a ground set of size N.*

Formally, the entropy set function $h : 2^{[N]} \to \mathbb{R}_+$, where $h(A) = H(X_A)$ for each $A \subseteq [N]$ is the entropy of the random variables X_A, obeys the axioms in Def. 5.

3) Dimension function of subspace arrangements: Consider a collection of N subspaces $\mathcal{S} = (S_1, \ldots, S_N)$ of a finite dimensional vector space, and define the set function $\mathrm{d} : 2^{\mathcal{S}} \to \mathbb{N}_+$ where $\mathrm{d}(A) = \dim\left(\sum_{i \in A} S_i\right)$ for each $A \subseteq [N]$ is the dimension of the vector space generated by the union of subspaces indexed by A. It is known that for any collection of subspaces $\mathcal{S}$, the function d obeys the polymatroid axioms in Def. 5 (and of course is also integer valued), but it is also known that these axioms are necessary but insufficient. That is, there exist additional inequalities beyond those in Def. 5 that describe the convex hull of all possible subspace dimension set functions. The following theorem due to Ingleton (1971) [15] addresses the case of $N = 4$ subspaces:

Theorem 7. *(Ingleton). For $N = 4$ subspaces $\mathcal{S} = (S_1, \ldots, S_4)$ the following are necessary and sufficient conditions for a set function $\mathrm{d} : 2^{\mathcal{S}} \to \mathbb{N}$ to capture the dimension of all possible subsets of $\mathcal{S}$:*

1) *Normalization: $\mathrm{d}(\emptyset) = 0$;*
2) *Monotonicity: if $A \subseteq B \subseteq [N]$, then $\mathrm{d}(A) \leq \mathrm{d}(B)$;*
3) *Submodularity: if $A \subseteq B \subseteq [N]$, then $\mathrm{d}(A \cup B) + \mathrm{d}(A \cap B) \leq \mathrm{d}(A) + \mathrm{d}(B)$;*
4) *Ingleton's inequality:*

$$\begin{aligned} \mathrm{d}(1,2) + \mathrm{d}(1,3,4) + \mathrm{d}(3) + \mathrm{d}(4) + \mathrm{d}(2,3,4) \quad &\leq \\ \mathrm{d}(1,3) + \mathrm{d}(1,4) + \mathrm{d}(2,3) + \mathrm{d}(2,4) + \mathrm{d}(3,4), \quad &(7) \end{aligned}$$

where the inequality must hold for all relabelings of the four subspaces.

Recent work by Kinser [9] and Dougherty, Freiling, and Zeger [8] addresses the case of $N = 5$ subspaces. In particular, [8] establishes 24 additional inequalities that hold (besides the Ingleton inequalities) and prove this set is irreducible and complete in that all inequalities are necessary and no additional non-redundant inequalities exist. The question of characterizing additional inequalities that hold for $N \geq 6$ subspace arrangements, and establishing that those additional inequalities fully characterize the set of achievable dimension functions is open.

C. Binary representable matroid inner bound Γ_N^{bin}

We observed the equivalence between rank functions for polymatroids and the Shannon outer bound, Γ_N, as well as the fact that the convex hull of the set of rank functions for matroids forms an inner bound on Γ_N, since all matroid rank functions are themselves polymatroid rank functions. The binary representable matroid inner bound Γ_N^{bin} is defined as the convex hull of all extreme rays of the Shannon outer bound Γ_N that are $i)$ matroid rank functions, and $ii)$ binary representable.

Indeed, such extreme rays that are binary representable must be entropic. To see this, let the matrix associated with the binary representable matroid M for the selected extreme ray be $\mathbf{A}$ a $\mathrm{rank}(M) \times N$ matrix. Define the random variables $U_1, \ldots, U_{\mathrm{rank}(M)}$ binary independent uniform random variables, and define the random variables $(X_1, \ldots, X_N) = (U_1, \ldots, U_{\mathrm{rank}(M)})\mathbf{A}$. The random variables $X_1, \ldots, X_N$ so defined have the desired entropies.

To fix ideas, consider the extreme rays of the Shannon outer bound at $N = 4$, i.e., Γ_4, which is a polyhedral cone in $\mathbb{R}_+^{15}$ formed by the intersection of the 60 Shannon information inequalities for four variables. This cone has 41 extreme rays listed in Table I, where the following notation is employed:

1) *Bin* (26): a binary representable matroid rank function;
2) *IgV* (6): an extreme ray violating Ingleton's inequality;
3) $U_{2,4}$ (1): the rank function of the $U_{2,4}$ matroid;
4) *Pol* (8): a polymatroid rank function that is not *Bin*, *IgV*, or $U_{2,4}$.

Thus Γ_4^{bin} is the convex hull of the 26 binary representable matroid rank functions, i.e., the extreme rays of Γ_4 listed as *Bin*. This polyhedral cone may equivalently be represented as the intersection of 28 halfspaces (inequalities).

For $N > 4$, say for $N = 5$, the natural generalization of this idea is to enumerate the extreme rays of the Shannon outer bound on 5 variables (hard), or alternatively all rank functions of matroids on a ground set of size 5 (easier), and categorize them as above, and form the binary representable matroid inner bound as the convex hull of those extreme rays that represent binary representable matroid rank functions. Unfortunately this idea suffers from significant computational overhead. Hassibi *et al.* [6] pointed out that since the forbidden minor $U_{2,4}$ being checked for has ground set 4, and the rank function of a minor is expressible in terms of the original matroids rank function, one only needs to carry out this program at $N = 4$. Plugging the rank function of each minor into the polyhedra Γ_4^{bin} yields the binary representable matroid rank function inner bound for

N variables, Γ_N^{bin}, without requiring an enumeration of the extreme rays of Γ_N or the rank functions of all matroids on a ground set of size N. The theorem provides a necessary and sufficient condition for membership in Γ_N^{bin} by adding to the halfspace representation of Γ_N a collection of inequalities. These additional inequalities ensure that the rank function of all size four subsets of the N random variables, obtained by either deletion or contraction of the remaining variables, lie in the set of binary representable matroid rank functions for four variables.

Theorem 8. *(Hassibi) A rank function r_N in $\mathbb{R}_+^{2^N-1}$ lies in the polyhedral cone Γ_N^{bin} iff:*

1) $r_N \in \Gamma_N$;
2) *for every size 4 subset of indices of $[N]$, say $\{i,j,k,l\}$, for every $\mathcal{A} \subseteq [N] \setminus \{i,j,k,l\}$ (a subset of the remaining indices) the rank function $r_4(\mathcal{B}) := r_N(\mathcal{A} \cup \mathcal{B}) - r_N(\mathcal{A}),\ \mathcal{B} \subseteq \{i,j,k,l\}$, i.e. the rank function of the minor created by deleting $[N] \setminus (\mathcal{A} \cup \{i,j,k,l\})$ and contracting on $\mathcal{A}$, lies in Γ_4^{bin}.*

Hassibi [6] states the theorem without proof. We provide a proof below.

Proof: We only need to exclude $U_{2,4}$ minor for a vector to be binary representable, according to Theorem 3. So, we need to show that satisfying the two conditions is excluding $U_{2,4}$ minor. The first condition is trivial. We consider the second one. We know that a minor is obtained by a series operations of deletion and/or contraction. Thus, for a fixed 4 elements $i,j,k,l \in [N]$, the minor is obtained by contracting $\mathcal{A} \subseteq [N] \backslash \{i,j,k,l\}$, and deleting the remaining elements $[N] \backslash (\mathcal{A} \cup \{i,j,k,l\}$. The rank function r_4 in the theorem is the rank function associated with this minor. It is in the convex cone of the entropic vectors of four binary random variables if and only if it is not the $U_{2,4}$ rank function. ∎

The theorem yields a halfspace representation of the binary representable matroid inner bound Γ_N^{bin} as follows.

1) Generate the 56 inequalities for the binary representable matroid inner bound for $N = 4$ variables, Γ_4^{bin}.
2) Generate the polymatroid rank function inequalities (i.e., the Shannon information inequalities) for N variables.
3) Enumerate all subsets of four variables from the set of N variables, say $\{i,j,k,l\}$. For each such subset, for every subset $\mathcal{A}$ of the remaining elements $[N] \setminus \{i,j,k,l\}$ express the rank function r_4 obtained by *deleting* $[N] \setminus (\mathcal{A} \cup \{i,j,k,l\})$, then contracting on $\mathcal{A}$ (using (5)). Apply this rank function to each of the 56 inequalities required for membership in Γ_4^{bin} to obtain 56 new inequalities for r_N for each distinct size 4 subset and each subset of the remaining elements.

The result is the complete set of inequalities for membership in Γ_N^{bin}. Application of this procedure produced (after removing redundant inequalities) a collection of 570 inequalities for $N = 5$ and 3420 inequalities for $N = 6$ variables.

Table I
EXTREME RAYS FOR 4 VARIABLES

#	r_1	r_2	r_{12}	r_3	r_{13}	r_{23}	r_{123}	r_4	r_{14}	r_{24}	r_{124}	r_{34}	r_{134}	r_{234}	r_{1234}	Property
1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	Bin
2	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	Bin
3	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	Bin
4	0	1	1	0	0	1	1	1	1	1	1	1	1	1	1	Bin
5	1	0	1	0	1	0	1	1	1	1	1	1	1	1	1	Bin
6	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	Bin
7	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	Bin
8	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	Bin
9	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	Bin
10	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	Bin
11	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	Bin
12	1	1	2	1	2	2	2	2	2	2	2	2	2	2	2	Pol
13	1	1	2	1	2	2	2	1	2	2	2	2	2	2	2	$U_{2,4}$
14	2	1	2	1	2	2	2	1	2	2	2	2	2	2	2	Pol
15	1	2	2	1	2	2	2	1	2	2	2	2	2	2	2	Pol
16	1	1	1	1	2	2	2	1	2	2	2	2	2	2	2	Bin
17	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	Bin
18	1	1	2	1	2	1	2	1	2	2	2	2	2	2	2	Bin
19	1	1	2	1	2	2	3	2	3	3	3	3	3	3	3	Pol
20	1	1	2	2	2	2	2	1	2	2	2	2	2	2	2	Pol
21	1	1	2	1	1	2	2	1	2	2	2	2	2	2	2	Bin
22	0	1	1	1	1	1	1	0	0	1	1	1	1	1	1	Bin
23	1	1	2	1	2	2	2	1	1	2	2	2	2	2	2	Bin
24	0	1	1	1	1	2	2	1	1	2	2	2	2	2	2	Bin
25	1	0	1	1	1	1	1	0	1	0	1	1	1	1	1	Bin
26	1	1	2	1	2	2	2	1	2	1	2	2	2	2	2	Bin
27	1	0	1	1	2	1	2	1	2	1	2	2	2	2	2	Bin
28	2	2	3	2	3	3	4	2	3	3	4	4	4	4	4	IgV
29	1	1	2	2	3	3	3	1	2	2	3	3	3	3	3	Pol
30	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	Bin
31	1	1	2	1	2	2	2	1	2	2	2	1	2	2	2	Bin
32	1	1	2	0	1	1	2	1	2	2	2	1	2	2	2	Bin
33	2	2	3	2	3	3	4	2	3	4	4	3	4	4	4	IgV
34	1	2	3	1	2	3	3	1	2	3	3	2	3	3	3	Pol
35	2	1	3	1	3	2	3	1	3	2	3	2	3	3	3	Pol
36	2	2	3	2	3	3	4	2	4	3	4	3	4	4	4	IgV
37	1	1	2	1	2	2	2	0	1	1	2	1	2	2	2	Bin
38	2	2	3	2	3	4	4	2	3	3	4	3	4	4	4	IgV
39	2	2	3	2	4	3	4	2	3	3	4	3	4	4	4	IgV
40	1	1	2	1	2	2	3	1	2	2	3	2	3	3	3	Bin
41	2	2	4	2	3	3	4	2	3	3	4	3	4	4	4	IgV

IV. Implicit Rate Regions from Γ_N^*

Multiple important rate regions in multi-terminal information theory can be directly expressed in terms of the region of entropic vectors. For completeness we review these expressions for multilevel diversity coding systems and multiple multicast network coding in this section. Since these regions are expressed in terms of Γ_N^* and close variants, we can use the inner and outer bounds for it reviewed in the previous sections to obtain inner and outer bounds on the rate regions, as well as other insights, which we will detail in the next section.

A. Multilevel Diversity Coding Systems

Multilevel diversity coding was introduced by Roche [1] and Yeung [2]. In a multilevel diversity coding system (MDCS), there are K *independent* sources $X_{1:K} \equiv (X_1, \ldots, X_K)$ where source k takes values in (has support) $\mathcal{X}_k$, and the sources are prioritized into K levels with X_1 (X_K) the highest (lowest) priority source, respectively. As is standard in source coding, each source X_k is in fact a sequence of random variables $\{X_k^t, t = 1, 2, \ldots\}$ i.i.d. in t, so that X_k is a representative random variable with this distribution.

All sources are made available to each of a collection of encoders indexed by the set E, the outputs of which are description/message variables $U_e, e \in E$. The message variables are mapped to a collection of decoders index by the set D that are classified into K levels, where a level k decoder must losslessly (in the typical Shannon sense) recover source variables $X_{1:k} = (X_1, \ldots, X_k)$, for each $k \in \{1, \ldots, K\}$. The mapping of encoders to decoders dictates the description variables that are available for each decoder in this recovery. Thus, different level k decoders must recover the same subset of source variables using possibly distinct subsets of description variables. Hau [3] enumerated 31 distinct configurations mapping descriptions to decoders for the case of $K = 2$ levels and $E = 3$ encoders, and 69 distinct configurations mapping descriptions to decoders for the case of $K = 3$ levels and $E = 3$ encoders, after symmetries are removed.

Many applications of this class of coding systems have been studied, such as distributed information storage[1], secret sharing [16], satellite communications [17], and peer-to-peer (P2P) networks [18].

1) Admissible Coding Rate Region: Let $\mathbf{R} = (R_1, \ldots, R_{|E|}) \in \mathbb{R}_+^E$ be the rate vector for the encoders, defined as follows. For each blocklength $n \in \mathbb{N}$ we consider a collection of E block encoders, where encoder e maps each block of n variables from each of the K sources to one of 2^{nR_e} different descriptions:

$$f_e^{(n)} : \prod_{i=1}^{K} \mathcal{X}_k^n \to \{1, \ldots, 2^{nR_e}\},\ e \in E. \tag{8}$$

The encoder outputs are indicated by $U_e = f_e^{(n)}(X_{1:K}^{1:n})$ for $e \in E$. The D decoders are characterized by their priority level and their available descriptions. Specifically, decoder j has an associated priority level $k \in [K]$ and an available subset of the descriptions, called fan of decoder j, in particular those with encoder indices $\mathcal{E}_j \subseteq E$ (i.e., decoder d has $\mathcal{U}_d = (U_e, e \in \mathcal{E}_d)$), and must asymptotically losslessly recover source variables $X_{1:k}$:

$$g_d^{(n)} : \prod_{e \in \mathcal{E}_d} \{1, \ldots, 2^{nR_e}\} \to \prod_{i=1}^{k} \mathcal{X}_k^n,\ d \in D. \tag{9}$$

The rate region $\mathcal{R}$ for the configuration specified by K, E, and the encoder to decoder mappings $(\mathcal{E}_j,\ j \in [D])$ consists of all rate vectors $\mathbf{R} = (R_1, \ldots, R_{|E|})$ such that there exist sequences of encoders $f^n = (f_e^n, e \in E)$ and decoders $g^n = (g_d^n, d \in D)$ for which the asymptotic probability of error goes to zero in n at all decoders. Specifically, define the probability of error for each level k decoder d as

$$p_{d,k}^{n,err}(\mathbf{R}) = \mathbb{P}(g_d(f_e(X_{1:K}^{1:n}), e \in \mathcal{E}_d) \neq X_{1:k}^{1:n}), \tag{10}$$

and the maximum over these as

$$p^{n,err}(\mathbf{R}) = \max_{k \in K} \max_{d \in \mathcal{D}_k} p_{d,k}^{n,err}, \tag{11}$$

where $\mathcal{D}_k \subseteq D$ is the set of level k decoders. A rate vector is in the rate region, $\mathbf{R} \in \mathcal{R}$, provided there exists $\{f_e^n\}$ and $\{g_d^n\}$ such that $p^{n,err}(\mathbf{R}) \to 0$ as $n \to \infty$.

The rate region $\mathcal{R}$ can be expressed in terms of the region of entropic vectors, Γ_N^* [4]. For the MDCS problem, collect random variables $\mathcal{N} = \{Y_k, k \in [K]\} \cup (U_e, e \in E\}, |\mathcal{N}| = N$, where $Y_k, k \in [K]$ are auxiliary random variables associated with $X_k, k \in [K]$. The inner bound $\mathcal{R}_{\text{in}}$ on region $\mathcal{R}$ is the set of rate vectors $\mathbf{R}$ such that there exists $\mathbf{h} \in \Gamma_N^*$ satisfying the following (see [17])

$$\begin{aligned} h_{Y_k} &> H(X_k), k \in [K] & (12)\\ h_{Y_{1:K}} &= \sum_{k=1}^{K} h_{X_k} & (13)\\ h_{U_e|Y_{1:K}} &= 0, e \in E & (14)\\ h_{Y_{1:k}|\mathcal{U}_d} &= 0,\ D_d \in \mathcal{D}_k & (15)\\ R_e &\geq h_{U_e}, e \in [E] & (16) \end{aligned}$$

The outer bound $\mathcal{R}_{\text{out}}$ is the set of rate vectors $\mathbf{R}$ such that there exists $\mathbf{h} \in \bar{\Gamma}_N^*$ satisfying equations (13) to (16) and the following inequality

$$h_{Y_k} \geq H(X_k), k \in [K] \tag{17}$$

where the conditional entropies $h_{A|B}$ are naturally equivalent to $h_{AB} - h_B$. These constraints can be interpreted as following: (13) represents that sources are independent; (14) represents that each description is a function of all sources available; (15) represents that the recovered source messages at a decoder are a function of the input descriptions available to it; and (16) represents the rate constraints.

Define the sets

$$\begin{aligned} \mathcal{L}_0 &= \{\mathbf{h} \in \mathbb{R}_+^{2^N-1} : h_{Y_k} > H(X_k), k \in [K]\} & (18)\\ \mathcal{L}_0' &= \{\mathbf{h} \in \mathbb{R}_+^{2^N-1} : h_{Y_k} \geq H(X_k), k \in [K]\} & (19)\\ \mathcal{L}_1 &= \{\mathbf{h} \in \mathbb{R}_+^{2^N-1} : h_{X_i, i \in [K]} = \sum_{j \in [K]} h_{X_j}\} & (20)\\ \mathcal{L}_2 &= \{\mathbf{h} \in \mathbb{R}_+^{2^N-1} : h_{U_e|X_{1:K}} = 0, e \in [E]\} & (21)\\ \mathcal{L}_3 &= \{\mathbf{h} \in \mathbb{R}_+^{2^N-1} : h_{X_{1:k}|\mathcal{U}_d} = 0, D_d \in \mathcal{D}_k\} & (22)\\ \mathcal{L}_4 &= \{\mathbf{h} \in \mathbb{R}_+^{2^N-1} : R_e \geq h_{U_e}, e \in [E]\}. & (23) \end{aligned}$$

Note that the sets $\mathcal{L}_0$ and $\mathcal{L}_0'$ are corresponding to the constraints (12) and (17), sets $\mathcal{L}_1, \ldots \mathcal{L}_4$ are corresponding to the constraints (13), (14), (15), (16), respectively. Then we see that

$$\mathcal{R}_{\text{in}} = \overline{\text{Ex}(\text{proj}_{h_{U_e, e \in [E]}}(\Gamma_N^* \cap \mathcal{L}_{0123}))} \tag{24}$$

and

$$\mathcal{R}_{\text{out}} = \text{Ex}(\text{proj}_{h_{U_e, e \in [E]}}(\bar{\Gamma}_N^* \cap \mathcal{L}_{123} \cap \mathcal{L}_0')) \tag{25}$$

where $\mathcal{L}_{0123} = \cap_{i=0}^{3} \mathcal{L}_i$, $\text{proj}_{h_{U_e, e \in [E]}}(\mathcal{B})$ is the projection of the set $\mathcal{B}$ on the coordinates $(h_{U_e} | e \in [E])$, and $\text{Ex}(\mathcal{B}) = \{\mathbf{h} \in \mathbb{R}_+^{2^N-1} : \mathbf{h} \geq \mathbf{h}' \text{ for some } \mathbf{h}' \in \mathcal{B}\}$, for $\mathcal{B} \subset \mathbb{R}_+^{2^N-1}$.

2) Results from Hau's thesis: It is shown in [3] that for all the 100 non-symmetrical 3-encoder MDCS configurations, all rate regions are achievable using linear coding of inputs and 86 out of them (29 out of 31 for 2-level-3-encoder and 57 out of 69 for 3-level-3-encoder) are achievable using superposition coding [2]. The rest can be achieved by linear combination or a hybrid of linear combination and superposition. When using superposition coding, the data streams are encoded separately and the output from an encoder is just the concatenation of those separated codewords. In this manner, each encoder can be viewed as a combination of several sub-encoders, and thus the coding rate of an encoder is the sum of coding rate of each sub-encoder.

B. Multiple Multicast Network Coding

The capacity region of network coding on a direct acyclic graph network with multiple multicasts can be directly expressed in terms of the region of entropic vectors with a similar series of linear constraints to those in the previous section. We will demonstrate the inner and outer bounding techniques in paper on MDCSs, however, the techniques are directly extensible to the network coding capacity region as well. The interested reader is referred to [4] for the expression for the associated rate region.

V. Computational Technique to Calculate Rate Regions and their Codes Via the Bounds $\Gamma_N, \Gamma_N^{\mathrm{BIN}}$

As shown in section IV-A1, the rate region for multilevel diversity coding systems, and more broadly multiple multicast network coding, can be expressed directly in terms of the region of entropic vectors. Since $\bar{\Gamma}_N^*$ is yet unknown for $N \geq 4$, to approach the goal of completely computing these rate regions, one must replace Γ_N^* with its inner and outer bounds. The polyhedral nature of Γ_N^{bin} and Γ_N, allow for a convenient computational technique for computing rate regions as follows.

Using Γ_N^{bin} in place of Γ_N^{**} yields an inner bound $\mathcal{R}_{\mathrm{in}} \subseteq \mathcal{R}$ on the achievable rate region $\mathcal{R}$, while using Γ_N in place of Γ_N^* yields an outer bound $\mathcal{R}_{\mathrm{out}} \supseteq \mathcal{R}$. If the expressions inner and outer bounds match $\mathcal{R}_{\mathrm{in}} = \mathcal{R}_{\mathrm{out}}$, then one has determined the rate region. Additionally, the matrix representation of the binary representable matroid associated with any extreme ray of Γ_N^{bin} chosen by selecting an extreme point of $\mathcal{R}$ gives a binary linear code for achieving the rate region at this extreme point.

The automated computational technique for calculating rate regions operates as follows:

1) Replace Γ_N^* by Γ_N^{bin} in (24) in order to obtain a polyhedron $\Gamma_N^{\mathrm{bin}} \cap \mathcal{L}_{123} \cap \mathcal{L}_0'$.
2) Using Fourier Motzkin or any equivalent alternative, eliminate all variables in the polyhedron $\Gamma_N^{\mathrm{bin}} \cap \mathcal{L}_{123} \cap \mathcal{L}_0'$ (i.e. all subset entropies) except for $R_e, e \in E$ and $H(X_k), k \in [K]$. The resulting polyhedron in $R_e, e \in E$ and $H(X_k), k \in [K]$ is the inner bound $\mathcal{R}_{in}$.
3) Replace Γ_N^* by Γ_N in (25) in order to obtain a polyhedron $\Gamma_N \cap \mathcal{L}_{0123}$.

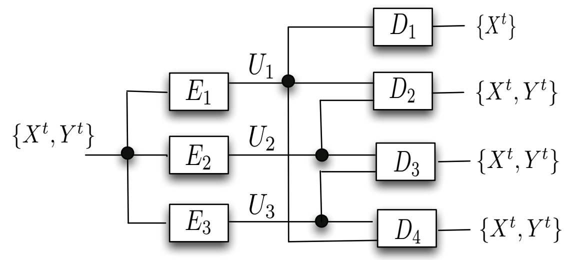

Figure 1. A 2-level-3-encoder MDCS: case 8

4) Using Fourier Motzkin or any equivalent alternative, eliminate all variables in the polyhedron $\Gamma_N \cap \mathcal{L}_{0123}$ (i.e. all subset entropies) except for $R_e, e \in E$ and $H(X_k), k \in [K]$. The resulting polyhedron in $R_e, e \in E$ and $H(X_k), k \in [K]$ is the outer bound $\mathcal{R}_{out}$.
5) Compare $\mathcal{R}_{\mathrm{in}}$ with $\mathcal{R}_{\mathrm{out}}$. If they are equal, they give $\mathcal{R}$. Additionally, the matrix representation of the binary representable matroid associated with any extreme ray of Γ_N^{bin} chosen by selecting an extreme point of $\mathcal{R}$ gives a binary linear code for achieving the rate region at this extreme point.

We have written software to implement this routine [19], and will demonstrate the results from using it on the MDCSs in Hau's thesis in the next section.

VI. Demonstration & Results

In this section, we demonstrate the computational technique for calculating rate regions and efficient codes described in §V by applying it to some small multilevel diversity coding systems. In this paper, as in [2], [3], we restrict our attention to $K = 2$ and $K = 3$ sources, and label the source variables as (X, Y) (for two levels) and (X, Y, Z) (for three levels). Hau [3] did not give all rate regions for all configurations and only gave one converse proof of the rate regions. The software we developed is able to generate achievable rate regions (or in some cases bounds) automatically and thus gives converse proofs.

1) 2-level-3-encoder: As mentioned above, after excluding permutation and symmetrical constructions, there are 31 configurations in total for 2-level-3-encoder MDCS, as shown in Table II. In the table, (*) indicates that superposition is not optimal for this configuration. The fan of a decoder is grouped in parentheses ( ) and each index is separated by a comma. The decoders that lie in the same level are grouped in braces { } and each grouped decoder is also separated by a comma. Level 1 decoders can reconstruct X and level 2 decoders can reconstruct X, Y.

Take case 8 for example (see Table II), it has configuration

level 1	level 2
{(1)}	{(1, 2), (1, 3), (2, 3)}

and the corresponding diagram is shown in Figure 1. The three encoded descriptions are noted as U_1, U_2 and U_3 and we use R_1, R_2, R_3 to denote the coding rates for the three encoders, respectively.

Table II
ALL 31 CASES FOR THE 2-LEVEL-3-ENCODER MDCS

Case	level 1 (X)	level 2 (XY)	Match
1	{(1)}	{(2)}	Y
2	{(1)}	{(2), (3)}	Y
3	{(1)}	{(1, 2)}	Y
4	{(1)}	{(2, 3)}	Y
5	{(1)}	{(1, 2), (3)}	Y
6	{(1)}	{(1, 2), (2, 3)}	Y
7	{(1)}	{(1, 2), (1, 3)}	Y
8*	{(1)}	{(1, 2), (1, 3), (2, 3)}	N
9	{(1)}	{(1, 2, 3)}	Y
10	{(1), (2)}	{(3)}	Y
11	{(1), (2)}	{(1, 2)}	Y
12	{(1), (2)}	{(2, 3)}	Y
13	{(1), (2)}	{(1, 2), (3)}	Y
14	{(1), (2)}	{(1, 2), (2, 3)}	Y
15	{(1), (2)}	{(1, 3), (2, 3)}	Y
16	{(1), (2)}	{(1, 2), (1, 3), (2, 3)}	N
17	{(1), (2)}	{(1, 2, 3)}	Y
18	{(1), (2), (3)}	{(1, 2)}	Y
19	{(1), (2), (3)}	{(1, 2), (2, 3)}	Y
20	{(1), (2), (3)}	{(1, 2), (1, 3), (2, 3)}	N
21	{(1), (2), (3)}	{(1, 2, 3)}	Y
22	{(1, 2)}	{(3)}	Y
23	{(1, 2)}	{(2, 3)}	Y
24	{(1, 2)}	{(1, 3), (2, 3)}	N
25	{(1, 2)}	{(1, 2, 3)}	Y
26	{(1), (2, 3)}	{(1, 2)}	Y
27*	{(1), (2, 3)}	{(1, 2), (1, 3)}	Y
28	{(1), (2, 3)}	{(1, 2, 3)}	Y
29	{(1, 2), (2, 3)}	{(1, 3)}	N
30	{(1, 2), (2, 3)}	{(1, 2, 3)}	Y
31	{(1, 2), (1, 3), (2, 3)}	{(1, 2, 3)}	N

We ran the technique discussed in §V on each of the cases in Table II. Comparing Shannon outer bound and binary representable inner bound for all 31 cases, we found only 6 cases where the Shannon outer bound and binary inner bound do not match. The cases are: 8, 16, 20, 24, 29 and 31. Table II also indicates the cases where the two bounds match. Also, we list the 6 cases where Shannon outer bound and binary representable inner bound do not match in Table III with explicit polyhedron inequality representations, where the differences between the two bounds are highlighted in bold.

2) 3-level-3-encoder: In 3-level-3-encoder MDCS, there are 3 source streams, X, Y and Z. Decoders lie in level 1 reconstruct X, level 2 reconstruct X, Y and level 3 reconstruct X, Y, Z. There are 69 non-symmetrical cases in total for 3-level-3-encoder MDCS. The full list of all the configurations are omitted here because of space limitations. Readers are referred to [3] from page 32-34, if interested.

Also take case 8 as an example in 3-level-3-encoder MDCS. It has configuration

level 1	level 2	level 3
{(1)}	{(2)}	{(1, 2), (1, 3), (2, 3)}

and the corresponding diagram is shown in Figure 2. Note that case 8 of 2-level-3-encoder MDCS is embedded in case 8 of 3-level-3-encoder MDCS by superimposing stream X onto a 2-level-3-encoder MDCS for streams Y, Z. Therefore, if the Shannon outer bound and binary inner bound for the former configuration do not match, they will not match in latter case, either. The inner and outer bound obtained from applying the rate region bounding technique to this problem is shown in Table IV, where the extra inequality for the inner bound is in bold.

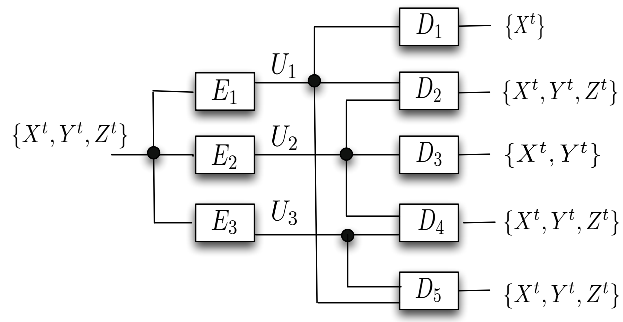

Figure 2. A 3-level-3-encoder MDCS: case 8

VII. Conclusions and Future Work

In this paper we demonstrated how to use inner bounds on the region of entropic vectors associated with representable matroids to obtain rate region expressions and optimal codes for multilevel diversity coding systems. The results are easily extensible to networks under network coding for multiple multicast. While the techniques focussed on binary representable matroids, similar excluded minor characterizations for representability for other classes of matroids can be applied to get other inner bounds. An additional promising direction we are looking into is to obtain bounds based on the ranks of subspaces by grouping variables in the representable matroid inner bounds. Future work will apply the developed computational techniques to a collection of small networks and look for characteristics when the outer bound created with the Shannon outer bound (known as the LP outer bound in this context) is tight.

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Table III
SIX CASES WHERE BINARY INNER BOUND AND SHANNON OUTER BOUND DO NOT MATCH. THE INEQUALITIES WHERE OUTER BOUND AND INNER BOUND DO NOT MATCH ARE IN BOLD.

Case	Outer Bound	Inner Bound
8*	$R_1 \geq H(X)$	$R_1 \geq H(X)$
	$R_1 + R_2 \geq H(X) + H(Y)$	$R_1 + R_2 \geq H(X) + H(Y)$
	$R_1 + R_3 \geq H(X) + H(Y)$	$R_1 + R_3 \geq H(X) + H(Y)$
	$R_2 + R_3 \geq H(X) + H(Y)$	$R_2 + R_3 \geq H(X) + H(Y)$
		$\mathbf{R_1 + R_2 + R_3} \geq \mathbf{H(X) + 2H(Y)}$
16	$R_1 \geq H(X)$	$R_1 \geq H(X)$
	$R_2 \geq H(X)$	$R_2 \geq H(X)$
	$R_1 + R_2 \geq 2H(X) + H(Y)$	$R_1 + R_2 \geq 2H(X) + H(Y)$
	$R_1 + R_3 \geq H(X) + H(Y)$	$R_1 + R_3 \geq H(X) + H(Y)$
	$R_2 + R_3 \geq H(X) + H(Y)$	$R_2 + R_3 \geq H(X) + H(Y)$
		$\mathbf{R_1 + R_2 + R_3} \geq \mathbf{2H(X) + 2H(Y)}$
20	$R_1 \geq H(X)$	$R_1 \geq H(X)$
	$R_2 \geq H(X)$	$R_2 \geq H(X)$
	$R_3 \geq H(X)$	$R_3 \geq H(X)$
	$R_1 + R_2 \geq 2H(X) + H(Y)$	$R_1 + R_2 \geq 2H(X) + H(Y)$
	$R_1 + R_3 \geq 2H(X) + H(Y)$	$R_1 + R_3 \geq 2H(X) + H(Y)$
	$R_2 + R_3 \geq 2H(X) + H(Y)$	$R_2 + R_3 \geq 2H(X) + H(Y)$
		$\mathbf{R_1 + R_2 + R_3} \geq \mathbf{3H(X) + 2H(Y)}$
24	$R_1 + R_2 \geq H(X)$	$R_1 + R_2 \geq H(X)$
	$R_1 + R_3 \geq H(X) + H(Y)$	$R_1 + R_3 \geq H(X) + H(Y)$
	$R_2 + R_3 \geq H(X) + H(Y)$	$R_2 + R_3 \geq H(X) + H(Y)$
		$\mathbf{R_1 + R_2 + R_3} \geq \mathbf{2H(X) + H(Y)}$
29	$R_1 + R_2 \geq H(X)$	$R_1 + R_2 \geq H(X)$
	$R_1 + R_3 \geq H(X) + H(Y)$	$R_1 + R_3 \geq H(X) + H(Y)$
	$R_2 + R_3 \geq H(X)$	$R_2 + R_3 \geq H(X)$
	$\mathbf{R_1 + 2R_2 + R_3} \geq \mathbf{2H(X) + H(Y)}$	$\mathbf{R_1 + R_2 + R_3} \geq \mathbf{2H(X) + H(Y)}$
31	$R_1 + R_2 \geq H(X)$	$R_1 + R_2 \geq H(X)$
	$R_1 + R_3 \geq H(X)$	$R_1 + R_3 \geq H(X)$
	$R_2 + R_3 \geq H(X)$	$R_2 + R_3 \geq H(X)$
	$\mathbf{R_1 + R_2 + 2R_3} \geq \mathbf{2H(X) + H(Y)}$	$\mathbf{R_1 + R_2 + R_3} \geq \mathbf{2H(X) + H(Y)}$
	$\mathbf{R_1 + 2R_2 + 2R_3} \geq \mathbf{2H(X) + H(Y)}$	
	$\mathbf{2R_1 + R_2 + R_3} \geq \mathbf{2H(X) + H(Y)}$	
	$\mathbf{3R_1 + 2R_2 + 2R_3} \geq \mathbf{3H(X) + 2H(Y)}$	

Table IV
COMPARISON OF SHANNON OUTER BOUND AND BINARY REPRESENTABLE INNER BOUND FOR CASE 8 OF 3-LEVEL-3-ENCODER MDCS

Outer Bound	Inner Bound
$R_1 \geq H(X)$	$R_1 \geq H(X)$
$R_2 \geq H(X) + H(Y)$	$R_2 \geq H(X) + H(Y)$
$R_1 + R_2 \geq 2H(X) + H(Y) + H(Z)$	$R_1 + R_2 \geq 2H(X) + H(Y) + H(Z)$
$R_1 + R_3 \geq H(X) + H(Y) + H(Z)$	$R_1 + R_3 \geq H(X) + H(Y) + H(Z)$
$R_2 + R_3 \geq H(X) + H(Y) + H(Z)$	$R_2 + R_3 \geq H(X) + H(Y) + H(Z)$
	$\mathbf{R_1 + R_2 + R_3} \geq \mathbf{2H(X) + H(Y) + 2H(Z)}$